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An equivalent linearization method for nonlinear systems under nonstationary random excitations using orthogonal functions

Amir Younespour*[1,2], Shaohong Cheng[2a] and Hosein Ghaffarzadeh[1b]

[1]*Department of Civil and Environmental Engineering, University of Tabriz, Tabriz, Iran*
[2]*Department of Civil and Environmental Engineering, University of Windsor, Windsor, Ontario, Canada*

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Abstract. Many practical engineering problems are associated with nonlinear systems subjected to nonstationary random excitations. Equivalent linearization methods are commonly used to seek for approximate solutions to this kind of problems. Compared to various approaches developed in the frequency and mixed time-frequency domains, though directly solving the system equation of motion in the time domain would improve computation efficiency, only limited studies are available. Considering the fact that the orthogonal functions have been widely used to effectively improve the accuracy of the approximated responses and reduce the computational cost in various engineering applications, an orthogonal-function-based equivalent linearization method in the time domain has been proposed in the current paper for nonlinear systems subjected to nonstationary random excitations. In the numerical examples, the proposed approach is applied to a SDOF system with a set-up spring and a SDOF Duffing oscillator subjected to stationary and nonstationary excitations. In addition, its applicability to nonlinear MDOF systems is examined by a 3DOF Duffing system subjected to nonstationary excitation. Results show that the proposed method can accurately predict the nonlinear system response and the formulation of the proposed approach allows it to be capable of handling any general type of nonstationary random excitations, such as the seismic load.

Keywords: equivalent linearization; nonstationary excitation; orthogonal functions; nonlinearity; random vibration

1. Introduction

Dynamic excitations originated from natural phenomena, such as earthquake and wind, are intrinsically nondeterministic. Thus, they are often modeled as random processes. Although linear dynamic problems can typically be solved using standard analytical approaches, of which the system response moments and statistics can be calculated in both time and frequency domains, in many practical applications, the system of concern would manifest nonlinear behavior, such as a system subjected to seismic load. Traditional analytical approaches are incapable of handling this kind of problems. A number of effective methods, such as the perturbation method (Crandall 1963), the Fokker-Planck-Kolmogorov (FPK) equation method (Zhu 2006), the moment equation method (Crandall 1980), the equivalent linearization method (Caughey 1956) and the Monte Carlo simulation method (Proppe, Pradlwarter *et al.* 2003), have been developed to conduct random vibration analysis of nonlinear systems. Unlike the equivalent linearization (EL) method and the Monte Carlo (MC) simulation method, the other three approaches have limitations in dealing with nonstationary excitation problems and relatively large scale multi-degree-of-freedom (MDOF) systems. Besides, the MC simulation method involves a large number of sample tests and is computationally very expensive. Hence, it is often used as a benchmark for other methods. The EL method does not have these restrictions and is applicable to a wide range of problems. In general, this technique consists of two main steps. The first step requires finding analytical formulas for linearization coefficients, which is based on the linearization criterion dependent of unknown response statistical terms such as variance and higher-order moments. Then, the actual set of nonlinear equations is replaced by an equivalent set of linear equations in the second step. It is important to bear in mind that the accuracy and feasibility of these solutions depend on the type of nonlinearity and the amplitude of the external excitation forces. In the EL method, the coefficients of the equivalent linear system can be found from a specified optimization criterion, such as the mean-square criterion (Caughey 1956), the spectral criteria (Apetaur and Opička 1983), the probability density criteria (Socha 1998) and the energy criteria (Zhang 2000), in some probabilistic sense.

Most of the real excitations are inherently nonstationary. The complexity of the problem would increase with the consideration of this practical aspect. Although nonstationary problems attract much interests, there are limited studies available on the analysis of nonstationary responses of nonlinear systems. Chaudhuri and Chakraborty (2004) evaluated the sensitivity of the structural response in the frequency domain when the structures were subjected to nonstationary seismic excitations. Garrè and Kiureghian

*Corresponding author, Ph.D. Candidate
E-mail: younespour@tabrizu.ac.ir
[a]Professor
E-mail: shaohong@uwindsor.ca
[b]Associate Professor
E-mail: ghaffar@tabrizu.ac.ir

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(2010) proposed an extension of the tail-equivalent linearization method to the frequency domain, which was particularly suitable when the input and the response processes were stationary. A comparison study between the time and frequency domain approaches can be found in Doughty *et al.* (2002). Ma *et al.* (2011) used the pseudo-excitation method to obtain the solution to the nonstationary random responses of MDOF nonlinear systems. In addition, based on the pseudo-excitation method, a new approach for the first and the second order sensitivity analysis of nonstationary random responses and evolutionary power spectral density functions was proposed (Xu *et al.* 2011, Liu 2012). However, when nonstationary random excitations are involved, numerical integrations in both time and frequency domains are needed in these methods, so the computational cost would be substantial. Besides, the existing analytical studies on nonlinear systems subjected to nonstationary random excitations were mainly dedicated to the development of approximate methods, since the exact and accurate approach requires significantly more computational efforts, which, sometimes is not available. Therefore, it is necessary to seek approximate solutions for nonlinear systems, such as using the stochastic equivalent linearization (EL) method. For nonlinear systems under nonstationary random excitations, though the EL method is a simple approach from the theoretical point of view, the implementation by numerical techniques needs lots of computational efforts. Time domain EL methods have much higher computational efficiency than the existing frequency domain and mixed time-frequency domain methods. Orabi and Ahmadi (1987) proposed a time-dependent method for a single degree-of-freedom (SDOF) nonlinear system under nonstationary excitations by the equivalent linearization method. An explicit time domain approach (Su and Xu 2014) was developed and applied to different random vibration problems of linear structures under nonstationary excitations. Hu *et al.* (2016) used direct differentiation method to analyze the sensitivity of nonstationary random vibration problem in the time domain. Su *et al.* (2016) further extended their earlier work in Su and Xu (2014) and proposed an efficient approach in time domain for the random vibration analysis of nonlinear MDOF structures subjected to nonstationary random excitations by combining the time-domain explicit formulation method and the EL method. However, it is worthy pointing out that many of the existing methods only focus on certain types of external excitations. For example, the approach proposed by Orabi and Ahmadi (1987) applies specifically to a nonlinear SDOF system subjected to stationary white noise excitation in the form of unit step envelope or exponential envelope functions, as well as nonstationary excitation of seismic ground motion defined by a model proposed in (Bogdanoff *et al.* 1961). Besides, although the time domain EL approach proposed by Su *et al.* (2016) manifested high computational efficiency via state space transformation of the equation of motion prior to its solution, its formulation requires the provision of the cross-correlation functions of the excitations in order to compute the correlation matrix of the displacement vector to obtain the second-order moment of response. Therefore, it is imperative to seek an alternative EL method in the time domain, which can not only accurately predict nonlinear system response with low computational cost, but is also applicable to any type of general nonstationary random excitations. This is achieved in the current study by introducing the orthogonal functions in the equivalent linearization.

The concept of orthogonal functions has been well described in the literature. For structural applications, the orthogonal functions may be assorted into three families, including the piecewise constant orthogonal functions, the orthogonal polynomials and the Fourier functions (Datta and Mohan 1995). These functions have been useful tools for the identification of dynamic systems since 1970s (Chen and Hsiao 1975). In recent years, these functions have been applied to controlling systems, as well as identification and sensitivity analysis (Pacheco and Steffen Jr 2004, Younespour and Ghaffarzadeh 2015, Younespour and Ghaffarzadeh 2016). The orthogonal functions play a prominent role in the numerical analysis and approximation theory for improving approximation accuracy. If an orthogonal function is converted to an orthonormal one, it would not only yield a more accurate approximation, but also simplify the mathematical operation. Among various orthogonal functions, the Block Pulse (BP) functions, which are a set of orthogonal functions with a unit pulse in each time step, are inherently orthonormal. Because of this property, in comparison with other orthogonal functions, the block pulse functions can lead more easily to recursive computations when solving concrete problems (Jiang and Schaufelberger 1992) and are usually used to reduce the problem to the solution of complex algebraic equations.

The methodology proposed in the present paper exploits the EL method by extending its applicability in the time domain to analyze nonlinear systems under nonstationary excitations, of which the BP function is used as an orthogonal function for reducing computational effort in the linearization procedures. In the proposed method, the statistical moments of the nonstationary system responses can be directly determined in the time domain. Thus, the proposed approach is more efficient when compared with the mixed time-frequency domain methods, of which a large number of time-history integrals are required at different frequency intervals when nonstationary random excitations are involved. The formulation of the proposed method allows it to be applicable to more general and realistic types of nonstationary excitations, such as the seismic load. A nonlinear Set-up spring under stationary excitation is considered first to evaluate the accuracy of the proposed method. Then both stationary and nonstationary excitations are applied to a SDOF nonlinear Duffing oscillator. Finally, a 3DOF nonlinear Duffing system is analyzed to demonstrate the applicability of the proposed method to MDOF nonlinear systems under nonstationary excitation. Results reveal that compared to the existing EL methods, the proposed orthogonal-function-based EL method can provide more accurate approximations. For SDOF nonlinear systems, the required time and number of iterations to satisfy the convergence criterion are almost the same as the existing EL approaches. However, the proposed approach has more predominant computational advantage in analyzing MDOF nonlinear systems, of which the time required for solving the nonstationary excitation problems is considerably less. Even with the presence of strong nonlinearity, the proposed method remains stable and the accuracy of the results is ensured. For a strongly nonlinear 3DOF Duffing oscillator subjected to nonstationary excitation, the proposed method is about 50 times faster when compared to MC simulation.

The rest part of this paper is organized as follows: A review of the orthogonal functions is presented in Section 2. Section 3 illustrates the equivalent linearization process using the orthogonal functions. For comparison, case studies are carried out in Section 4. Section 5 concluded the paper by summarizing the main findings and highlighting the contributions.

2. Orthogonal function review

A set of functions $\phi_i(t)$ $(i = 1, 2, 3, \ldots)$ is said to be orthogonal over the interval $[a, b]$ if

$$\int_a^b \phi_m(t)\,\phi_n(t)dt = K_{mn} \tag{1}$$

where K_{mn} is a nonzero positive constant, which satisfies

$$\begin{cases} K_{mn} = 0 & \text{if} \quad \mathrm{m} \neq \mathrm{n} \\ K_{mn} \neq 0 & \text{if} \quad \mathrm{m} = \mathrm{n} \end{cases}$$

If K_{mn} is the Kronecker delta function, the set of functions $\phi_i(t)$ is said to be orthonormal. The following property, related to the successive integration of the vector basis, holds for a set of r orthonormal functions

$$\underbrace{\int_0^t \cdots \int_0^t \{\phi(\tau)\}(d\tau)^n}_{n\ times} \cong [P]^n\{\phi(t)\} \tag{2}$$

where $[P] \in \Re^{r,r}$ is a square matrix with constant elements, which is called operator or operational matrix and is dependent on the type of orthogonal function; and $\{\phi(t)\} = [\phi_0(t), \phi_1(t), \ldots, \phi_{r-1}(t)]^T$ is the vector basis of the orthonormal series. This operator plays a key role in the methodology. The operators give a proper mathematical frame for the orthogonal functions and are advantageous to the convergence analysis of their series expansions. In other words, the operators would produce an image matrix or vector of function $f(t)$ in the orthogonal function domain.

A set of BP functions over a unit time interval $[0,1)$ is defined as (Jiang and Schaufelberger 1992)

$$\phi_i(t) = \begin{cases} 1 & \dfrac{i}{m} \le t \le \dfrac{i+1}{m} \\ 0 & \text{otherwise} \end{cases} \tag{3}$$

where $i = 0, 1, 2, \ldots, m-1$ and m is a positive integer, and ϕ_i is the i^{th} BP function.

The block pulse operator $\mathcal{B}$ is determined in the BP domain as

$$\mathcal{B}\{f(t)\} = F^T \tag{4}$$

where vector F is evaluated from

$$F = \frac{1}{q}\int_0^T f(t)\phi(t)dt = [f_1, f_2, \ldots, f_m] \tag{5}$$

where $q = 1/m$. The BP operator has numerous operation rules, of which those will be applied in the next section are listed below (Jiang and Schaufelberger 1992).

(a) For a real constant k, we have

$$\mathcal{B}\{k\} = kE^T \tag{6}$$

where E^T is a constant vector with all entries being one.

(b) For addition and subtraction of functions $f(t), g(t) \in [0, T)$, we have

$$\mathcal{B}\{f(t) \pm g(t)\} = F^T \pm G^T \tag{7}$$

This relation can be derived directly from the linearity of the BP operator.

(c) For integration of a function $f(t) \in [0, T)$, we have

$$\mathcal{B}\left\{\int_0^T f(t)dt\right\} = F^T P \tag{8}$$

where P is a conventional integration operational matrix defined as

$$P = \frac{q}{2}\begin{bmatrix} 1 & 2 & 2 & \cdots & 2 \\ 0 & 1 & 2 & \cdots & 2 \\ 0 & 0 & 1 & \cdots & 2 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix} \tag{9}$$

(d) For Convolution integral of functions $f(t), g(t) \in [0, T)$, we have

$$\mathcal{B}\left\{\int_0^T f(\tau)g(t-\tau)d\tau\right\} \cong \frac{q}{2}F^T J_G \cong \frac{q}{2}G^T J_F \tag{10}$$

where J_G and J_F are the convolution operational matrices defined in Eqs. (11) and (12).

$$J_F = \frac{q}{2}\begin{bmatrix} f_1 & f_1+f_2 & f_2+f_3 & \cdots & f_{m-1}+f_m \\ 0 & f_1 & f_1+f_2 & \cdots & f_{m-2}+f_{m-1} \\ 0 & 0 & f_1 & \cdots & f_{m-3}+f_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & f_1 \end{bmatrix} \tag{11}$$

$$J_G = \frac{q}{2}\begin{bmatrix} g_1 & g_1+g_2 & g_2+g_3 & \cdots & g_{m-1}+g_m \\ 0 & g_1 & g_1+g_2 & \cdots & g_{m-2}+g_{m-1} \\ 0 & 0 & g_1 & \cdots & g_{m-3}+g_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & g_1 \end{bmatrix} \tag{12}$$

(e) For multiple integrals, we have the following rule

$$\mathcal{B}\left\{\underbrace{\int_0^t \cdots \int_0^t f(t)dt \cdots dt}_{k}\right\} = F^T P^k \tag{13}$$

3. Equivalent linearization technique based on orthogonal functions

3.1 SDOF system

The equation of motion of a SDOF nonlinear system is given as

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2 x(t) + g[x(t), \dot{x}(t)] = w(t) \tag{14}$$

with

$$\beta = \xi\omega \tag{15}$$

where ξ and ω are respectively the damping coefficient and the system frequency, $x(t), \dot{x}(t)$ and $\ddot{x}(t)$ are respectively the displacement, the velocity and the acceleration vectors, $g[x(t), \dot{x}(t)]$ is a nonlinear function of displacement and velocity, $w(t)$ is the excitation, which is assumed to be a zero-mean nonstationary random process.

In accord with the equivalent linearization method, Eq. (14) can be replaced by the following equation of motion as

$$\ddot{y}(t) + 2\beta_{eq}\dot{y}(t) + \omega_{eq}^2 y(t) = w(t) \tag{16}$$

where the coefficients of linearization, β_{eq} and ω_{eq}, can be found by the equivalent linearization approach. When the excitation to the original nonlinear system is a Gaussian function, assuming the response of the nonlinear system is also Gaussian, the equivalent linearization coefficients can be calculated by the simplified expressions proposed by Atalik and Utku (1976).

The coefficients β_{eq} and ω_{eq} are determined as follow

$$2\beta_{eq} = 2\beta + E\left[\frac{\partial g(x,\dot{x})}{\partial \dot{x}}\right] \tag{17}$$

$$\omega_{eq}^2 = \omega^2 + E\left[\frac{\partial g(x,\dot{x})}{\partial x}\right] \tag{18}$$

where $E[\cdot]$ stands for the mathematical expectation.

For nonstationary analysis, the equivalent damping and frequency are functions of time. For a system which is initially at rest $(x(0) = \dot{x}(0) = 0)$ and by assuming that these coefficients are constants as in stationary analysis, the solution to Eq. (16) in the time domain can be expressed by the Duhamel's integral as follow

$$\begin{aligned} x(t) &= \int_{-\infty}^{\infty} h(\tau_1) w(t-\tau_1) d\tau_1 \\ &= \int_{-\infty}^{\infty} h(t-\tau_1) w(\tau_1) d\tau_1 \end{aligned} \tag{19}$$

where $h(t)$ is the impulse response of the linearized system and is defined as follows

$$h(t) = \begin{cases} \dfrac{1}{\omega_d} e^{-\beta_{eq} t} \sin(\omega_d t) \ ; \ t \geq 0 \\ 0 \qquad\qquad\qquad ; \ t < 0 \end{cases} \tag{20}$$

where

$$\omega_d^2 = \omega_{eq}^2 - \beta_{eq}^2 \tag{21}$$

By using Eq. (19) to evaluate the mean square response or variance of the displacement and velocity responses, we have (Lutes and Sarkani 2004)

$$E[x^2] = \iint_{-\infty}^{\infty} h(t-\tau_1) E[w(\tau_1)w(\tau_2)] h(t - \tau_2) d\tau_1\, d\tau_2 \tag{22}$$

$$E[\dot{x}^2] = \iint_{-\infty}^{\infty} \dot{h}(t-\tau_1) E[w(\tau_1)w(\tau_2)] \dot{h}(t - \tau_2) d\tau_1\, d\tau_2 \tag{23}$$

The linearization coefficients can be determined by using the values calculated from Eqs. (22) and (23). The solution of the mean square response as given is valid for constant values of β_{eq} and ω_{eq}. However, as is obvious from Eqs. (17) and (18), the equivalent damping and frequency are, in general, functions of time in the nonstationary random process. The common assumption is to use the constant stationary limits with large duration for these coefficients. However, this assumption only gives the first order approximate solutions for the nonstationary responses. To overcome this limitation, an iterative solution procedure is introduced to improve the accuracy of the solutions (see (Iwan and Yang 1972, Orabi and Ahmadi 1987)).

1. Assign initial estimations of c_{eq} and k_{eq} in order to obtain the mean square response of displacement and velocity $(E[x^2]\ ,\ E[\dot{x}^2])$.
2. Substitute the obtained values into Eqs. (17) and (18) to obtain new estimations for β_{eq} and ω_{eq}.
3. In order to find new estimation for the mean square response, substitute the new values of β_{eq} and ω_{eq} into Eq. (20) and then Eqs. (22) and (23).
4. Use the obtained $E[x^2]$ and $E[\dot{x}^2]$ values and return to step (2).
5. Repeat steps (2), (3) and (4) until the results satisfy the following convergence criterion

$$\frac{E[x^2]_{i+1} - E[x^2]_i}{E[x^2]_i} < \varepsilon \ ; \ \frac{E[\dot{x}^2]_{i+1} - E[\dot{x}^2]_i}{E[\dot{x}^2]_i} < \varepsilon \tag{24}$$

where $\varepsilon = 0.001$ is used in the current study.

To reduce the computational complexity, the mean square response of the linearized system is calculated using the operational rules of orthogonal functions in this paper, i.e., Eqs. (10)-(13). By applying the Convolution integral (Eq. (10)) and the multiple integrals (Eq. (13)) operators of the BP functions, we have

$$\begin{aligned} E[x^2] &= \iint_{-\infty}^{\infty} h(t-\tau_1) E[w(\tau_1)w(\tau_2)] h(t-\tau_2) d\tau_1\, d\tau_2 \\ &= \int_{-\infty}^{\infty} h^2(t-\tau) E[w^2(\tau)] d\tau = \int_{-\infty}^{\infty} r(t-\tau) E[w^2(\tau)] d\tau = \frac{q^2}{4} R^T J_w{}^2 \\ &= \frac{q^2}{4} [r_1, r_2, \dots, r_m] \begin{bmatrix} w_1 & w_1 + w_2 & w_2 + w_3 & \cdots & w_{m-1} + w_m \\ 0 & w_1 & w_1 + w_2 & \cdots & w_{m-2} + w_{m-1} \\ 0 & 0 & w_1 & \cdots & w_{m-3} + w_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w_1 \end{bmatrix}^2 \end{aligned} \tag{25}$$

where $r(t-\tau) = h^2(t-\tau)$ and

$$\begin{aligned} r_i &= \frac{1}{q}\int_{iq}^{(i+1)q} r(t)\phi(t) dt; \\ w_i &= \frac{1}{q}\int_{iq}^{(i+1)q} w(t)\phi(t) dt \end{aligned} \tag{26}$$

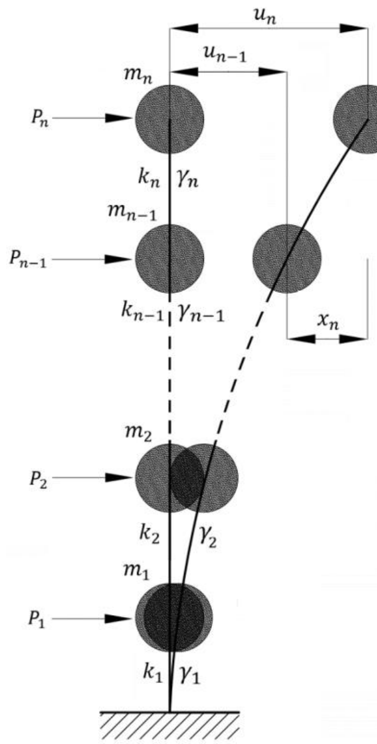

Fig. 1 An n-degree-of-freedom shear-type duffing system

and

$$
\begin{aligned}
E[\dot{x}^2] &= \iint_{-\infty}^{\infty} \dot{h}(t-\tau_1)E[w(\tau_1)w(\tau_2)]\dot{h}(t-\tau_2)d\tau_1\, d\tau_2 \\
&= \int_{-\infty}^{\infty} \dot{h}^2(t-\tau)E[w^2(\tau)]d\tau = \int_{-\infty}^{\infty} l(t-\tau)E[w^2(\tau)]d\tau = \frac{q^2}{4}L^T {J_w}^2 \\
&= \frac{q^2}{4}[l_1, l_2, \ldots, l_m]\begin{bmatrix} w_1 & w_1+w_2 & w_2+w_3 & \cdots & w_{m-1}+w_m \\ 0 & w_1 & w_1+w_2 & \cdots & w_{m-2}+w_{m-1} \\ 0 & 0 & w_1 & \cdots & w_{m-3}+w_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w_1 \end{bmatrix}^2
\end{aligned}
\tag{27}
$$

where $l(t-\tau) = \dot{h}^2(t-\tau)$ and

$$
l_i = \frac{1}{q}\int_{iq}^{(i+1)q} l(t)\phi(t)dt \tag{28}
$$

It is important to point out that the formulation of the proposed orthogonal-functions-based EL method, as illustrated above, clearly indicates that only one time-history analysis of the equivalent linear system from $t = 0$ to $t = \tau$ is required in the proposed approach to determine the dynamic responses of the equivalent system, which can be further used to directly derive the statistical moments of responses. Namely, this is completely different from the frequency domain methods, of which a large number of time-history analyses at different frequency intervals need to be performed to obtain the responses. Thus, the proposed method could enhance the efficiency of the EL method, especially for nonstationary problems associated with nonlinear systems. Besides, it is applicable to any general type of nonstationary random excitations.

3.2 MDOF system

The equation of motion of an n degree-of-freedom nonlinear system is given as (Su *et al.* 2016)

$$
\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \boldsymbol{C}\dot{\boldsymbol{x}}(t) + \boldsymbol{K}\boldsymbol{x}(t) + \boldsymbol{G}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)) = \boldsymbol{\phi}\boldsymbol{W}(t) \tag{29}
$$

where $\boldsymbol{M}, \boldsymbol{C}$ and $\boldsymbol{K}$ are the $n \times n$ mass matrix, damping matrix and elastic stiffness matrix of the considered system, respectively; $\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)$ and $\ddot{\boldsymbol{x}}(t)$ denote the nodal displacement vector, and the corresponding velocity vector and acceleration vector, respectively; $\boldsymbol{G}(\boldsymbol{x}(t), \dot{\boldsymbol{x}}(t)) = [g_1(t)\ g_2(t) \ldots\ g_n(t)]^T$ is an n-dimension nonlinear vector function of the coordinate displacement and velocity, $\boldsymbol{\phi}$ is an orientation matrix of the nonstationary zero-mean Gaussian random loading vector $W(t) = [W_1(t)\ W_2(t) \ldots W_n(t)]^T$, where the superscript T denotes matrix transposition.

By assuming linear behavior for the mass matrix, Eq. (29) can be replaced by the following equalized linear equation of motion as

$$
\begin{aligned}
\boldsymbol{M}\ddot{\boldsymbol{x}}(t) + \left[\boldsymbol{C} + \boldsymbol{C_{eq}}(\tau)\right]\dot{\boldsymbol{x}}(t) + \left[\boldsymbol{K} + \boldsymbol{K_{eq}}(\tau)\right]\boldsymbol{x}(t) \\
= \boldsymbol{\phi}\boldsymbol{W}(t)
\end{aligned}
\tag{30}
$$

where $\boldsymbol{C_{eq}}(\tau)$ and $\boldsymbol{K_{eq}}(\tau)$ are respectively the $n \times n$ equivalent matrices at time instant τ. By assuming the Gaussian excitation and using the simplified expressions proposed in Atalik and Utku (1976), the elements of the equivalent linearization matrices can be obtained by the following equations

$$
K_{eq,ij}(\tau) = E\left[\frac{\partial g_i(\tau)}{\partial x_j(\tau)}\right] \quad i,j = 1,2,\ldots,n \tag{31}
$$

$$
C_{eq,ij}(\tau) = E\left[\frac{\partial g_i(\tau)}{\partial \dot{x}_j(\tau)}\right] \quad i,j = 1,2,\ldots,n \tag{32}
$$

where $K_{eq,ij}(\tau)$ and $C_{eq,ij}(\tau)$ are the elements of $\boldsymbol{K_{eq}}(\tau)$ and $\boldsymbol{C_{eq}}(\tau)$, respectively. It can be seen that the equivalent parameters in Eqs. (31) and (32) depend on the statistical responses. Therefore, an iterative procedure proposed in the previous section is required to determine the accurate equivalent matrices.

Consider an n degree-of-freedom shear type Duffing system shown in Fig. 1. The equation of motion of this MDOF nonlinear system is

$$
\boldsymbol{M}\ddot{\boldsymbol{U}}(t) + \boldsymbol{C}\dot{\boldsymbol{U}}(t) + \boldsymbol{K}\boldsymbol{U}(t) + \boldsymbol{G}(t) = \boldsymbol{P}(t) \tag{33}
$$

where $\boldsymbol{U}(t), \dot{\boldsymbol{U}}(t)$ and $\ddot{\boldsymbol{U}}(t)$ denote respectively the horizontal displacement vector, the corresponding velocity vector and the acceleration vector, i.e.,

$$
\ddot{\boldsymbol{U}}(t) = \begin{bmatrix} \ddot{u}_1 \\ \ddot{u}_2 \\ \vdots \\ \ddot{u}_n \end{bmatrix};\ \dot{\boldsymbol{U}}(t) = \begin{bmatrix} \dot{u}_1 \\ \dot{u}_2 \\ \vdots \\ \dot{u}_n \end{bmatrix};\ \boldsymbol{U}(t) = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \tag{34}
$$

where $\ddot{u}_i$, $\dot{u}_i$ and u_i $(i = 1, 2, \ldots, n)$ are respectively the horizontal acceleration, velocity and displacement of the i^{th} floor. $\boldsymbol{P}(t) = \boldsymbol{\phi}\boldsymbol{W}(t)$, where $\boldsymbol{\phi}$ is the orientation matrix of the nonstationary zero-mean Gaussian random excitation $\boldsymbol{W}(t)$, and the nonlinear term $\boldsymbol{G}(t)$ can be expressed as

$$\boldsymbol{G}(t) = \begin{bmatrix} \gamma_1 k_1 x_1^3 - \gamma_2 k_2 x_2^3 \\ \gamma_2 k_2 x_2^3 - \gamma_3 k_3 x_3^3 \\ \gamma_3 k_3 x_3^3 - \gamma_4 k_4 x_4^3 \\ \vdots \\ \gamma_{n-1} k_{n-1} x_{n-1}^3 - \gamma_n k_n x_n^3 \\ \gamma_n k_n x_n^3 \end{bmatrix}; \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ \vdots \\ x_{n-1} \\ x_n \end{bmatrix} = \begin{bmatrix} u_1 \\ u_2 - u_1 \\ u_3 - u_2 \\ \vdots \\ u_{n-1} - u_{n-2} \\ u_n - u_{n-1} \end{bmatrix} \tag{35}$$

where x_i is the relative displacement between the (i-1)th and the ith floor and can be expressed as $x_i = u_i - u_{i-1}$ $(i = 1,2,\dots,n)$ with $u_0 = 0$; k_i $(i = 1,2,\dots,n)$ is the linear stiffness of the ith story and γ_i $(i = 1,2,\dots,n)$ is the coefficient reflecting the nonlinearity at the ith story.

Substitute Eq. (35) into Eqs. (17) and (18), the equivalent matrices can be obtained at the time instant of interest, τ, in Eq. (30) as

$$\boldsymbol{C_{eq}}(\tau) = 0 \tag{36}$$

$$\boldsymbol{K_{eq}}(\tau) = 3\begin{bmatrix} \chi_1(\tau)+\chi_2(\tau) & -\chi_2(\tau) & 0 & \dots & 0 \\ -\chi_2(\tau) & \chi_2(\tau)+\chi_3(\tau) & -\chi_3(\tau) & & \\ 0 & -\chi_3(\tau) & & & \vdots \\ \vdots & & & \ddots & -\chi_n(\tau) \\ 0 & \dots & & -\chi_n(\tau) & \chi_n(\tau) \end{bmatrix} \tag{37}$$

where

$$\chi_i(\tau) = \gamma_i k_i E[x_i^2(\tau)] = \gamma_i k_i (E[u_i^2(\tau)] - E[u_i(\tau) u_{i-1}(\tau)] + E[u_{i-1}^2(\tau)]) \tag{38}$$

Accordingly, the time-invariant equivalent linear system for the studied MDOF Duffing system at time instant τ can be described using the following linear equation of motion as

$$\boldsymbol{M}\ddot{\boldsymbol{U}}(t) + \boldsymbol{C}\dot{\boldsymbol{U}}(t) + [\boldsymbol{K} + \boldsymbol{K_{eq}}(\tau)]\boldsymbol{U}(t) = \boldsymbol{\phi}\boldsymbol{W}(t) \tag{39}$$

where $\boldsymbol{U}(t) = [u_1\, u_2 \dots u_n]$.

It can be seen from Eqs. (37) and (38) that at the interested time instant τ, the equivalent stiffness matrix $\boldsymbol{K_{eq}}(\tau)$ is dependent on the second-order moment of the responses at the same time instant, which, in turn, needs to be determined through the nonstationary random vibration analysis of the equivalent linear system based on Eq. (39). Therefore, an iterative procedure based on a series of nonstationary linear random vibration analyses at each concerned time instant is required.

Again, we assume that the considered MDOF system is initially at rest. By conducting modal analysis, the solution to Eq. (30) for every degree of freedom in the time domain can be evaluated using Eqs. (19)-(23). Therefore, in the case of a MDOF system, the concept of orthogonal functions can be used to approximate the linearization coefficients and the mean square values of the system response. The application of the proposed method will be illustrated in the next section by considering examples of SDOF and MDOF systems subjected to stationary and nonstationary random excitations.

4. Numerical examples

4.1 SDOF nonlinear system with a set-up spring

We first consider a SDOF nonlinear system subjected to stationary excitation. The system consists of a concentrated mass m, which is connected to a set-up spring and a linear viscous damper. The system equation of motion has the form of

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2[x(t) + \gamma sgn(x)] = w(t) \tag{40}$$

where γ is a positive real constant corresponds to the strength of the nonlinearity and $w(t)$ is the stationary excitation which is assumed to be a zero mean Gaussian white noise process with the following statistical properties

$$E[w(t)] = 0; \quad E[w(t_1)w(t_2)] = 2\pi S_0 \delta(t_1 - t_2) \tag{41}$$

where S_0 is a constant power spectrum and $\delta(\cdot)$ is the Dirac delta function. The sampling rate of the considered random excitation is 0.01 second. In this study, the mean-square response of the linearized system is calculated up to 20 seconds. Thus, the positive integer m in Eq. (25) is 2000.

By applying this assumption, the parameters of the equivalent linear system can be determined from Eqs. (17) and (18). It can be seen from Eq. (40) that the equivalent linearization coefficient becomes (Crandall 1962)

$$\omega_{eq}^2 = \omega^2\left(1 + \gamma\sqrt{\frac{2}{\pi E[x^2]}}\right) \tag{42}$$

Therefore, the equivalent linear equation of Eq. (40) can be rewritten as

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2\left(1 + \gamma\sqrt{\frac{2}{\pi E[x^2]}}\right)x(t) = w(t) \tag{43}$$

For a stationary response, the exact transition probability density function was obtained from the Fokker-Planck-Kolmogorov equation by Crandall (Crandall 1962). The corresponding stationary mean-square response is given by

$$E[x^2] = \sigma_0^2\left(1 - \frac{\gamma}{\sigma_0}\sqrt{\frac{2}{\pi}}\frac{\exp\left(-\frac{\gamma^2}{2\sigma_0^2}\right)}{\operatorname{erfc}\left(\frac{\gamma}{\sigma_0\sqrt{2}}\right)} + \frac{\gamma^2}{\sigma_0^2}\right); \quad \sigma_0^2 = \frac{\pi S_0}{2\beta\omega^2} \tag{44}$$

where $\operatorname{erfc}(\cdot)$ is the complementary error function.

Now by using the proposed method and the iteration technique described earlier, the following equation can be solved for $E[x^2]$ at each time step.

$$\begin{aligned} E[x^2] &= 2\pi S_0 \iint_{-\infty}^{\infty} h(t-\tau_1)E[w(\tau_1)w(\tau_2)]h(t-\tau_2)d\tau_1 d\tau_2 \\ &= 2\pi S_0 \int_{-\infty}^{\infty} h^2(t-\tau)E[w^2(\tau)]d\tau \\ &= 2\pi S_0 \int_{-\infty}^{\infty} r(t-\tau)E[w^2(\tau)]d\tau = 2\pi S_0\left(\frac{q^2}{4}R^T {J_w}^2\right) \\ &= 2\pi S_0 \\ &\times \frac{q^2}{4}[r_1, r_2, \dots, r_m]\begin{bmatrix} w_1 & w_1+w_2 & w_2+w_3 & \cdots & w_{m-1}+w_m \\ 0 & w_1 & w_1+w_2 & \cdots & w_{m-2}+w_{m-1} \\ 0 & 0 & w_1 & \cdots & w_{m-3}+w_{m-2} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & w_1 \end{bmatrix}^2 \end{aligned} \tag{45}$$

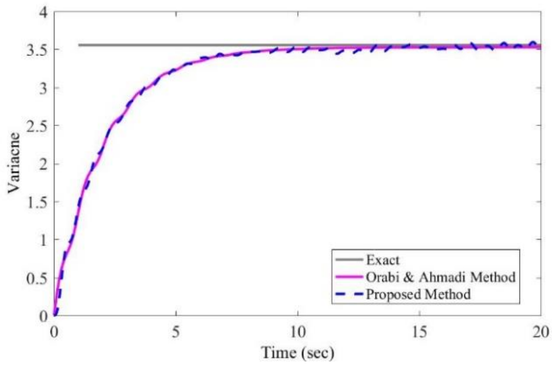

Fig. 2 Mean square displacement response of a nonlinear system with a Set-Up spring due to stationary excitation. (γ=1.0)

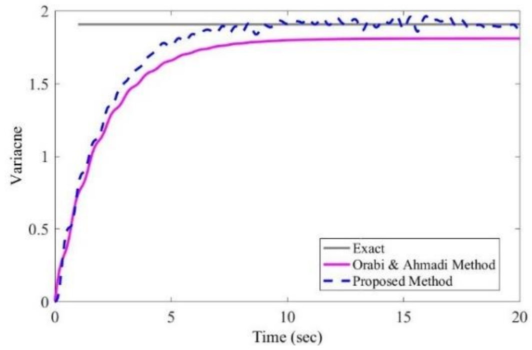

Fig. 3 Mean square displacement response of a nonlinear system with a Set-Up spring due to stationary excitation. (γ=3.0)

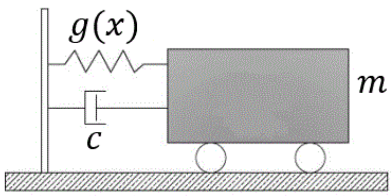

Fig. 4 A SDOF Duffing oscillator

where $h(t)$ is the impulse response of the linearized system defined by Eq. (20). For $S_0 = 20$, $\omega = 5$, $\xi = 0.05$ and two different values of nonlinearity strength (γ=1.0 and 3.0), the mean-square responses were evaluated by the proposed method. As a comparison, the problem was also solved using the iteration method proposed by Orabi and Ahmadi (1987). These two sets of results are shown in Figs. 2 and 3, along with the exact value of variances, for the cases of nonlinearity strength of γ=1.0 and γ=3.0, respectively.

It can be clearly seen from Figs. 2 and 3 that for a SDOF nonlinear system subjected to stationary excitation, results yielded from the proposed equivalent linearization method, which is based on orthogonal functions, agree well with those obtained by the iterative EL method (Orabi and Ahmadi 1987) and the exact solution when γ=1.0, and have better accuracy than the iteration method when the nonlinearity increases to γ=3.0.

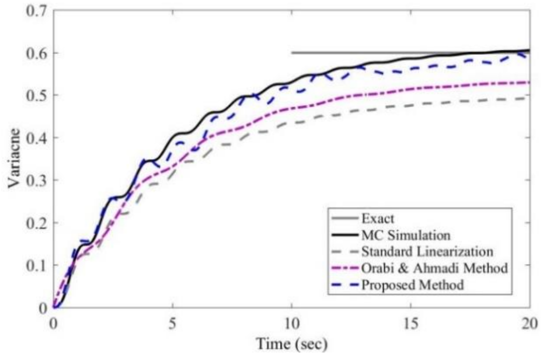

Fig. 5 Mean square displacement response due to stationary excitation. (γ=0.1, ξ=0.05)

4.2 SDOF duffing oscillator

The Duffing oscillator has been successfully used to model a wide range of physical process of which the response has nonlinear dynamical nature. In the current section, the proposed orthogonal- function-based equivalent linearization method is applied to study the behavior of a SDOF Duffing oscillator.

Consider a SDOF Duffing system shown in Fig. 4. The system equation of motion has the form of (Kovacic and Brennan 2011)

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2[x(t) + \gamma x^3(t)] = w(t) \tag{46}$$

where γ is a positive real constant representing the strength of the nonlinearity and $w(t)$ is the excitation. Based on Eq. (18) and the following formula for the Gaussian process $x(t)$, i.e.,

$$E[x^{2n}] = (2n-1)!\,(E[x^2])^n\,; \quad n = 1, 2, 3, \ldots \tag{47}$$

the linearization coefficient for the equivalent linear system becomes

$$\omega_{eq}^2 = \omega^2(1 + 3\gamma E[x^2]) \tag{48}$$

Therefore, the equivalent linear equation of Eq. (46) can be expressed as follows

$$\ddot{x}(t) + 2\beta\dot{x}(t) + \omega^2(1 + 3\gamma E[x^2])\,x(t) = w(t) \tag{49}$$

The exact value of the mean-square displacement response, $E[x^2]$, of Eq. (46) is evaluated by

$$E[x^2]_{exact} = \frac{\int_{-\infty}^{\infty} x^2 exp\left[\frac{4\xi\omega}{2\pi S_0\sigma^2}\left(\frac{1}{2}\omega^2x^2 + \frac{1}{4}\gamma x^4\right)\right]dx}{\int_{-\infty}^{\infty} exp\left[\frac{4\xi\omega}{2\pi S_0\sigma^2}\left(\frac{1}{2}\omega^2x^2 + \frac{1}{4}\gamma x^4\right)\right]dx} \tag{50}$$

By using the iterative procedure illustrated in the previous section and the formula of the proposed method (Eq. (45)), the mean-square displacement response of the linearized system can be computed. In this example, we consider both stationary and nonstationary excitations.

4.2.1 Stationary excitation

For stationary excitation, the excitation force function in Eq. (46) is a Gaussian white noise process, i.e.,

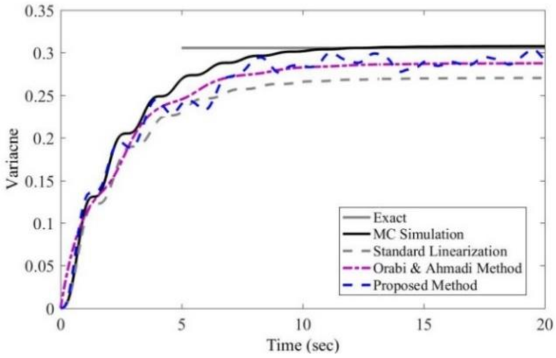

Fig. 6 Mean square displacement response due to stationary excitation. (γ=0.1, ξ=0.1)

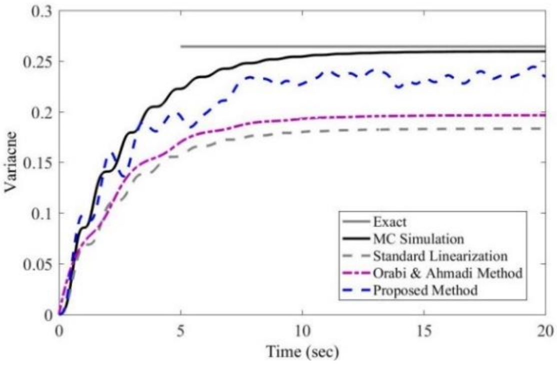

Fig. 7 Mean square displacement response due to stationary excitation. (γ=1.0, ξ=0.1)

$$w(t) = e(t)n(t) \tag{51}$$

where $n(t)$ is a zero mean stationary white noise process with the following statistical properties

$$E[n(t)] = 0\,;\; E[n(t_1)n(t_2)] = 2\pi S_0 \delta(t_1 - t_2) \tag{52}$$

Again, here S_0 is a constant power spectrum and $\delta(\cdot)$ is the Dirac delta function. Besides, $e(t)$ is a unit function, i.e.,

$$e(t) = u(t) = \begin{cases} 1\;, & t \geq 0 \\ 0\;, & t < 0 \end{cases} \tag{53}$$

Now by using the iteration technique described earlier, Eq. (45) could be solved for $E[x^2]$ at each time step. The mean-square response of the SDOF Duffing oscillator was evaluated by the proposed orthogonal-function-based equivalent linearization method under the assumptions of $S_0 = \frac{1}{2\pi}$ and $\omega = 2$ for two different nonlinearity strength of γ=0.1 and 1.0, and two different damping levels of ξ=0.05 and 0.1. The standard equivalent linearization method (i.e., with the stationary constant value and without iteration) and the iteration method proposed by Orabi and Ahmadi (1987) were also applied to analyze the response of the studied Duffing oscillator. In addition, a MC simulation with 1000 samples was exploited to estimate the transient responses. The results obtained from the above four different approaches are portrayed in Figs. 5 to 7, along with the exact value of variances determined by Eq. (50).

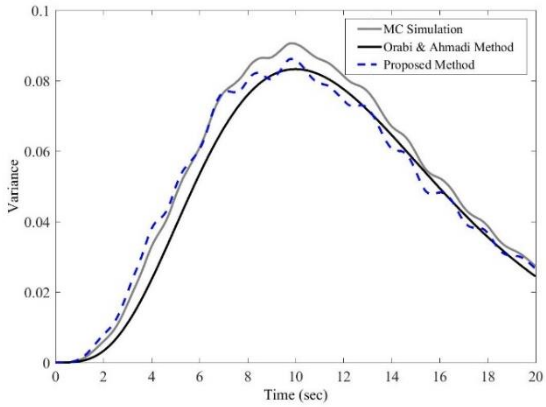

Fig. 6 Mean square displacement response due to nonstationary excitation. (γ=1.0, ξ=0.1)

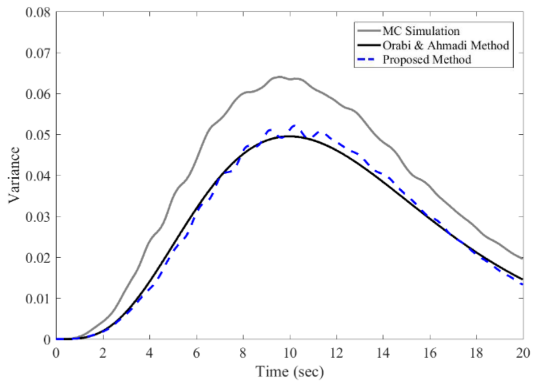

Fig. 7 Mean square displacement response due to nonstationary excitation. (γ=5.0, ξ=0.1)

It can be seen from Figs. 5 to 7 that the system responses determined by the standard equivalent linearization method, the iterative EL method (Orabi and Ahmadi 1987) and the proposed orthogonal-function-based equivalent linearization method are always smaller than the exact solution and the prediction by the MC simulation, especially when the system has relatively stronger nonlinearity and lower damping ratio. However, when compared with the former two methods, the responses determined by the proposed approach show better agreement with the exact solution and the MC simulation. It is worth pointing out that the accuracy of an EL method is mainly affected by the linearization criteria and the Gaussian response assumption. Since the proposed approach and Orabi and Ahmadi's method (1987) use the same assumptions for the mean-square criterion and Gaussian response, it is reasonable to expect that the results yielded from these two different approaches should have the same accuracy level. Thus, the small discrepancies between these two sets of results are believed to be caused by the differences in the procedures of computing the mean-square response.

In other words, for a SDOF Duffing oscillator under stationary excitation, the proposed approach outperforms the standard equivalent linearization method and the iterative EL method by Orabi and Ahmadi (1987) when the system is lightly damped and has higher nonlinearity. In addition, the proposed approach can give a more accurate prediction on the structural response should the system behave less nonlinearly. In all cases, the respective error between the three linear equivalent methods and the exact solution gradually decreases as time proceeds.

4.2.2 Nonstationary excitation

Two types of nonstationary excitation are considered for the same SDOF Duffing oscillator, i.e., a non-white noise function and the El Centro (1940) earthquake record.

(a) Non-white noise forcing function

This forcing function can be expressed as

$$f(t) = \sum_{j=1}^{m} t a_j exp\{-\beta_j t\} \cos(\omega_j t + \theta) \tag{54}$$

where a_j, β_j, ω_j are constant system parameters, and θ is a random variable uniformly distributed over $[0, 2\pi]$. The forcing function in Eq. (54) was proposed by Bogdanoff *et al.* (1961) as a model to describe ground acceleration induced by earthquake. Again, by using the proposed method, Eq. (45) can be solved and the response variance due to this nonstationary excitation can be computed at each time step. If assume $\mu_1 = 0.1$ and $\omega_1 = 1$, the system parameters in Eq. (54) would become

$$a_j = 1, \qquad \omega_j = j\omega_1, \qquad \beta_j = \mu_1\omega_j \tag{55}$$

Under this set of system parameters, the mean-square response of the Duffing oscillator was evaluated for two different nonlinearity strength of γ=1.0 and γ=5.0. For the nonstationary excitation, the sampling rate and positive integer *m* are taken as 0.01 second and 2000, respectively.

The results are shown in Figs. 8 and 9. For the convenience of comparison, the results predicted by the iterative EL method (Orabi and Ahmadi 1987) and the MC simulation are also shown in these two figures.

Again, Figs. 8 and 9 depict clearly that the results yielded from the proposed orthogonal-function-based equivalent linearization method, agree well with those obtained by Orabi and Ahmadi's method. As the system behavior becomes less nonlinear, the responses predicted by the proposed method become more agreeable with that by the MC simulation. These suggest that the proposed approach is applicable to a SDOF nonlinear system subjected to either stationary or nonstationary excitations.

(b) El Centro (1940) earthquake record with $S_0 = 55.44$

Fig. 10 illustrates the mean-square displacement response of the studied SDOF Duffing oscillator when subjected to a nonstationary excitation in terms of the El Centro (1940) earthquake record with $S_0 = 55.44$ at three different nonlinearity strengths of γ=0.0 (linear), 3.0 and 10.0. The damping coefficient is assumed to be $\xi = 0.05$. The sampling rate of this seismic record is 0.02 second and its duration is 30 seconds.

It can be seen from Fig. 10 that the mean square displacement time histories of all three nonlinearity strength scenarios manifest the same pattern. The responses reach the peak value at about 5 seconds and then decrease gradually. As expected, the linear system (γ=0) has the largest variance and the response amplitude decreases as the strength of nonlinearity increases.

The Set-up spring and the Duffing oscillator examples discussed in the previous two sections demonstrated that the proposed method is applicable to a nonlinear SDOF system subjected to either stationary or nonstationary excitations with high accuracy. The introduction of orthogonal functions can considerably reduce computational effort in the linearization

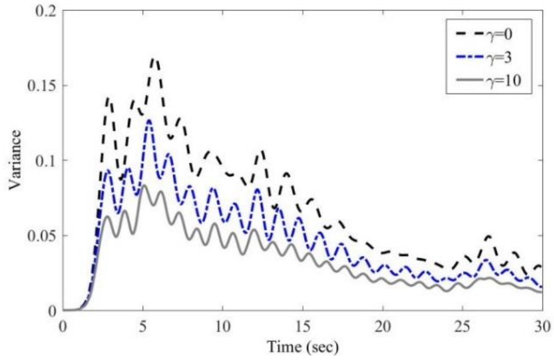

Fig. 8 Mean square displacement response due to El Centro (1940) earthquake ground motion excitation

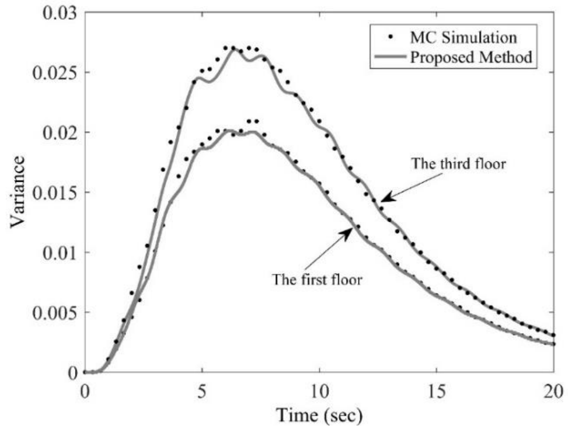

Fig. 9 Mean square displacement response of a 3DOF Duffing system under nonstationary excitation. (γ=1.0)

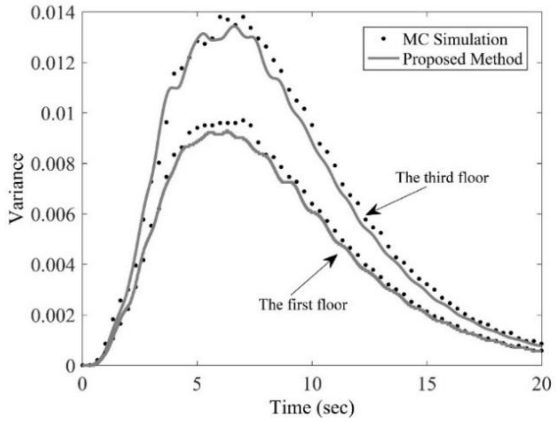

Fig. 10 Mean square displacement response of a 3DOF Duffing system under nonstationary excitation. (γ=20.0)

procedures. The application of the proposed method to a MDOF system subjected to nonstationary excitation will be illustrated in the next section.

4.3 MDOF duffing oscillator

As an illustrative example, we now consider the number of degree-of-freedom as $n=3$ and use the nonlinear terms in Eq. (35) for the Duffing system. The mass and stiffness matrices of this nonlinear Duffing system are assumed as follows

$$\boldsymbol{M}=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}kg\,;\qquad \boldsymbol{K}=\begin{bmatrix}50&-20&0\\-20&40&-20\\0&-20&20\end{bmatrix}N/m \tag{56}$$

The system damping ratios are assumed to be 5% for all the modes, i.e., $\xi_1=\xi_2=\xi_3=0.05$. Two nonlinearity cases are considered with the nonlinear strength coefficients being $\gamma_1=\gamma_2=\gamma_3=1$ and $\gamma_1=\gamma_2=\gamma_3=20$, respectively. The system is subjected to a nonstationary random process defined by Eq. (54) for the same excitation parameters and with $\mu_1=0.3$.

The orthogonal-function-based equivalent linearization method is used to determine the variance of displacements at each floor. Again, to check the accuracy of the proposed method, the MC simulation method with 1000 samples was exploited. The mean-square lateral displacements of the first and the third floor of the studied 3DOF Duffing system are shown in Figs. 11 and 12.

It can be seen from Figs. 11 and 12 that for the two studied nonlinearity strength cases, the responses predicted by the proposed approach are in good agreement with those by the MC simulation, with the maximum relative difference between the two sets of results being 2.1% and 5.8%, respectively. This not only demonstrates the accuracy of the proposed method, but also indicates that as the nonlinearity of a system becomes stronger, the application of the proposed equivalent linearization method would cause relatively larger error in the predicted response. This fact is understandable, since the accuracy of the equivalent linearization method decreases for strongly nonlinear systems.

Besides, to better understand the impact of the degree of nonlinearity on the system behavior, the response of the studied MDOF Duffing system, in terms of its dynamic eigenvalues ω_i^2 $(i=1,2,3)$, are computed for two different nonlinearity strength of $\gamma_i=1$ and 20 $(i=1,2,3)$, and compared with those of the reference linear system with $\gamma_i=0$ $(i=1,2,3)$. The results are summarized in Table 1.

As can be seen from Table 1, compared to the reference linear system, the dynamic eigenvalues of the equivalent linear system with $\gamma_i=1$ $(i=1,2,3)$ increase respectively by 0.45%, 0.23%, and 0.41%; whereas in the case of $\gamma_i=20$ $(i=1,2,3)$, a much significant increase of the three dynamic eigenvalues, i.e., 24.27%, 15.24%, and 30.89%, is observed. Results obtained from this example suggest that the proposed method is applicable to a nonlinear problem with both weak and strong degree of nonlinearity.

The computation time by the MC simulation method and the proposed method are listed in Table 2 for the two studied nonlinearity strength cases. Results show that the total computation time needed by the MC simulation method is respectively 140 and 49 times longer than that by the proposed method for $\gamma_i=1$ and $\gamma_i=20(i=1,2,3)$. This indicates that the proposed method has a very high computational efficiency for nonlinear systems. It should be mentioned, all computations were performed on a computer with Intel Core i7 2600, 2.0 GHz processor and 4 G of RAM.

Table 1 Three dynamic eigenvalues of the studied MDOF nonlinear system

Mode	Reference Linear system (*rad/sec*)	Equivalent linear system (*rad/sec*) ($\gamma_i=1.0$)	Relative change (%)	Equivalent linear system (*rad/sec*) ($\gamma_i=20.0$)	Relative change (%)
1	2.193	2.203	0.45	2.896	24.27
2	6.007	6.021	0.23	7.087	15.24
3	8.312	8.346	0.41	12.028	30.89

Table 2 Comparison of the computation time required by MC simulation and the proposed method

γ_i	MC simulation method (T_1)	proposed method (T_2)	T_1/T_2
1.0	2230 sec	15.93 sec	140
20.0	2470 sec	50.72 sec	49

5. Conclusions

An orthogonal-function-based equivalent linearization method has been proposed in the current paper for the analysis of nonlinear systems subjected to stationary and nonstationary excitations in the time domain. The formulation of the proposed method has been presented first, and its validity and accuracy have been verified through numerical examples, of which it has been applied to a SDOF nonlinear system with a set-up spring under stationary excitation, a SDOF nonlinear Duffing system subjected to either stationary or nonstationary excitation, and eventually extended to analyzing a MDOF nonlinear Duffing system subjected to nonstationary excitation. Results show that compared to other existing equivalent linearization methods, the system responses predicted by the proposed method are in better agreement with the exact solution and those by the MC simulation. In addition, it is applicable to any general type of nonstationary random excitations. These advantages of the proposed equivalent linearization method become more pronounced when a system has higher nonlinearity and lower damping ratio.

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