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Nonlinear vibration of oscillatory systems using semi-analytical approach

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Abstract. In this paper, He's Variational Approach (VA) is used to solve high nonlinear vibration equations. The proposed approach leads us to high accurate solution compared with other numerical methods. It has been established that this method works very well for whole range of initial amplitudes. The method is sufficient for both linear and nonlinear engineering problems. The accuracy of this method is shown graphically and the results tabulated and results compared with numerical solutions.

Keywords: variational approach (VA); nonlinear oscillators; Runge-Kutta's algorithm

1. Introduction

Many engineering phenomena can be modeled mathematically. The mathematical models could be in linear and nonlinear partial differential equations. Generally, it's not an easy task to prepare analytical solution for nonlinear partial differential equations. Therefore in recent years, scientific has been working on approximate analytical solution for nonlinear problems such as: Homotopy perturbation method (Baki *et al*. 2011); Variational approach method (He 2007); Energy balance method (He 2002, 2010, Mehdipour *et al*. 2010) Galerkin method (Chen 1987); Variational Iterational method (He 1999); harmonic balance method (Lau 1983) and many other proposed methods (Akgoz *et al*. 2011, Bayat 2015a, b, c, 2016, 2012, Pakar *et al*. 2015, Shen *et al*. 2009, Wu 2011, Öziş *et al*. 2017, Hashemietal 2013, Kaya 2013, Bayat 2017, Zhifeng 2013, Radomirovic 2015, Filobello-Nino 2015).

In this paper, variational approach method has been used to solve high nonlinear vibration equations. The method first was proposed by He (2007). It has been shown that the first iteration of the variational approach leads us to a high accuracy of the solution and has an excellent agreement with the numerical solutions. To show the efficiency and accuracy of the methods some comparisons have done with the results obtained by the variatioanl approach method and Runge-kutta. The energy balance method has an excellent agreement with the Runge-kutta and are valid for whole domain.

2. Mathematical formulation

A conservative nonlinear single degree of freedom is shown in Fig. 1. The governing equation of the oscillation is as follows

$$\left(m_1+\frac{m_2 x^2}{l^2-x^2}\right)\ddot{x}+\frac{m_2 l^2 x\dot{x}^2}{\left(l^2-x^2\right)^2}+k\,x+m_2 g\,\frac{x}{\sqrt{l^2-x^2}}=0 \tag{1}$$

Where m_1 is the first mass of system, m_2 is the second mass of system, k spring stiffness, l distance between two mass and g is gravity.

For the simplicity of the system let consider $R=m_2/m_1$ and $u=x/l$. Then expanding for $|u|\ll 1$, obtain.

$$\left(1+Ru^2\right)\ddot{u}+R\,u\,\dot{u}^2+\omega_0^2\,u+\frac{Rg}{2l}u^3=0 \tag{2}$$

Where

$$\omega_0^2=\frac{k}{m_1}+\frac{Rg}{l} \tag{3}$$

With boundary condition

$$u(0)=A \qquad \dot{u}(0)=0 \tag{4}$$

3. Basic idea of variational approach (VA)

He suggested a variational approach which is different from the known variational methods in open literature (He 2007). Hereby we give a brief introduction of the method

$$\ddot{u}+f\ (u)=0 \tag{5}$$

Its variational principle can be easily established utilizing the semi-inverse method (He 2007)

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$$J(u)=\int_0^{T/4}\left(-\frac{1}{2}\dot{u}^2+F(u)\right)dt \tag{6}$$

Where T is period of the nonlinear oscillator, $\partial F/\partial u = f$. Assume that its solution can be expressed as

$$u(t)=A\cos(\omega t) \tag{7}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (3) into Eq. (2) results in

$$\begin{aligned}J(A,\omega)&=\int_0^{T/4}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F(A\cos\omega t)\right)dt\\&=\frac{1}{\omega}\int_0^{\pi/2}\left(-\frac{1}{2}A^2\omega^2\sin^2\omega t+F(A\cos\omega t)\right)dt\\&=-\frac{1}{2}A^2\omega\int_0^{\pi/2}\sin^2 t\,dt+\frac{1}{\omega}\int_0^{\pi/2}F(A\cos t)dt\end{aligned} \tag{8}$$

Applying the Ritz method, we require

$$\frac{\partial J}{\partial A}=0 \tag{9}$$

$$\frac{\partial J}{\partial \omega}=0 \tag{10}$$

But with a careful inspection, for most cases we find that

$$\frac{\partial J}{\partial \omega}=-\frac{1}{2}A^2\int_0^{\pi/2}\sin^2 t\,dt-\frac{1}{\omega^2}\int_0^{\pi/2}F(A\cos t)dt<0 \tag{11}$$

Thus, we modify conditions Eqs. (5) and (6) into a simpler form

$$\frac{\partial J}{\partial \omega}=0 \tag{12}$$

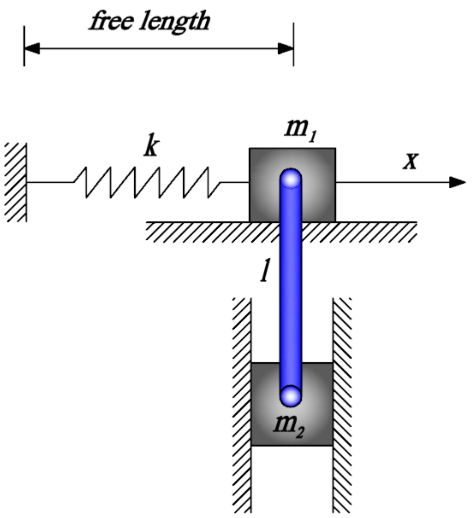


Fig. 1 A conservative Nonlinear single degree of freedom

Table 1 Comparison of frequency corresponding to various parameters of system

A	m_1	m_2	L	g	K	ω_{VA}	ω_{NM}	Error(%)
0.1	4	2	2	10	5	3.74657	3.77425	0.73876
0.5	3	1	5	10	10	3.92683	3.95378	0.68637
1	4	5	4	10	20	6.43010	6.51424	1.30861
2	2	4	4	10	5	3.57071	3.59453	0.66710
3	1	3	5	10	15	5.64007	5.69443	0.96397
4	2	5	4	10	10	2.79508	2.80374	0.30956
5	1	4	5	10	30	5.45750	5.52437	1.22534
10	3	4	10	10	20	1.29797	1.29453	0.26533

4. Application of variational approach

Variational formulation can be readily obtained Eq. (11) as follows

$$J(u)=\int_0^{t}\left(-\frac{1}{2}\dot{u}^2-\frac{1}{2}Ru^2\dot{u}^2+\frac{1}{2}\omega_0^2u^2+\frac{1}{8}\frac{Rgu^4}{l}\right)dt \tag{13}$$

Choosing the trial function $u(t)=A\cos(\omega t)$ into Eq. (13) we obtain

$$J(A)=\int_0^{T/4}\left(\begin{aligned}&-\frac{1}{2}A^2\omega^2\sin^2(\omega t)-\frac{1}{2}A^4R\omega^2\sin^2(\omega t)\cos^2(\omega t)\\&+\frac{1}{2}\omega_0^2A^2\cos^2(\omega t)+\frac{1}{8}\frac{RgA^4}{l}\cos^4(\omega t)\end{aligned}\right)dt \tag{14}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\left(\begin{aligned}&-A\omega^2\sin^2(\omega t)-2A^3\omega^2R\sin^2(\omega t)\cos^2(\omega t)\\&+\omega_0^2A\cos^2(\omega t)+\frac{1}{2}\frac{RgA^3\cos^4(\omega t)}{l}\end{aligned}\right)dt=0 \tag{15}$$

Or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\left(\begin{aligned}&-A\omega^2\sin^2 t-2A^3\omega^2R\sin^2 t\cos^2 t\\&+\omega_0^2A\cos^2 t+\frac{1}{2}\frac{R\,g\,A^3\cos^4 t}{l}\end{aligned}\right)dt=0 \tag{16}$$

Solving Eq. (16), according to ω, we have

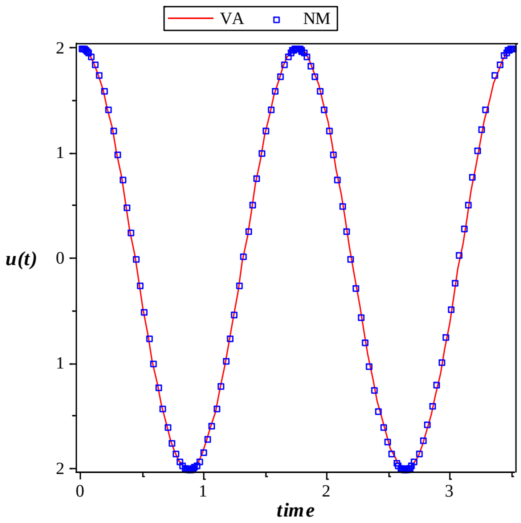
$$\omega^2=\frac{\int_0^{\frac{\pi}{2}}\left(\omega_0^2A\cos^2 t+\frac{1}{2}\frac{RgA^3\cos^4 t}{l}\right)dt}{\int_0^{\frac{\pi}{2}}\left(A\omega^2\sin^2 t+2A^3\omega^2R\sin^2 t\cos^2 t\right)dt} \tag{17}$$

Then we have

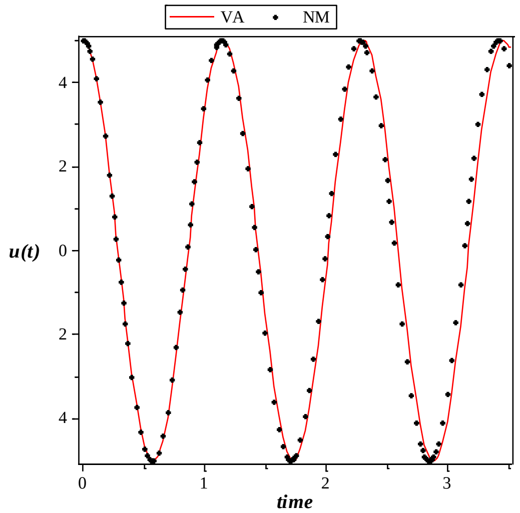
$$\omega_{VA}=\frac{1}{2}\sqrt{\frac{3R\,g\,A^2+8\omega_0^2\,l}{l\left(A^2R+2\right)}} \tag{18}$$

According to $u(t)=A\cos(\omega t)$ and (18), we can obtain the following approximate solution

$$u(t)=A\cos\left(\frac{1}{2}\sqrt{\frac{3R\,g\,A^2+8\omega_0^2\,l}{l\left(A^2R+2\right)}}\,t\right) \tag{19}$$



(a)



(b)

Fig. 2 Comparison of VA solution of $u(t)$ based on time with the numerical solution for
(a): $m_1 = 2, \quad m_2 = 4, \quad l = 4, \quad g = 10, \quad k = 5, \quad A = 2$
(b): $m_1 = 1, \quad m_2 = 4, \quad l = 5, \quad g = 10, \quad k = 30, \quad A = 5$

5. Results and discussions

In this section, to illustrate the accuracy of variational approach (VA), the results obtained with this method are compared with numerical solutions. It has been shown that the methods work well for whole range of amplitudes.

Comparisons of angular frequencies for different parameters via numerical is presented in Table 1. The maximum relative error between the Variational approach results and numerical results is 1.3%.

Fig. 2 is the displacements results of VA compared with numerical solution for two different cases:

(a): $m_1 = 2, \quad m_2 = 4, \quad l = 4, \quad g = 10, \quad k = 5, \quad A = 2$

(b): $m_1 = 1, \quad m_2 = 4, \quad l = 5, \quad g = 10, \quad k = 30, \quad A = 5$

Fig. 3 is the variation of frequency respect to various parameters of amplitude and *l*. Fig. 4 is the variation of frequency respect to various parameters of two masses of

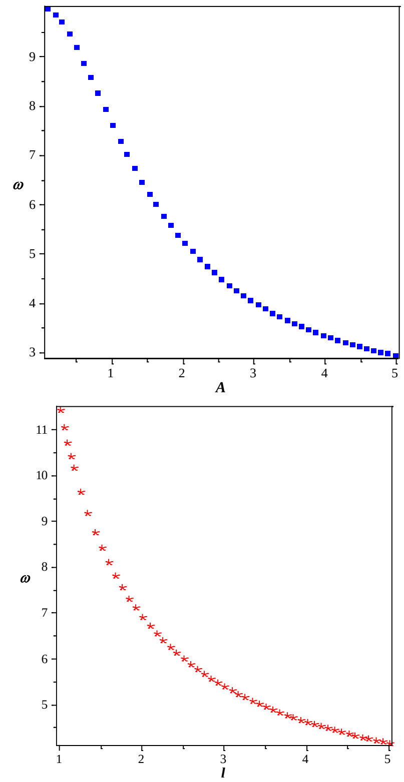


Fig. 3 Variation of frequency respect to various parameters of amplitude and *l*

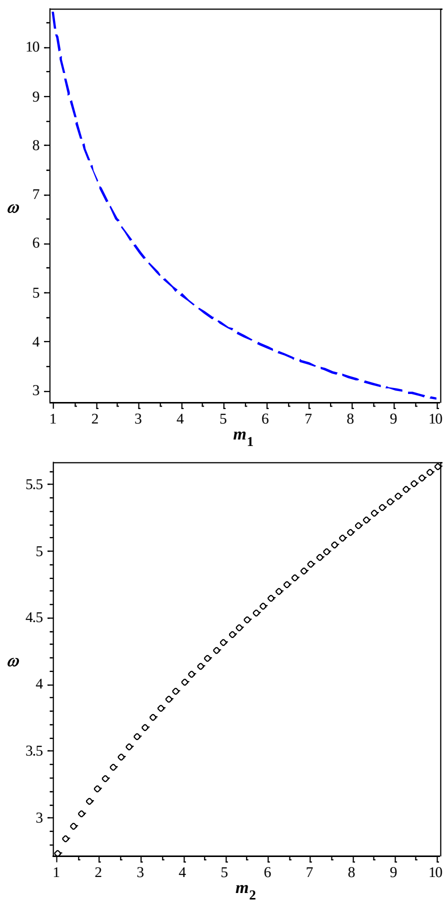


Fig. 4 Variation of frequency respect to various parameters of two masses of the system

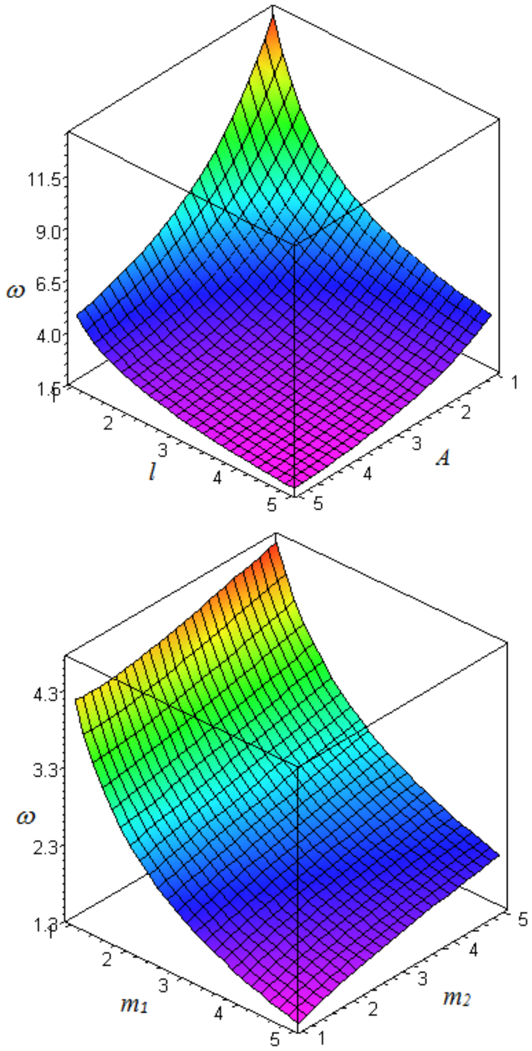


Fig. 5 Sensitivity analysis of nonlinear frequency

the system. Fig. 5 shows the Sensitivity analysis of nonlinear frequency for different important parameters. As shown in Figs. 2 to 5 and Table 1, it is apparent that the Variatioanl approach (VA) has an excellent agreement with the numerical solution using Rung-Kutta and these expressions are valid for a wide range.

6. Conclusions

In this paper, Varitioanl approach method was applied successfully for high nonlinear vibration equations. The validity of the proposed method was compared with numerical solution using Runge-kuttas algorithm. An excellent agreement of the Varitioanl Approach (VA) solutions and the Runge-Kutta solutions shows the reliability and the efficiency of the method. Only one iteration of the proposed approach leads to high accurate solution for whole domain. The method is useful to obtain analytical solution for all oscillators and vibration problems. The Varitioanl Approach (VA) is a well-established method for analyzing nonlinear systems, which can be easily extended to any nonlinear equation.

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