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Vibration analysis of a Timoshenko beam carrying 3D tip mass by using differential transform method

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Abstract. Dynamic behaviour of beam carrying masses has attracted attention of many researchers and engineers. Many studies on the analytical solution of beam with concentric tip mass have been published. However, there are limited works on vibration analysis of beam with an eccentric three dimensional object. In this case, bending and torsional deformations of beam are coupled due to the boundary conditions. Analytical solution of equations of motion of the system is complicated and lengthy. Therefore, in this study, Differential Transform Method (DTM) is applied to solve the relevant equations. First, the Timoshenko beam with 3D tip attachment whose centre of gravity is not coincident with beam end point is considered. The beam is assumed to undergo bending in two orthogonal planes and torsional deformation about beam axis. Using Hamilton's principle the equations of motion of the system along with the possible boundary conditions are derived. Later DTM is applied to obtain natural frequencies and mode shapes of the system. According to the relevant literature DTM has not been applied to such a system so far. Moreover, the problem is modelled by Ansys, the well-known finite element method, and impact test is applied to extract experimental modal data. Comparing DTM results with finite element and experimental results it is concluded that the proposed approach produces accurate results.

Keywords: differential transform method (DTM); bending and torsional vibration; tip mass; Timoshenko beam; natural frequencies; mode shapes

1. Introduction

It is well-known that the analysis of dynamic behaviors of mechanical or structural systems is an essential research area for providing successful design of structures, machines or mechanisms (Salarieh and Ghorashi 2006, Auciello 1996). Among these systems vibrating structures have significant place. Many vibrating structures can be modeled as beam with mass attachments such as robot arms, tall buildings, towers, mast antennas, space crafts, satellites, aircraft wings, the Space Shuttle Remote Manipulator System and the Space Station Mobile Manipulator System (Auciello 1996, Abramovich and Hamburger 1991, Joshi 1995, Kirk and Wiedemann 2002). Furthermore, dynamical behaviors of systems would be improved because of the increased flexibility properties of the system with the attachment of the tip mass (Esmailzadeh and Nakhaire-Jazar 1998). In majority of the studies reported in the literature, the center of tip mass is assumed to be coincident with the beam end point while the dimensions of tip mass are neglected. However, Oguamanam (2003) studied the vibration of an Euler-Bernoulli beam with an eccentric 3D tip mass, and defined the closed form expressions for the orthogonality of modes, and obtained the frequency equation. The eccentricity of the payload was presented in three spatial coordinates for the first time in his study. Later, Oguamanam and Arshad (2005) investigated the similar system considering bending of the beam in two orthogonal planes. Salarieh and Ghorashi (2006) reconsidered the study of Oguamanam (2003) by changing the Euler-Bernoulli Beam with a Timoshenko beam. In another study Gökdağ and Kopmaz (2005) extended the Oguamanam's work to monosymmetric open cross section beam with tip mass and springs, and analyzed the forced and free vibration of the system using Euler-Bernoulli beam theory. Later other studies on beam with mass attachment appeared though none of them considered the tip body exactly as in the previous cases. For instance, Vakil *et al.* (2013) modeled a Timoshenko beam with eccentric tip mass mounted on a cart in order to explain the behavioral analysis of flexible manipulators used in robots and machines. Hamilton's Principle was used to derive the equations of motion and the method of separation of variables was implemented to obtain the closed-form (analytical) expressions. Matt (2013) developed a theoretical model for the transverse vibration of a cantilever beam carrying an axially eccentric tip mass. The governing equations were solved by an integral transform approach based on implicit filter scheme and eigenfunction expansion.

In the relevant literature, the governing equations have been solved analytically although the analytical solution is generally complicated and lengthy. Alternatively, approximate/numerical techniques such as He's variational iteration technique, Galerkin, Frobenius, Adomian decomposition and Rayleigh-Ritz methods have been

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Table 1 Some fundamental DTM rules (Yesilce 2010, Arikoglu and Ozkol 2006)

Original Function	Transformed Function
$f(x)=g(x)\pm h(x)$	$F[k]=G[k]\pm H[k]$
$f(x)=cg(x)$	$F[k]=cG[k]$
$f(x)=\dfrac{d^n g(x)}{dx^n}$	$F[k]=\dfrac{(k+n)!}{k!}G[k+n]$
$f(x)=g(x)h(x)$	$F[k]=\sum_{k_1=0}^{k} G[k_1]H[k-k_1]$
$f(x)=x^n$	$F[k]=\delta(k-n)=\begin{cases}1 & k=n \\ 0 & k\neq n\end{cases}$

Table 2 Boundary condition transform rules (Yesilce 2010, Arikoglu and Ozkol 2006)

$x=0$		$x=L$	
Original Boundary Conditions	Transformed Boundary Conditions	Original Boundary Conditions	Transformed Boundary Conditions
$f(0)=0$	$F[0]=0$	$f(L)=0$	$\sum_{k=0}^{\infty} L^k F[k]$
$\dfrac{df(0)}{dx}=0$	$F[1]=0$	$\dfrac{df(L)}{dx}=0$	$\sum_{k=0}^{\infty} kL^{k-1}F[k]$
$\dfrac{d^2 f(0)}{dx^2}=0$	$F[2]=0$	$\dfrac{d^2 f(L)}{dx^2}=0$	$\sum_{k=0}^{\infty} k(k-1)L^{k-2}F[k]$
$\dfrac{d^3 f(0)}{dx^3}=0$	$F[3]=0$	$\dfrac{d^3 f(L)}{dx^3}=0$	$\sum_{k=0}^{\infty} k(k-1)(k-2)L^{k-3}F[k]$

employed. In recent years, much attention has been devoted to solve vibration problems by using a semi-analytical/numerical method called DTM because of its simplicity and accuracy. For instance, Kaya and Ozgumus (2007) adopted DTM to solve governing equations of motion of flexural-torsional-coupled axially loaded composite Timoshenko beam. Yesilce (2010) investigated vibration characteristics of a moving Bernoulli beam with axial force solving governing equations with DTM. Çatal (2008) implemented DTM in order to solve equations of a beam on elastic soil. Chen and Ho (1998) analysed eigenvalue problems of a rotating twisted Timoshenko beam subjected to the axial loading by DTM. DTM has been used to solve some other boundary value problems, vibration problems, differential-difference equations, control theory, etc. by many researchers (Salehi *et al*. 2012, Liu *et al*. 2013, Ho and Chen 2006, Balkaya *et al*. 2008, Rajasekaran and Tochaei 2014, Ebrahimi and Mokhtari 2015).

In this study, after a brief history and theory of DTM, governing equations of motion and boundary conditions of a Timoshenko beam carrying 3D eccentric tip mass are obtained by Hamilton's principle. Later DTM solution is introduced for the first time to solve the equations of motion of this beam-tip mass system. Then, DTM results are compared with those obtained by ANSYS and experimental solutions.

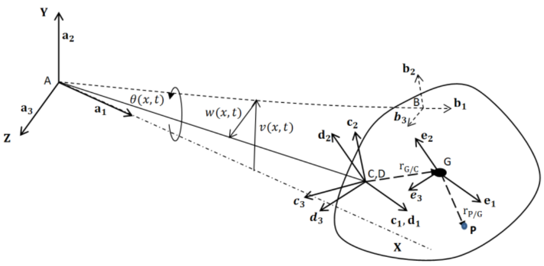


Fig. 1 Clamped-Free Timoshenko beam with 3D tip mass

2. Differential transform method (DTM)

The DTM was introduced by Pukhov (1981, 1982) to solve nonlinear and linear initial value problems, and then it is developed by Zhou (1986) to study electrical circuit problems (Hwang *et al*. 2009, Liu *et al*. 2015). DTM is based on the Taylor's series expansion. By this method some transformation theorems are applied to convert differential equations from the space or time domain into a transformed domain. DTM is an effective method to solve linear/nonlinear boundary and initial value problems, and gives accurate results provided sufficient number of terms in the series expansion is employed. In addition to its accurateness, DTM is easy to code in computer environment.

The differential transform of the k^{th} derivative of a function $f(x)$ at $x=x_0$ is (Yesilce 2010)

$$F\left[k\right]=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{1a}$$

where $f(x)$ is the original function while $F[k]$ is the transformed function. The differential inverse transformation of $F[k]$ is given as

$$f(x)=\sum_{k=0}^{\infty} F\left[k\right]\left(x-x_0\right)^k \tag{1b}$$

Substituting Eq. (1a) into (1b)

$$f(x)=\sum_{k=0}^{\infty}\frac{\left(x-x_0\right)^k}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=x_0} \tag{1c}$$

In practical applications the first N terms in the series expansion is considered:

$$f(x)=\sum_{k=0}^{N} F\left[k\right]x^k,\ F[k]=\frac{1}{k!}\left[\frac{d^k f(x)}{dx^k}\right]_{x=0} \tag{1d}$$

Tables 1 and 2 include other properties of DTM.

3. Formulation

3.1 The governing equations

The Timoshenko beam model shown in Fig. 1 performs two flexural deformations denoted by $v(x,t)$ and $w(x,t)$ in the orthogonal planes along with torsional deformation $\theta(x,t)$ about the longitudinal axis X. There are several frames with unit vectors denoted by $\mathbf{a}_i$, $\mathbf{b}_i$, $\mathbf{c}_i$, $\mathbf{d}_i$, $\mathbf{e}_i$, i=1, 2, 3 to describe the motion of tip mass and beam deformation.

The total kinetic (*T*) and potential (*V*) energies of the whole system as follows

$$\begin{aligned} T &= \frac{1}{2}\int_0^L \rho A\dot{v}^2 dx + \frac{1}{2}\int_0^L \rho A\dot{w}^2 dx + \frac{1}{2}\int_0^L I_P\dot{\theta}^2 dx \\ &+ \frac{1}{2}\int_0^L I_1\rho\dot{\varphi}_1^{\,2} dx + \frac{1}{2}\int_0^L I_2\rho\dot{\varphi}_2^{\,2} dx + \frac{1}{2}\int_{(C)} \dot{\vec{r}}_p^{\,2} dM \end{aligned} \tag{2a}$$

$$\begin{aligned} V &= \frac{1}{2}\int_0^L EI_1(\varphi_1^{'})^2 dx + \frac{1}{2}\int_0^L EI_2(\varphi_2^{'})^2 dx + \frac{1}{2}\int_0^L GJ\theta'^2 dx \\ &+ \frac{1}{2}\int_0^L kGA\left(v'-\varphi_1\right)^2 + \frac{1}{2}\int_0^L kGA\left(w'-\varphi_2\right)^2 \end{aligned} \tag{2b}$$

in which ρ is density, $\vec{r}_P$ is the position vector of a differential element P of the 3D tip mass, E is Young modulus, A is the cross section area of the beam, I_1 and I_2 are area moments of inertia with respect to relevant axes, I_P is the mass moment of inertia per unit length with respect to beam axis, k is shape factor, GJ is torsional stiffness.

Using total kinetic and potential energies of the system the Lagrangian is formed. Then, applying Hamilton's principle, i.e., $\int_{t_1}^{t_2}\delta\left(T-V\right)dt=0$, and after some algebra (see Oguamanam 2003, Oguamanam and Arshad 2005 for similar details) the governing equations of motion and boundary conditions of a uniform Timoshenko beam with tip attachment are obtained as follows:

$$\rho A\ddot{v} - (kGA(v'-\varphi_1))' = 0 \tag{3a}$$

$$\rho A\ddot{w} - (kGA(w'-\varphi_2))' = 0 \tag{3b}$$

$$f_1\ddot{\varphi}_1 - EI_1\varphi_1'' - kGA\left(v'-\varphi_1\right) = 0 \tag{3c}$$

$$f_2\ddot{\varphi}_2 + EI_2\varphi_2'' - kGA\left(w'-\varphi_2\right) = 0 \tag{3d}$$

$$I_P\ddot{\theta} - GJ\theta'' = 0 \tag{3e}$$

where $f_1 = I_1\rho$ and $f_2 = I_2\rho$. Prime and overhead dot symbols denote derivatives with respect to *x* and *t*, respectively.

The boundary conditions depending on the left end of the beam being clamped (C) and free (F) are

$$\bar{I}_{zz}\ddot{\alpha} + M\ddot{v}_L\bar{x} - \bar{I}_{xz}\ddot{\gamma} + \bar{I}_{yz}\ddot{\beta} + EI_1\varphi_1'\left(L,t\right) = 0 \tag{4a}$$

$$\bar{I}_{yy}\ddot{\beta} + M\ddot{w}_L\bar{x} + \bar{I}_{xy}\ddot{\gamma} + \bar{I}_{yz}\ddot{\alpha} + EI_2\varphi_2'\left(L,t\right) = 0 \tag{4b}$$

$$\begin{aligned} &\bar{I}_{xx}\ddot{\gamma} - M\ddot{v}_L\bar{z} + M\ddot{w}_L\bar{y} + \bar{I}_{xy}\ddot{\beta} - \bar{I}_{xz}\ddot{\alpha} \\ &+GJ\theta'\left(L,t\right) = 0 \end{aligned} \tag{4c}$$

$$\begin{aligned} &M\ddot{v}_L - M\ddot{\gamma}\bar{z} + kGA\left(v'\left(L,t\right) - \varphi_1\left(L,t\right)\right) \\ &+M\ddot{\alpha}\bar{x} = 0 \end{aligned} \tag{4d}$$

$$\begin{aligned} &M\ddot{w}_L + M\ddot{\beta}\bar{x} + kGA\left(w'\left(L,t\right) - \varphi_2\left(L,t\right)\right) \\ &+M\ddot{\gamma}\bar{y} = 0 \end{aligned} \tag{4e}$$

$$\varphi_1\left(0,t\right) = 0 \ \text{ or } \ EI_1\varphi'_1(0,t) = 0 \tag{4f}$$

$$\varphi_2\left(0,t\right) = 0 \ \text{ or } \ EI_2\varphi'_2(0,t) = 0 \tag{4g}$$

$$\theta\left(0,t\right) = 0 \ \text{ or } \ GJ\theta'(0,t) = 0 \tag{4h}$$

$$v\left(0,t\right) = 0 \ \text{ or } \ kGA((v'-\varphi_1)(0,t)) = 0 \tag{4i}$$

$$w\left(0,t\right) = 0 \ \text{ or } \ kGA((w'-\varphi_2)(0,t)) = 0 \tag{4j}$$

where

$\bar{I}_{xx} = I_{xx} + M(\bar{z}^2+\bar{y}^2)$, $\bar{I}_{yy} = I_{yy} + M(\bar{x}^2+\bar{z}^2)$

$\bar{I}_{zz} = I_{zz} + M(\bar{x}^2+\bar{y}^2)$, $\bar{I}_{xy} = I_{xy} + M\overline{xy}$, $\bar{I}_{xz} = I_{xz} + M\overline{xz}$

$\bar{I}_{yz} = I_{yz} + M\overline{yz}$ .M denotes tip mass I_{xx} , I_{xz} , I_{yy} , I_{zz} , I_{xy} , I_{yz} are components of the tip mass inertia tensor. $\bar{x},\bar{y},\bar{z}$ are the coordinates of the tip mass center of gravity G with respect to the beam end point C.

The method of separation of variables ($v(x,t) = Ve^{i\omega t}$, $w(x,t) = We^{i\omega t}$, $\theta(x,t) = \Theta e^{i\omega t}$, $\varphi_1(x,t) = \psi_1 e^{i\omega t}$, $\varphi_2(x,t) = \psi_2 e^{i\omega t}$) is applied to rewrite the equations of motions and boundary conditions

$$V'' + \lambda_1 V - \psi_1' = 0 \tag{5a}$$

$$\psi''_1 + \left(\lambda_2 - \lambda_3\right)\psi_1 + \lambda_3 V' = 0 \tag{5b}$$

$$W'' + \lambda_1 W - \psi_2' = 0 \tag{5c}$$

$$\psi''_2 + \left(\lambda_4 - \lambda_5\right)\psi_2 + \lambda_5 W' = 0 \tag{5d}$$

$$\Theta'' + \lambda_6^2\Theta = 0 \tag{5e}$$

where

$$\begin{aligned} &\lambda_1 = \frac{\rho\omega^2}{kG},\ \lambda_2 = \frac{f_1\omega^2}{EI_1},\ \lambda_3 = \frac{kGA}{EI_1},\ \lambda_4 = \frac{\rho\omega^2}{EI_2}, \\ &\lambda_5 = \frac{kGA}{EI_2},\ \lambda_6 = \left(\frac{I_P w^2}{GJ}\right)^{1/2} \end{aligned} \tag{6}$$

with the associated boundary conditions for the C and F beam at $x = L$;

$$\begin{aligned}&\bar{I}_{zz}\omega^2\psi_1(L)+M\bar{x}\omega^2V(L)-\bar{I}_{xz}\omega^2\Theta(L)\\&+\bar{I}_{yz}\omega^2\psi_2(L)-EI_1(x)\psi'_1(L)=0\end{aligned}\tag{7a}$$

$$\begin{aligned}&\bar{I}_{yy}\omega^2\psi_2(L)+M\bar{x}\omega^2W(L)+\bar{I}_{xy}\omega^2\Theta(L)\\&+\bar{I}_{yz}\omega^2\psi_1(L)-EI_2(x)\psi'_2(L)=0\end{aligned}\tag{7b}$$

$$\begin{aligned}&\bar{I}_{xx}\omega^2\Theta(L)-M\bar{z}\omega^2V(L)+M\bar{y}\omega^2W(L)\\&+\bar{I}_{xy}\omega^2\psi_2(L)-\bar{I}_{xz}\omega^2\psi_1(L)-GJ\Theta'(L)=0\end{aligned}\tag{7c}$$

$$\begin{aligned}&M\omega^2V(L)+\left(M\bar{x}\omega^2+kGA(x)\right)\psi_1(L)\\&-M\bar{z}\omega^2\Theta(L)-kGA(x)V'(L)=0\end{aligned}\tag{7d}$$

$$\begin{aligned}&M\omega^2W(L)+\left(M\bar{x}\omega^2kGA(x)\right)\psi_2(L)\\&+M\bar{y}\omega^2\Theta(L)-kGA(x)W'(L)=0\end{aligned}\tag{7e}$$

at $x=0$;

$$\psi_1(0)=0 \text{ or } EI_1\psi_1'(0)=0 \tag{7f}$$

$$\psi_2(0)=0 \text{ or } EI_2\psi_2'(0)=0 \tag{7g}$$

$$\Theta(0)=0 \text{ or } GJ\Theta'(0)=0 \tag{7h}$$

$$V(0)=0 \text{ or } kGA(V'(0)-\psi_1(0))=0 \tag{7i}$$

$$W(0)=0 \text{ or } kGA(W'(0)-\psi_2(0))=0 \tag{7j}$$

3.2 DTM formulation

DTM theorems presented in Tables 1 and 2 are applied to the governing Eqs. (5a)-(5e) and boundary condition Eqs. (7a)-(7j) in order to obtain recurrence expressions and transformed boundary conditions, respectively.

$$\bar{V}[k+2]=\frac{k!}{(k+2)!}\left\{\frac{(k+1)!}{k!}\bar{\psi}_1[k+1]-\lambda_1\bar{V}[k]\right\}\tag{8a}$$

$$\bar{\psi}_1[k+2]=\frac{k!}{(k+2)!}\left\{(\lambda_3-\lambda_2)\bar{\psi}_1[k]-\frac{(k+1)!}{k!}\lambda_3\bar{V}[k+1]\right\}\tag{8b}$$

$$\bar{W}[k+2]=\frac{k!}{(k+2)!}\left\{\frac{(k+1)!}{k!}\bar{\psi}_2[k+1]-\bar{\lambda}_1W[k]\right\}\tag{8c}$$

$$\bar{\psi}_2[k+2]=\frac{k!}{(k+2)!}\left\{(\lambda_5-\lambda_4)\bar{\psi}_2[k]-\frac{(k+1)!}{k!}\lambda_5\bar{W}[k+1]\right\}\tag{8d}$$

$$\bar{\Theta}[k+2]=-\frac{k!}{(k+2)!}\lambda_6^2\bar{\Theta}[k]\tag{8e}$$

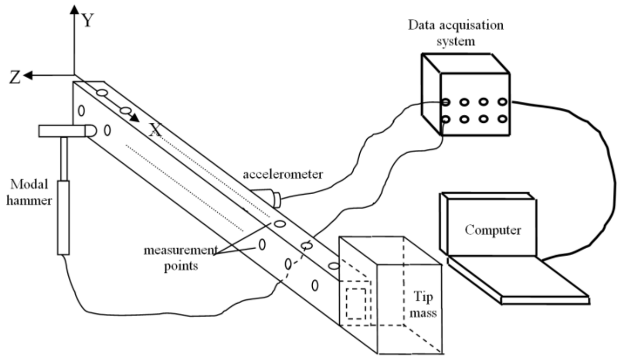


Fig. 2 The schematic view of the experimental setup

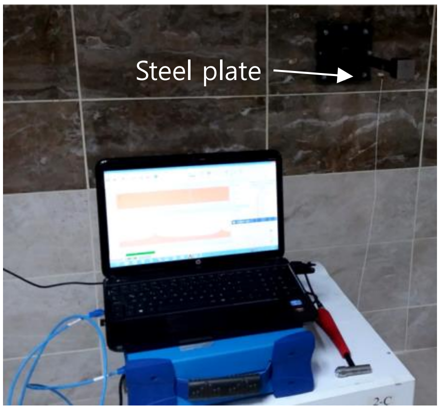


Fig. 3 Photograph of the experimental setup

Boundary conditions for the C beam;
at $x=L$;

$$\begin{aligned}&\bar{I}_{zz}\omega^2\sum_{k=0}^{N}L^k\bar{\psi}_1[k]+M\bar{x}\omega^2\sum_{k=0}^{N}L^k\bar{V}[k]\\&-\bar{I}_{xz}\omega^2\sum_{k=0}^{N}L^k\bar{\Theta}[k]+\bar{I}_{yz}\omega^2\sum_{k=0}^{N}L^k\bar{\psi}_2[k]\\&-EI_1\sum_{k=1}^{N}kL^{k-1}\bar{\psi}_1[k]=0\end{aligned}\tag{9a}$$

$$\begin{aligned}&\bar{I}_{yy}\omega^2\sum_{k=0}^{N}L^k\bar{\psi}_2[k]+M\bar{x}\omega^2\sum_{k=0}^{N}L^k\bar{W}[k]\\&+\bar{I}_{xy}\omega^2\sum_{k=0}^{N}L^k\bar{\Theta}[k]+\bar{I}_{yz}\omega^2\sum_{k=0}^{N}L^k\bar{\psi}_1[k]\\&-EI_2\sum_{k=1}^{N}kL^{k-1}\bar{\psi}_2[k]=0\end{aligned}\tag{9b}$$

$$\begin{aligned}&\bar{I}_{xx}\omega^2\sum_{k=0}^{N}L^k\bar{\Theta}[k]-M\bar{z}\omega^2\sum_{k=0}^{N}L^k\bar{V}[k]\\&-M\bar{y}\omega^2\sum_{k=0}^{N}L^k\bar{W}[k]+\bar{I}_{xy}\omega^2\sum_{k=0}^{N}L^k\bar{\psi}_2[k]\\&-\bar{I}_{xz}\omega^2\sum_{k=0}^{N}L^k\bar{\psi}_1[k]-GJ\sum_{k=1}^{N}kL^{k-1}\bar{\Theta}[k]=0\end{aligned}\tag{9c}$$

$$\begin{aligned}&M\omega^2\sum_{k=0}^{N}L^k\bar{V}[k]-M\bar{z}\omega^2\sum_{k=0}^{N}L^k\bar{\Theta}[k]\\&+(M\bar{x}\omega^2+kGA)\sum_{k=0}^{N}L^k\bar{\psi}_1[k]-kGA\sum_{k=1}^{N}kL^{k-1}\bar{V}[k]=0\end{aligned}\tag{9d}$$

$$\begin{aligned} &M\omega^2 \sum_{k=0}^{N} L^k \overline{W}[k] - M\overline{y}\omega^2 \sum_{k=0}^{N} L^k \overline{\Theta}[k] \\ &+(M\overline{x}\omega^2 + kGA)\sum_{k=0}^{N} L^k \overline{\psi}_2[k] - kGA\sum_{k=1}^{N} kL^{k-1}\overline{W}[k] = 0 \end{aligned} \tag{9e}$$

at $x = 0$;

$$\begin{aligned} &\overline{V}[0] = 0,\ \overline{W}[0] = 0,\ \overline{\Theta}[0] = 0,\ \overline{\psi}_1[0] = 0, \\ &\overline{\psi}_2[0] = 0 \end{aligned} \tag{9f}$$

Boundary conditions at x=0 for the F beam;

$$\begin{aligned} &\overline{\Theta}[1] = 0,\ \overline{\psi}_1[1] = 0,\ \overline{\psi}_2[1] = 0 \\ &\overline{V}[1] - \overline{\psi}_1[0] = 0,\ \overline{W}[1] - \overline{\psi}_2[0] = 0 \end{aligned} \tag{10}$$

in which $\overline{V},\ \overline{W},\ \overline{\psi}_1,\ \overline{\psi}_2$ and $\overline{\Theta}$ are transformed functions and $V,\ W,\ \psi_1,\ \psi_2, \Theta$ are original functions.

By incorporating boundary conditions, a set of equations is obtained.

$$\begin{bmatrix} A_{11}(\omega) & A_{12}(\omega) & A_{13}(\omega) & A_{14}(\omega) & A_{15}(\omega) \\ A_{21}(\omega) & A_{22}(\omega) & A_{23}(\omega) & A_{24}(\omega) & A_{25}(\omega) \\ A_{31}(\omega) & A_{32}(\omega) & A_{33}(\omega) & A_{34}(\omega) & A_{35}(\omega) \\ A_{41}(\omega) & A_{42}(\omega) & A_{43}(\omega) & A_{44}(\omega) & A_{45}(\omega) \\ A_{51}(\omega) & A_{52}(\omega) & A_{53}(\omega) & A_{54}(\omega) & A_{55}(\omega) \end{bmatrix} \begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{Bmatrix} \tag{11a}$$

where c_i (i=1, 2, 3, 4, 5) are undetermined constants. The natural frequencies make the determinant of the coefficient matrix equal to zero.

$$\begin{vmatrix} A_{11}(\omega) & A_{12}(\omega) & A_{13}(\omega) & A_{14}(\omega) & A_{15}(\omega) \\ A_{21}(\omega) & A_{22}(\omega) & A_{23}(\omega) & A_{24}(\omega) & A_{25}(\omega) \\ A_{31}(\omega) & A_{32}(\omega) & A_{33}(\omega) & A_{34}(\omega) & A_{35}(\omega) \\ A_{41}(\omega) & A_{42}(\omega) & A_{43}(\omega) & A_{44}(\omega) & A_{45}(\omega) \\ A_{51}(\omega) & A_{52}(\omega) & A_{53}(\omega) & A_{54}(\omega) & A_{55}(\omega) \end{vmatrix} = 0 \tag{11b}$$

The obtained natural frequencies are inserted into Eq. (11b) to find constants c_i (i=1, 2, 3, 4, 5) which is necessary for the mode shapes.

4. Experimental procedure

It is difficult to measure angular displacements, while linear motions can easily be measured by ordinary modal accelerometers. Thus, in this study, only lateral vibration modes, i.e., V_i and W_i, are considered. Impact test is applied, since it is the cheapest and easiest way of modal data acquisition method. Figs. 2 and 3 illustrate the configuration used in the experiments. The modal hammer (Endevco 2302-10) with steel tip is employed to apply impact-like force, and the response of the system along the same direction is measured by a single axis accelerometer (Dytran 3097A2T). For the modal data V_i and corresponding frequencies impact and response directions are along Y axis, whereas impact and response directions are along Z axis for W_i and corresponding frequencies. Input and response analog signals are sent to the data acquisition system (OROS OR36), where signals are filtered and conditioned. Then, accelerance type frequency response functions (FRFs) are computed in the OROS NVGate Version 8.00.002 software environment. Finally, peak picking method (He and Fu 2001), which is the easiest way of modal parameter estimation, is implemented to extract modal data such as natural frequencies, damping ratios and modal constants. During the experiments a rectangular tubular cross section beam with length 0.5 m is employed. Twenty measurement points with 1 inch spacing are determined on the beam. Thus, twenty FRF curves are measured for each Y and Z directions (see Fig. 4).

Table 3 The first three natural frequencies (Hz) of C beam with tip mass

Along Z axis			
Method	ω_1	ω_2	ω_3
DTM (D)	32.43	341.19	979.44
ANSYS (A)	32.49	339.62	920.83
Experiment (E)	29.41	311.88	914.37
Along Y axis			
Method	ω_1	ω_2	ω_3
DTM (D)	45.06	423.11	512.43
ANSYS (A)	45.5	427.69	514.38
Experiment (E)	38.75	392.69	469.59

5. Results and discussions

The recursive expressions and the transformed boundary conditions in Eqs. (8), (9) are coded in Matlab environment to obtain natural frequencies ω_i (i=1, 2, 3) and mode shapes. As an example, a uniform beam with section sizes 20 mm (width), 30 mm (height) and thickness 2 mm is considered. Other properties are as follows L=0.5 m, E=205 GPa, ρ=7850 kg/m^3, ν=0.3 (Poisson's ratio). The 3D tip mass is a cubic block with 50 mm side length and made of the same material as beam. The center of gravity coordinates of the tip mass with respect to the attachment point of the beam are 25mm; 10mm; -15mm (X, Y, Z axes respectively). Clamped end conditions are satisfied by welding a steel plate of thickness 8mm and sizes $20\text{x}20\ cm^2$ to the left end of the beam. Then the plate is screwed to the wall (Fig. 4). For the free end condition the beam-tip mass structure is hanged by strings. In Ansys environment SOLID187 elements are used to mesh the structure. For the DTM computations the first thirty terms in the series expansions are regarded (i.e., N=30 in Eq. (9)) since sufficient convergence is provided. It was observed that DTM results are the same as the analytical results, thus analytical results are not given in the following tables and plots.

Table 3 includes the natural frequencies of the system by the three approaches, i.e., DTM, Ansys, experiment. It is clear that DTM results are close to the others. The difference between the experimental and the other results may be attributed to the difficulty of satisfying clamped end conditions in laboratory conditions. Moreover, the element

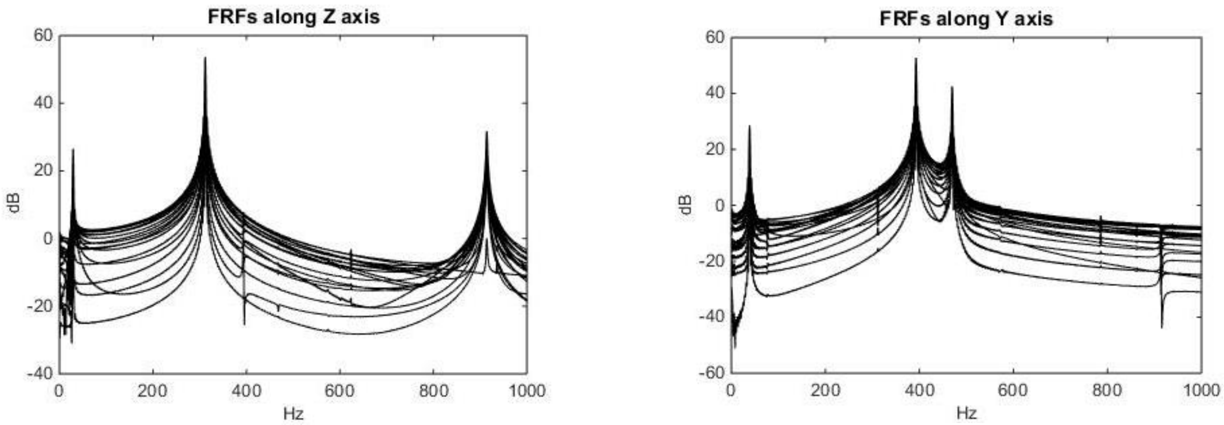
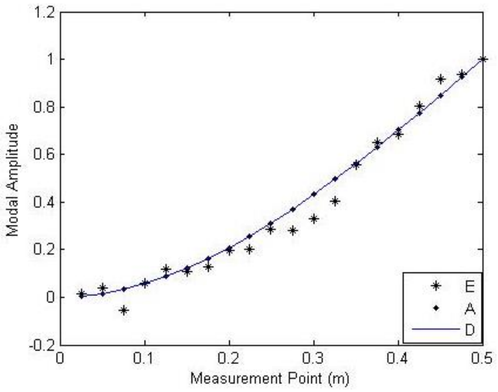
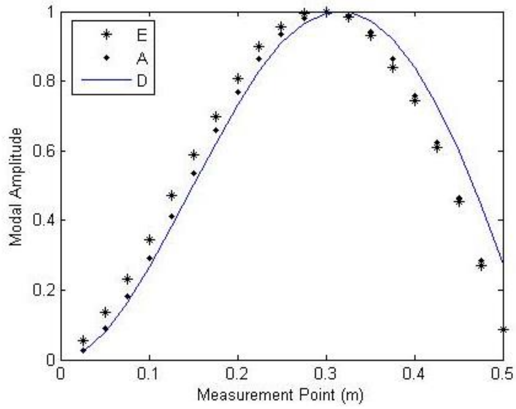


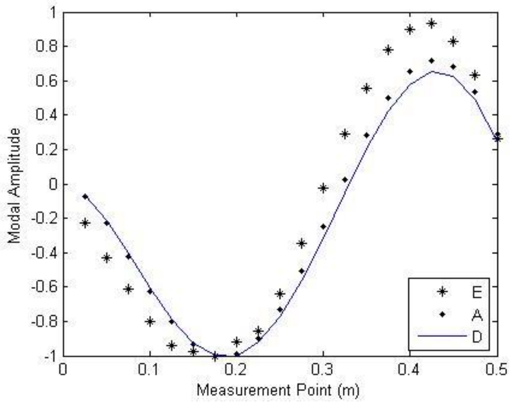
Fig. 4 FRFs of the C beam



(a) First Mode Shapes

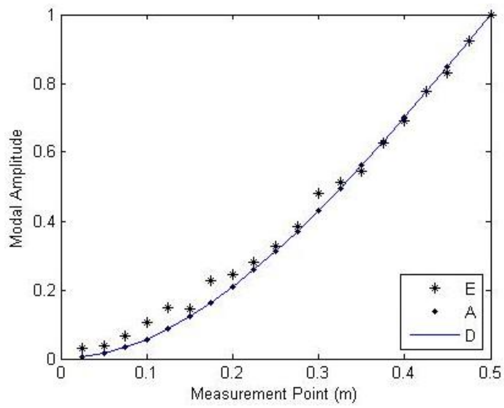


(b) Second Mode Shapes

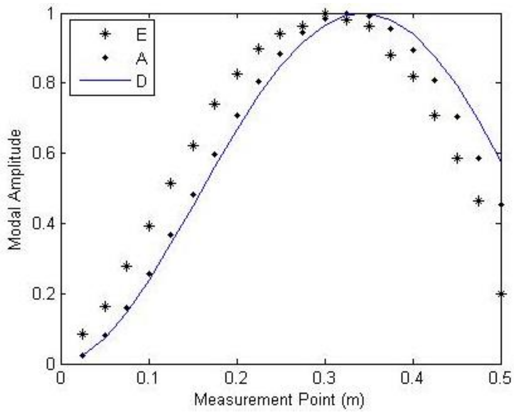


(c) Third Mode Shapes

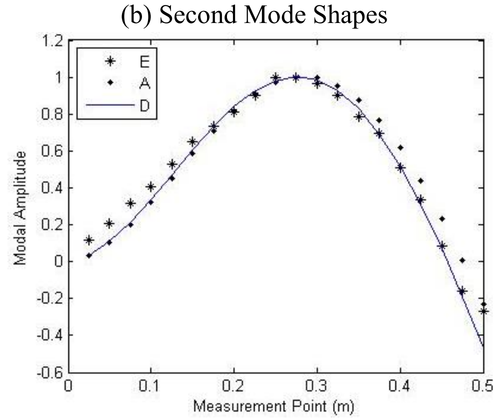
Fig. 5 The first three mode shapes of the C beam-tip mass system along Z axis



(a) First Mode Shapes



(b) Second Mode Shapes



(c) Third Mode Shapes

Fig. 6 The first three mode shapes of the C beam-tip mass system along Y axis

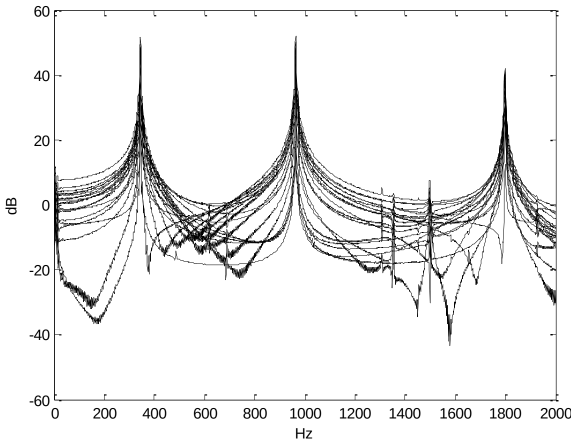
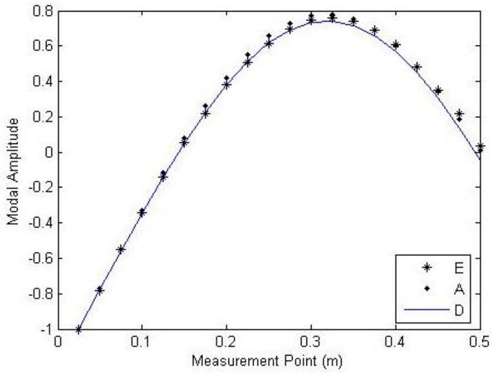
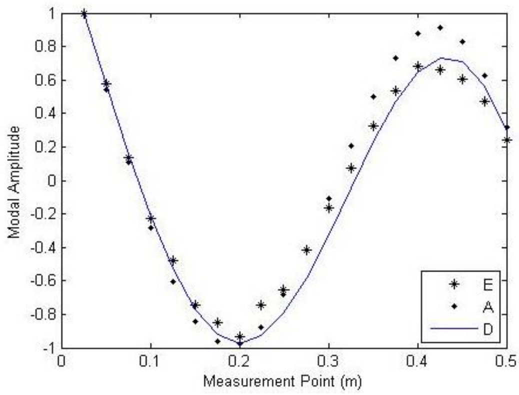


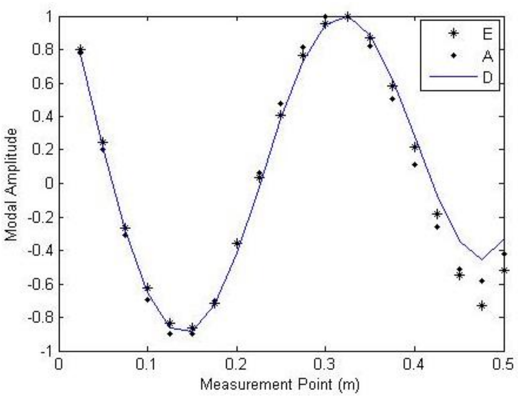
Fig. 7 FRFs of the F beam along Z axis



(a) First Mode Shapes



(b) Second Mode Shapes



(c) Third Mode Shapes

Fig. 8 Vibration modes of the F beam with-tip mass along Z axis

Table 4 The first three natural frequencies (Hz) of F beam with tip mass

Along Z axis			
Method	ω_1	ω_2	ω_3
DTM (D)	351.8	986.6	1801.5
ANSYS (A)	354.6	999.8	1851.2
Experiment (E)	345.0	967.5	1798.5
Along Y axis			
Method	ω_1	ω_2	ω_3
DTM (D)	496.7	1390.7	1711.7
ANSYS (A)	497.9	1394.9	1518.0

type in the finite element model and the degree of freedom of the theoretical model are the probable issues that lead to the difference between Ansys and DTM results.

Figs. 5 and 6 show the comparison of the first three normalized mode shapes of the C beam along Z and Y axes. It is clear that generally experimental and numerical modes are sufficiently compatible. If the bare beam was considered, experimental and numerical results would be more compatible. However, the complicated effect of the tip mass seems lead to some difference between numerical and experimental results.

Another experiment is implemented for the F beam along Z axis. Fig. 7 demonstrates the measured FRFs, and Table 4 includes numerical results as well as the experimental along Z axis. Besides, mode shapes are compared in Fig. 8. This time it is observed the experimental results are closer to the numerical results, since free end condition can accurately be modeled by just hanging the beam from two points. The experimental modes along the Y axis were not be measured accurately, which may be due to the deficiency of the experimental set up, thus comparison of DTM with experiment along this axis is not given. Only comparison of DTM and ANSYS results was given along Y axis.

6. Conclusions

In this paper, the free vibration analysis of a Timoshenko beam carrying 3D tip mass whose center of gravity is not coincident with beam end is considered. The governing equations and boundary conditions of the system are derived by using Hamilton's Principle. Later, DTM is applied in order to solve differential equations, and obtain natural frequencies and normalized mode shapes for C and F end conditions. Although, there are some studies on vibration analysis of beam where DTM is used, it is the first time DTM is used to solve this beam-tip mass system. Moreover, ANSYS and vibration experiment are performed to compare natural frequency and mode shape results. Experimental values are obtained by the impact testing, and peak picking approach is employed to extract modal data. It is observed that experimental and ANSYS results are generally close to DTM results, though some of them include remarkable errors. These errors mainly due to the ineffectiveness of impact testing and dissimilarity of element type of beam model between theoretical and ANSYS.

It is concluded that, based on the presented results, DTM is very efficient and accurate method to solve high order differential equations and boundary value problems.

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