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Static and dynamic analysis of cable-suspended concrete beams

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Abstract. A new theory of weightless sagging planer elasto-flexible cables under point loads is developed earlier by the authors and used for predicting the nonlinear dynamic response of cable-suspended linear elastic beams. However, this theory is not valid for nonlinear elastic cracked concrete beams possessing different positive and negative flexural rigidity. In the present paper, the flexural response of simply supported cracked concrete beams suspended from cables by two hangers is presented. Following a procedure established earlier, rate-type constitutive equations and third order nonlinear differential equations of motion for the structures undergoing small elastic displacements are derived. Upon general quasi-static loading, negative nodal forces, moments and support reactions may be introduced in the cable-suspended concrete beams and linear modal frequencies may abruptly change. Subharmonic resonances are predicted under harmonic loading. Uncoupling of the nodal response is proposed as a more general criterion of crossover phenomenon. Significance of the bilinearity ratio of the concrete beam and elasto-configurational displacements of the cable for the structural response is brought out. The relevance of the proposed theory for the analysis and the design of the cable-suspended bridges is critically evaluated.

Keywords: cable-suspended concrete beams; configurational nonlinearity; rate-type constitutive equations; third-order equations of motion; crossover phenomenon

1. Introduction

In contrast to the conventional elastic structures, the configuration of the suspension cables of cable-suspended bridges under dead loads has also to be determined. Structural behavior of cable-suspended bridges with specified configuration is affected by the mechanical properties of the suspension cables and the bridge deck apart from the cable-deck interaction. The relevant available literature dealing with these aspects critically evaluated by the authors elsewhere (Kumar *et al*. 2016, 2017) is summarized here. Adopting Lagrangian kinematical description, the constitutive relations for elastic sagging cables are stated in terms of their local axial tension and elastic extension. The elastic suspension cables are rendered nonlinear because of their nonlinear stress-strain relations and their finite elastic displacements under forces transferred through the suspenders. Also, the presence of the self-weight introduces nonlinearity in the nodal force-displacement relations of the cable catenary segments between the suspenders. Discrete formulations are based upon tangent elastic and geometric stiffness matrices (Rega 2004, Antman 2005, Lacarbonara 2013).

Linear modal frequencies and the mode shapes of a single shallow elastic planer cable are known to depend upon a geometric- elastic parameter depending upon the sag/span ratio, axial elastic stiffness and tensile forces in it. The lowest frequency pertains to the first symmetric mode at lower values of the above parameter, while it pertains to the first anti-symmetric mode at higher values of the parameter. The crossover point corresponding to the critical value of the parameter implies the equality of these symmetric and anti-symmetric modal frequencies (Irvine and Caughey 1974).

Modern theories of cable-suspended bridges attempt to incorporate the effect of spatial and flexural-torsional behavior, axial and shear deformations, and rotatory inertia of the deck as well as the elasticity of the suspenders. Complex cable-deck interaction models are developed to establish the incremental constitutive equations and second-order nonlinear differential equations of motion of the structure. These equations are then used to determine its dead load configuration as well as its static and dynamic response (Kim and Lee 2001, Santos and Paulo 2011, Lacarbonara 2013, Coarita and Flores 2015). Spatial catenary cable element have been used for nonlinear analysis of cable-supported structures (Vu *et al*. 2012). Substantial temperature variation experienced by the space-deployed cable-beam structures are shown to introduce mechanical vibrations (Deng *et al*. 2015). Methodology for economic evaluation of the cable-supported structures is also investigated (Sun *et al*. 2016).

Recently, the authors have proposed a new theory of single weightless sagging planer elasto-flexible cables with linear stress-strain relations and undergoing small elastic displacements when subjected to point loads. Obviously,

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none of the above-mentioned causes of nonlinearity of cables are present. Still, the cables exhibit peculiar configurational nonlinearity associated with the dependence of their natural state upon the applied nodal forces. These structures lack unique natural state configuration and their elastic displacements are defined in reference to their instantaneous natural state obtained by proportional unloading from the equilibrium state. These structures are shown to belong to the class of homogeneous mechanical systems. Their configuration-defining nodal coordinates and nodal elastic displacements are functions homogeneous of order zero and unity of the equilibrated forces. Rate-type constitutive equations and third order nonlinear differential equations of motion are derived. New energy and contra-gradient principles are also deduced (Menon 2009, Kumar *et al*. 2016).

A simple structure consisting of a simply-supported linear elastic beam suspended from two parallel sagging cables has been investigated by the authors to model the cable-suspended bridges (Kumar *et al*. 2017). Dead load configuration of the structure with vertical suspenders is established by using the Beam on Elasto-Flexible Support (BEFS) Model. Compatibility condition between the vertical elastic nodal displacements of the beam under load increments and the total configurational and elastic nodal displacements of the cable is satisfied. The rate-type constitutive equations and third order nonlinear differential equations of motion are derived for the cable-suspended beam capable of undergoing small flexural and torsional displacements and vibrations. Typical static and dynamic response predicted for the 4-DOF structure reveals the significance of the configurational response of the suspension cable.

It is well-known that dead load of the cable-suspended bridges is resisted mainly by the cables. Simply-supported deck beam suspended from the suspension cables is subjected to small positive flexural moments at all the points. During vibrations, the flexural moments are expected to exhibit periodic variations resulting in change of sense of nodal flexural moments. Flexural-torsional deck vibrations are found to be coupled with vibrations of the suspension cables (Lepidi and Gattulli 2014). Generally, the geometrical and reinforcement details of sections of fully cracked concrete beams, e.g. T-beams and box-girders, are asymmetrical about the horizontal axis of flexure. Thus, their flexural rigidity depends upon the sense of the applied flexural moment. Flexural rigidity distribution of such concrete beams is determined by finding the location of the point of contra-flexure on the beam. Despite the nonlinearity associated with sliding of their point of contra-flexure, their tangent stiffness matrices are shown by Benipal to play the role of secant stiffness matrices (Benipal 1992, 1994). Effect of bilinearity ratio and damping on the vibration response of SDOF cracked concrete beams supporting a point mass has been investigated (Pandey and Benipal 2006, 2011).

The objective of the present paper is to investigate the flexural behavior of cable-suspended nonlinear elastic concrete beams. The stiffness matrix of the nonlinear elastic cracked concrete beam is established for different cases

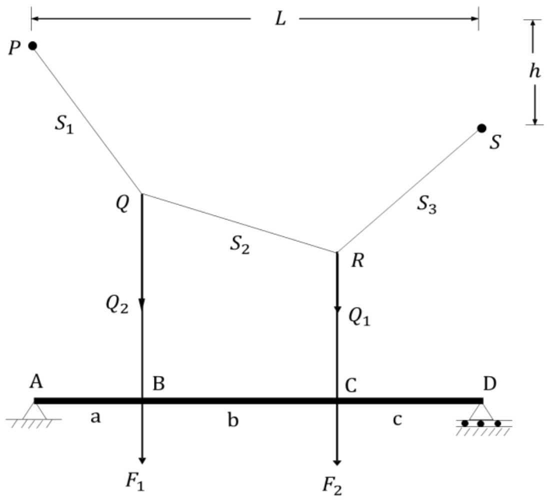


Fig. 1 Cable-suspended concrete beam

defined by senses of the two nodal moments. Thereafter, the methodology developed earlier by the authors (Kumar *et al*. 2017), for steel beam with constant stiffness matrix is followed. The same dead load configuration is adopted. Rate-type constitutive equations and third order differential equations on motion are derived for this 2-DOF structure undergoing flexural displacements. Static analysis confirms the change in the sense of the nodal moments with variation of nodal loads. Different aspects of typical dynamic response to harmonic loading are explored. The relevance of the paper for the design of cable-suspended bridges with concrete decks is discussed.

2. Theoretical formulation

The derivation of the governing third order nonlinear differential equation of motion for cable-suspended linear elastic beams is presented elsewhere by the authors (Kumar *et al*. 2017). Theoretical formulation also valid for the cable-suspended nonlinear elastic cracked concrete beams is presented here. Consider a simply supported concrete beam AD supported at two sections, *B* and *C*, on its span from a cable by two vertical suspenders as shown in Fig. 1

Nodal vertical loads applied at points *B* and *C* are resisted by the combined action of the beam and the cable arranged in parallel. The dead load configuration is determined by BEFS Model. The third order differential equation of motion of the cable-suspended beam is stated below in terms of its vertical displacements (v) as

$$M\dddot{v} + C\ddot{v} + K\dot{v} = \dot{F} \tag{1}$$

The tangent stiffness matrix $(K = k_c + k_d)$ of the structure is obtained as the sum of tangent stiffness matrices k_c and k_d of cable and the beam respectively. The constant damping matrix $(C = a_0\, M + a_1\, K_0)$ is determined from the mass matrix (M) and the initial stiffness matrix (K_0). The above equation of motion requires following nodal load-displacements relation in rate form as

$$\dot{F} = \dot{Q} + \dot{F}_d = k_c\dot{v} + k_d\dot{v} = K\dot{v} \tag{2}$$

The following dynamic equilibrium condition have also to be satisfied at all instants

$$Q(t) = F(t) - M\ddot{v} - C\dot{v} - k_d v \tag{3}$$

For specified nodal initial displacements $v(0) = \bar{v}$ and velocities $\dot{v}(0)$, the required nodal initial accelerations are determined as follows

$$\ddot{v} = M^{-1}\left[F(0) - C\dot{v}(0) - \bar{F}\right] \tag{4}$$

In Eq. (4), $\bar{F}$ represents the force corresponding to the initial nodal displacements $\bar{v}$ via equations of static equilibrium. The compatibility condition expressed as $(x - x_0) + u = v$ and in the rate form as $\dot{x} + \dot{u} = \dot{v}$ has also to be satisfied. Here, x_0 and x represent the nodal coordinates defining the natural state configuration of the cable under dead loads and the current loads respectively. The tangent stiffness matrix k_c of the cable is evaluated as follows from the constitutive equations for the cable stated in the rate form (Kumar *et al*. 2017) as

$$\begin{gathered} \dot{x} = D\dot{Q} \qquad \dot{u} = f\dot{Q} \\ \dot{x} + \dot{u} = \dot{v} = (D + f)\dot{Q} = N\dot{Q} \qquad \dot{Q} = k_c\dot{v} \end{gathered} \tag{5}$$

The tangent flexibility matrix for the cracked concrete beam is obtained here by following the procedure developed earlier (Benipal 1992). The nodal forces acting downwards and the flexural moments causing tension at the bottom of the beam are considered positive. Depending upon the relative magnitudes of the nodal forces (F_{d1}, F_{d2}) acting on the concrete beam and sense of introducing nodal moments (M_B, M_C) following four elastically-distinct cases are identified:

Case (I) $M_B \geq 0$ and $M_C \geq 0$

If $F_{d1} \geq 0$: $\frac{F_{d2}}{F_{d1}} \geq -\left(\frac{a}{a+b}\right)$

If $F_{d1} < 0$: $\frac{F_{d2}}{F_{d1}} \geq -\left(\frac{b+c}{c}\right)$

Case (II) $M_B \geq 0$ and $M_C < 0$

If $F_{d1} \geq 0$: $-\left(\frac{b+c}{c}\right) < \frac{F_{d2}}{F_{d1}} < -\left(\frac{a}{a+b}\right)$

If $F_{d1} < 0$: Impossible

Case (III) $M_B < 0$ and $M_C \geq 0$

If $F_{d1} > 0$: Impossible

If $F_{d1} < 0$: $-\left(\frac{a}{a+b}\right) < \frac{F_{d2}}{F_{d1}} < -\left(\frac{b+c}{c}\right)$

Case (IV) $M_B < 0$ and $M_C < 0$

If $F_{d1} \geq 0$: $\frac{F_{d2}}{F_{d1}} \geq -\left(\frac{a}{a+b}\right)$,

If $F_{d1} < 0$: $\frac{F_{d2}}{F_{d1}} \geq -\left(\frac{b+c}{c}\right)$

It can be observed that the case (II) does not arise when the nodal force F_{d1} is negative, while case (III) arises only when F_{d1} is negative. The bilinear cracked concrete beams generally possess two values of flexural rigidity due to asymmetry in section geometry and different reinforcement details. Which of these two values of flexural rigidity will be operational at a section depends upon the sense of the flexural moment acting at that section. Here, EI_1 and EI_2 represent the flexural rigidity values of the prismatic concrete beam under positive and negative flexural moments. Bilinearity ratio (r) is defined as $EI_2 = r\,EI_1$. In case (I) flexural rigidity is EI_1 and in case (IV) flexural rigidity is EI_2 remains same at all sections of the beam. However, the sense of both the nodal moments are different in the case (II) and case (III) and the point of contraflexure lies in the segment BC. The beam possesses different flexural rigidity on both sides of the point of contraflexure. Its location is determined by the ratio of the nodal forces. Shifting of the point of contraflexure with variation of nodal forces results in change in the stiffness thus rendering it nonlinear. In contrast, the beam exhibits linear elastic behavior in load case (I) and case (IV).

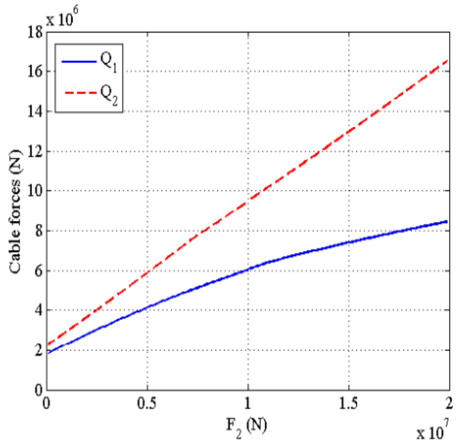
The cracked concrete beams are shown to belong to the class of First Order Homogeneous Mechanical (FOHM) Systems as their nodal displacements are functions homogeneous of order unity of the nodal forces and vice-versa. Also, the location of the point of contraflexure as well as tangent flexibility matrix coefficients in cases II and III are zero order functions homogeneous of the nodal forces. In linear cases (I) and (IV), the flexibility matrix is independent of the loads. In all the cases mentioned, the complementary energy is second order homogeneous function of the nodal forces. The linear elastic systems also belong to the class of FOHM Systems. For developing the constitutive equations for the 2-DOF determinate concrete beam in all these cases, expression for the complementary energy (Ω) is formulated in each case. Application of Castigliano theorem yields the nodal force-displacement relations for these conservative mechanical systems. Then, the coefficients of the symmetric tangent flexibility matrix (f_{dij}) are obtained as

$$\begin{gathered} \Omega = \Omega(F_{d1}, F_{d2}) \quad v = \partial\Omega / \partial F_{di} \quad F_{dij} = \partial v_i / \partial F_{dj} \\ F_{di} = k_{dij} v_j \qquad k_{dij} = f_{dij}^{-1} \end{gathered} \tag{6}$$

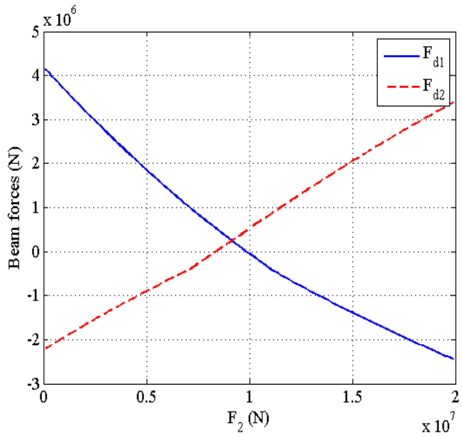
The constitutive equations can also be stated in the incremental and the rate form. For these FOHM Systems, the tangent flexibility and stiffness matrices also play the role of secant flexibility and stiffness matrices respectively for relating the total nodal forces and the displacements (Benipal 1992, 1994). This is the justification for using the same stiffness matrix for stating the equation of motion (Eq. (1)), rate-type constitutive equation (Eq. (3)) and the equilibrium equation (Eq. (4)) of the structure. In the earlier investigation (Kumar *et al*. 2017), the beam was assumed to be linear elastic and so the same stiffness matrix k_d could be used to represent both its tangent and secant stiffness matrices. Such is not the case with general nonlinear structures. However, the tangent and the secant stiffness matrices are same for the first order homogeneous cracked concrete beams.

3. Static analysis

The predictions of the static and dynamic behaviour presented here pertain to a structure with particular numerical details with bilinearity ratio (r) equals 0.2. To recapitulate, the nodal forces (F) applied on the cable-suspended structure introduce suspender forces (Q) transferred to the suspension cable and the nodal forces

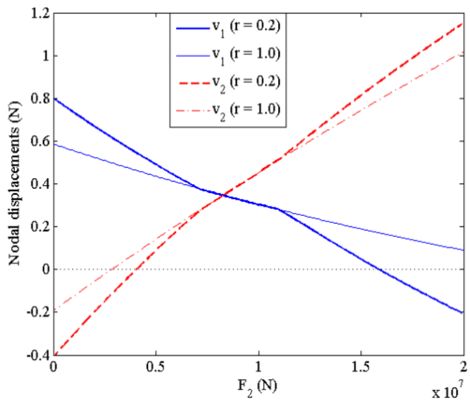


(a) Suspender forces

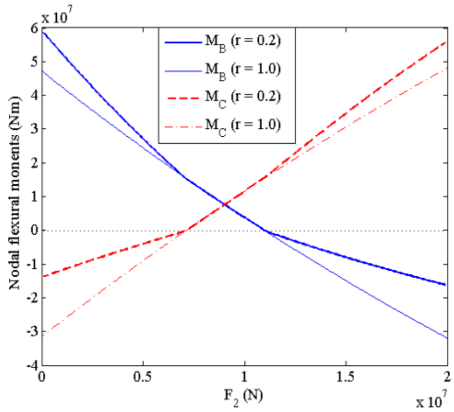


(b) Beam forces

Fig. 2 Variation of suspender and beam forces with non-proportional loading



(a) Displacements



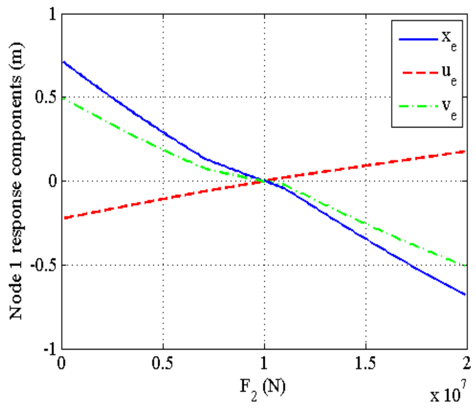
(b) Moments

Fig. 3 Effect of bilinearity ratio with non-proportional loading

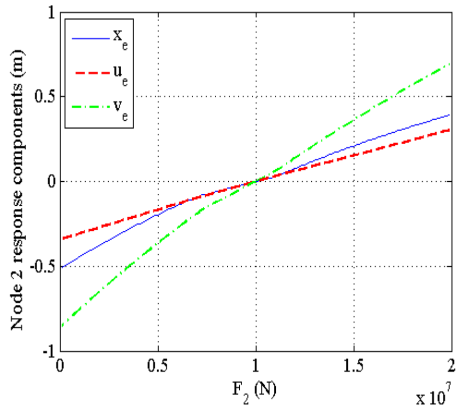
(F_d) being resisted by the suspended beam. Here, proportional loading implies variations of applied nodal forces while maintaining their relative magnitudes as for the applied dead loads. In contrast, non-proportional loading is achieved by keeping the applied nodal force F_1 constant as under dead load case while varying the nodal force F_2. It should be noted that the cable-suspended structure possesses unique natural passive state. In the absence of all the loads, the internal forces and elastic displacements of the cable, beam and suspenders are zero. Static analysis shows that, under proportional loading, the suspender forces as well as beam forces vary linearly. Both the suspender forces increase with loading, while the beam force F_{d1} decreases and F_{d2} increases. From Fig. 2, these conclusions hold for non-proportional loading as well. In this case, both the beam nodal forces can attain negative values, though not simultaneously, even under positive applied nodal forces. The beam nodal force F_{d1} is negative even under dead loads $(F_1 = 6000\text{ kN}, F_2 = 10000\text{ kN})$. When a beam force is negative, the corresponding suspender force has absolute magnitude higher than the applied nodal force.

It has been verified, though not shown here, that both the nodal displacements and the nodal moments in the suspended beam increase linearly with proportional loading. However, such is not the case under non-proportional loading as depicted in Fig. 3. Here, both the displacement and the moment at the second node increase nonlinearly as the force F_2 applied at that node increases, while the displacement and moment at the other node decrease in a nonlinear manner. For a certain range $(0.71 - 1.10) \times 10^4$ kN of the load F_2 near dead load, the concrete beam continues to be in case (I). An increase in its magnitude beyond its upper limit takes the beam to case (III), while case (II) obtains when F_2 decreases below its lower limit. Thus, negative nodal moments can be introduced in the cable-suspended simply supported beam even by positive nodal loads introducing positive nodal displacements.

It can also be observed from Fig. 3 that the sense of the nodal moments in the beam is determined only by the applied nodal loads and it is not affected by the bilinearity ratio of the beam. Also, negative moments at any node implies negative reaction at the adjacent support of the beam. However, the sense of the nodal displacements does depend upon the bilinearity ratio. Since the flexibility matrices of the beam and so of the structure differ from case to case and vary with loads on the nonlinear cases (II) and (III), the nodal displacements are observed to vary nonlinearly with F_2 but without any abrupt changes. Even negative displacements are introduced at nodes B and C respectively at very high and very low magnitudes of force F_2.



(a) Displacements



(b) Moments

Fig. 4 Effect of bilinearity ratio with non-proportional loading

Cable nodes Q and R are located at the top ends of the suspenders at beam nodes B and C respectively. The total vertical nodal displacements of the cable equal those of the beam. However, under general loading in addition to dead load, the suspension cable experiences configurational (x_e), elastic (u_e) and total nodal displacements (v_e) from the equilibrium state of the structure under dead loads. It can be observed from Fig. 4(a) that the total cable displacement at Q decreases with F_2, even though its elastic component increases. The negative total nodal displacements (v_e) at Q result when the negative configurational displacements (x_e) dominate the always positive elastic displacement (u_e), of the cable from the equilibrium state. Thus, the suspenders remain in tension even for negative total nodal displacements. The decrease in the total vertical displacement at node Q is caused by the dominant configurational component which decreases with F_2. In contrast, Fig. 4(b) shows that displacement components as well as total displacement at cable node R increase with F_2. It should be remembered that proportional load variations from the dead load do not cause any configurational displacements at cable nodes and so, the vertical nodal elastic displacements of the cable and the beam are equal.

The tensile forces T_1, T_2 and T_3 respectively in the cable segments PQ, QR and RS have been predicted to increase monotonically but linearly with proportional and

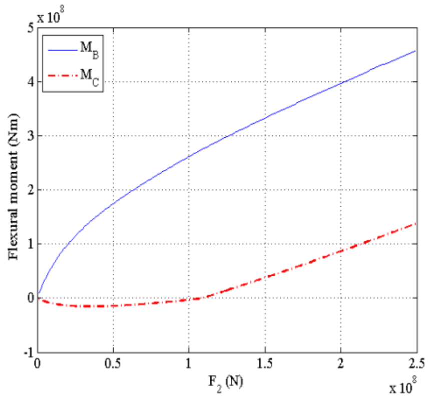


Fig. 5 Variation of flexural moment with general proportional loading

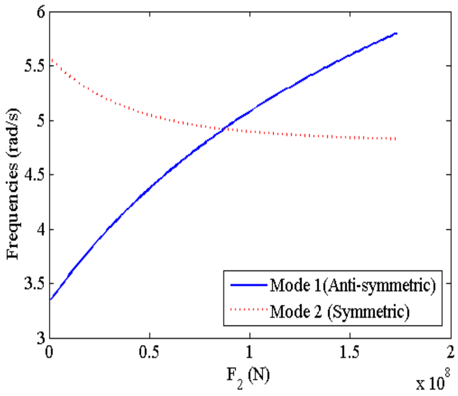
nonlinearly with non-proportional loading. As discussed above, the internal forces and displacements do vary linearly upon proportional loading but only when the nodal load ratio $(p = F_2/F_1)$ equals that (1.67) for dead load for which the configuration is determined. However, the cable-suspended structures exhibit nonlinear response under general proportional loading with different constant load ratios.

For example, the nodal moments M_B and M_C in the cable-suspended concrete beams are shown in Fig. 5 to vary nonlinearly with loading with load ratio of 0.75. Specifically, the nodal moment M_C registers non-monotonic variation. The negative moment first decreases to attain minimum magnitude before it starts to increase and change sense at sufficiently higher loads.

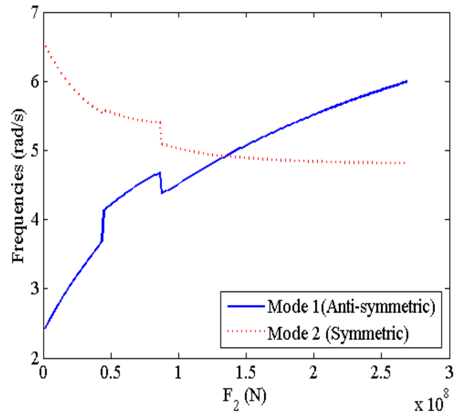
3. Linear modal frequencies and crossover phenomenon

The stiffness matrix, and so the linear modal frequencies, of the cable-suspended concrete beam in static equilibrium depends upon the applied loads. As applied loads are increased proportional to the dead loads, the suspender forces and so the cable stiffness increases. The stiffness of the concrete beam remaining same, proportional loading results in higher stiffness of the cable-suspended structure. Though not shown here, both the modal frequencies increase with proportional increase in applied loads. However, the variation of modal frequencies is more complex under non-proportional loading.

Fig. 6 depicts the effect of bilinearity ratio upon the variation of modal frequencies for a particular case of non-proportional loading with constant load $F_1 = 43710$ kN. The symmetric modal frequency decreases, while the anti-symmetric modal frequency increases with F_2 for all bilinearity ratios. As shown in Fig. 6(a), for linear elastic beam $(r = 1)$, the modal frequencies vary continuously with F_2. However for the nonlinear elastic concrete beams $(r = 0.2)$ represented in Fig. 6(b), the modal frequencies vary continuously as long as the sense of nodal moments remains unchanged. Change in sense of any nodal moment results in an abrupt change in the flexural rigidity



(a) $r = 1.0$



(b) $r = 0.2$

Fig. 6 Variation of modal frequencies with non-proportional loading

distribution in the concrete beam, the stiffness matrix of the structure and so its modal frequencies. It can also be observed from Fig. 6 that, for certain nodal loads, the two modal frequencies equal each other. As discussed in the Introduction, such a ‘cross-over phenomenon’ has earlier been predicted for elastic catenaries (Irvine and Caughey 1974).

The cable-suspended beams investigated here are 2-DOF conservative structures with symmetric stiffness matrices. Two equivalent forms of expressions for their linear modal frequencies $(\omega_n = \sqrt{M^{-1}K})$ are expressed in Eq. (7). These expressions imply that the linear modal frequencies of these structures are always real and positive. These two modal frequencies can be equal if and only if the stiffness matrix is diagonal. However, the diagonal stiffness matrix does not always imply equality of the modal frequencies.

$$\left[(\omega_{1,2})^2 = \left(\frac{1}{2}\right)\left\{\left(\frac{K_{11}}{M_1} + \frac{K_{22}}{M_2}\right) \pm \sqrt{\left(\frac{K_{11}}{M_1} + \frac{K_{22}}{M_2}\right)^2 - 4\left(\frac{K_{11}}{M_1}\frac{K_{22}}{M_2} - \frac{K_{12}}{M_1}\frac{K_{21}}{M_2}\right)}\right\}\right]$$

or

$$\left[(\omega_{1,2})^2 = \left(\frac{1}{2}\right)\left\{\left(\frac{K_{11}}{M_1} + \frac{K_{22}}{M_2}\right) \pm \sqrt{\left(\frac{K_{11}}{M_1} - \frac{K_{22}}{M_2}\right)^2 + 4\frac{K_{12}}{M_1}\frac{K_{21}}{M_2}}\right\}\right] \tag{7}$$

To recapitulate, the tangent stiffness matrix coefficients of these nonlinear structures depend upon the applied nodal loads. It has been observed that, upon proportional loading, the stiffness matrices of the cable-suspended linear elastic beams $(r = 1.0)$ get diagonalised at certain magnitudes of loads. In the first quadrant of load space, there exists one point with diagonal stiffness matrix for each load ratio. The locus of such points constitutes a continuous curve in the load space. It can be observed from Fig. 7 that, for linear elastic cable-suspended beams $(r = 1.0)$, the first quadrant of the load space is indeed partitioned by this curve into two regions. Within the interior region of the load space, the lower frequencies pertains to positive values of K_{12} and anti-symmetric vibration mode. In contrast, in the outer region, K_{12} is negative and the symmetric mode has lower frequency. For this reason, this curve **abcd** can be called the cross-over curve in the sense described earlier (Irvine and Caughey, 1974). However, such is not the case with nonlinear elastic cracked concrete beams $(r = 0.2)$. For some range of load ratios (0.7079 to 0.7869 and 2.0235 to 2.0963), diagonal stiffness may never be obtained for any absolute magnitudes of nodal loads.

Further, for the cable-suspended concrete beams, both the nodal moments M_B and M_C are positive for segment **bc** of the crossover curve. This is why this segment of the cross-over curve nonlinear elastic beam coincides with that of the linear elastic beam. The nodal moments M_B and M_C respectively attain negative magnitudes for the segments **ef** and **gh** of the cross-over curve.

Both equality of modal frequencies and diagonal stiffness matrix represent cross-over phenomenon for the special case of two-DOF systems. On both sides of the cross-over curve, the lowest modal frequency pertains to different vibration modes. On each curve, there exists only one point at which the modal frequencies are equal. Points A (18510, 87140) kN and B (43710, 135080) kN represent load combinations resulting in equal modal frequencies respectively for linear and nonlinear $(r = 0.2)$ suspended

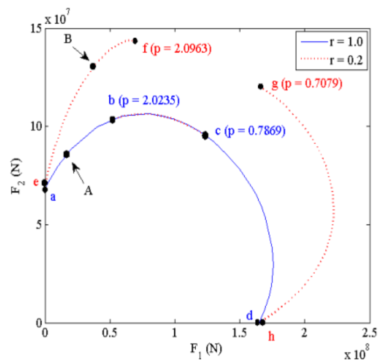
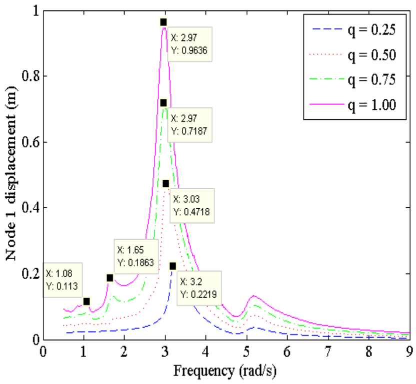
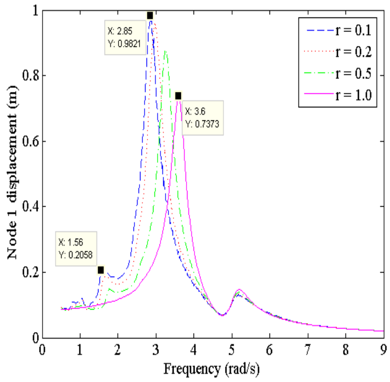


Fig. 7 Partitioning of load-space by cross-over curves

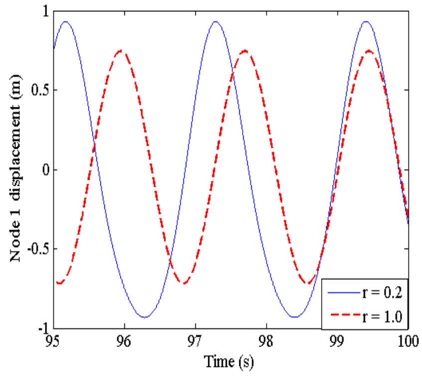


(a) Different peak sinusoidal forces

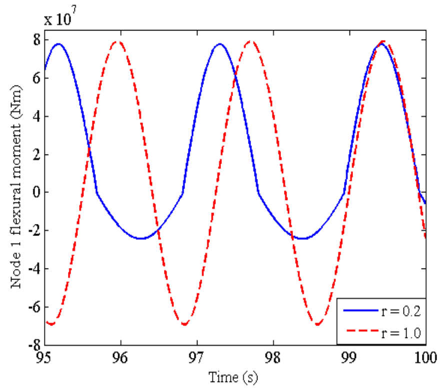


(b) Different bilinearity ratios

Fig. 8 FDR for different conditions



(a) Displacements



(b) Flexural moments

Fig. 9 Typical waveforms for beam node 1

beams. When the modal frequencies are equal, one cannot identify which vibration mode pertains to the 'lower' frequency. Similarly, in the case of diagonal stiffness and mass matrices, the nodal vibrations are uncoupled and so it is meaningless to define and identify symmetric and anti-symmetric modes. In view of the above, for the present case, the diagonal stiffness matrix constitutes more general criterion than Irvine's criterion of equality of modal frequencies.

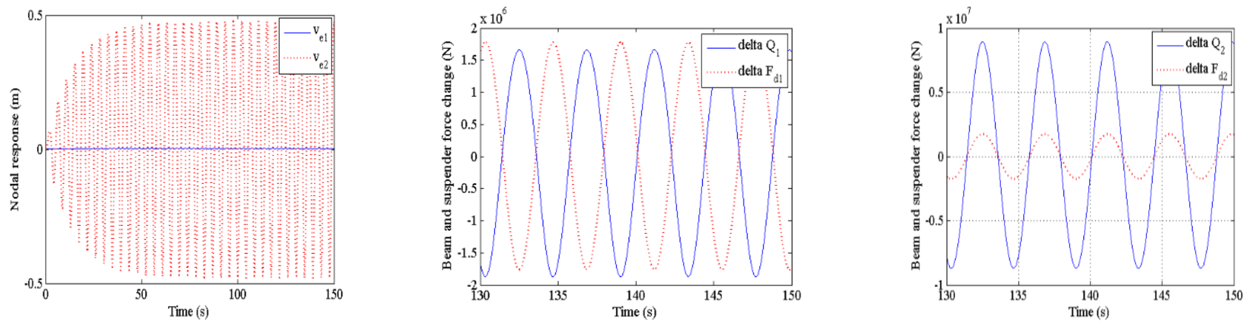
4. Predicted dynamic response

Forced vibrations of cable-suspended beam are investigated for the forcing function of the form $F(t) = F_0 + F_L \sin \omega t$. Since the suspension cables are known to possess little damping, the damping matrix of the structure depends only upon the damping properties of the concrete deck carrying nodal masses. The constant damping matrix is determined by assuming the modal damping ratios as 0.05.

4.1 Loading case A

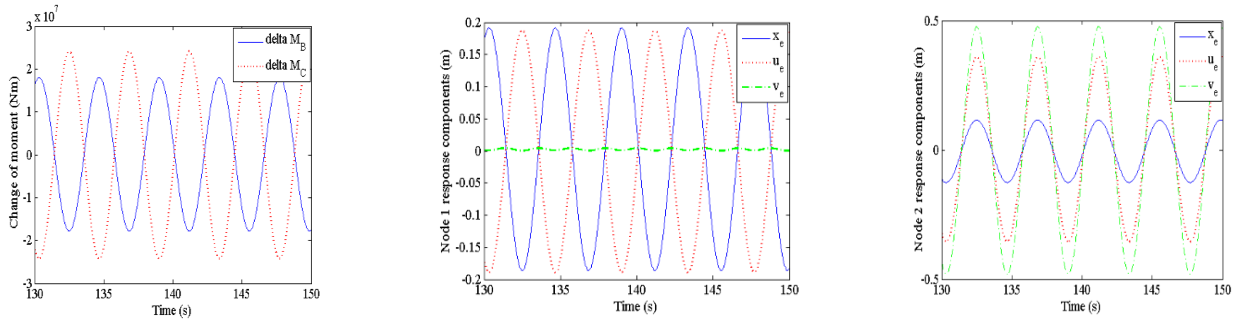
The frequency domain response (FDR) plots for the structure under sustained dead load and subjected to peak sinusoidal forces proportional (q times) to the first mode shape is investigated. The effect of the magnitude of the peak sinusoidal forces and bilinearity ratio (r) on the dynamic response is presented. As shown in Fig. 8(a), the first modal resonance frequency of the cracked concrete beam ($r = 0.2$) decreases with increase in peak sinusoidal forces. Also, the first modal resonance peak amplitudes increase, but disproportionately, with sinusoidal forces. Two irregular subharmonic peaks occur at forcing frequencies 1.65 rad/s and 1.08 rad/s respectively which are about one-half and one-third of the fundamental frequency. These subharmonic resonance peaks become more pronounce at higher peak sinusoidal forces. It can be observed from Fig. 8(b) that the fundamental resonance peak frequency decreases but the peak amplitude increases with decrease in bilinearity ratio. However, the second modal resonance response is relatively small and is not affected much by change in bilinearity ratio. Linear elastic beams ($r = 1$) do not exhibit any subharmonics. Irregular subharmonic resonance peaks observed for nonlinear elastic cracked concrete beams become more pronounced at lower bilinearity ratios.

The effect of bilinearity ratio on the time domain response of the structure at their fundamental frequencies is presented in Fig. 9. The displacement amplitude at first node is slightly higher for lower bilinearity ratio. However, the effect of bilinearity ratio on the nodal flexural moment M_B is substantial and more complex. The maximum value of the negative flexural moment is considerably lesser at lower bilinearity ratio, but the positive moment is not affected. Such an effect is perhaps caused by the lesser flexural rigidity (EI_2) associated with negative moment at lower bilinearity ratio. Similarly, the same positive moments are predicted because of the same positive flexural



(a) Total displacements (b) Node 1 response components (c) Node 2 response components

Fig. 10 Displacements waveforms



(a) Nodal moments (b) Suspender and beam force of node 1 (c) Suspender and beam force of node 2

Fig. 11 Reaction and moment waveforms

rigidity (EI_1) irrespective of the bilinearity ratio.

4.2 Loading case B

The time domain response of a particular cable-suspended cracked concrete beam structure with diagonal stiffness matrix is investigated here. When the sinusoidal force is applied only at node 2, the nodal vibration response turns out to be uncoupled. To be specific, node 1 does not experience any vibrations as shown in Fig. 10(a). Such is not the case with the corresponding cable node Q. It experiences equal and opposite configurational and elastic vibration amplitudes. As shown in Fig. 10(b), these out of phase configurational and elastic vibrations cancel out resulting in vanishing total response. In contrast, Fig. 10(c) shows that, these cable response components being in phase, add up to result in much higher vibration amplitude at the node where sinusoidal force is applied.

It is interesting to note that none of the beam moments, beam forces and suspender forces vanish at any of the nodes. It can be observed from Fig. 11(a) that the nodal beam moments are out of phase. In the absence of any applied force at node 1, the suspender force and beam force at that node are out of phase and cancel each other as depicted in Fig. 11(b). Such is not the case with node 2 where the external sinusoidal force is applied as shown in Fig. 11(c).

5. Discussion

In this paper, authors' earlier theory of cable-suspended linear elastic beams is extended to nonlinear cracked concrete beams. The tangent stiffness matrix of the cracked concrete beam modelled as a FOHM system is used here as the secant stiffness matrix as well in the equation of dynamic equilibrium of the structure. Using the respective tangent stiffness matrices of the cable and the beam, the rate-type constitutive equations and third order differential equations of motion are derived for the indeterminate cable-suspended structure.

Generally, simply supported beams under positive nodal loads acting on their span are subjected only to positive nodal moments. In view of this, concrete bridge deck beams designed with lesser top reinforcement possess different flexural rigidity under positive and negative flexural moments. Such concrete beams exhibit nonlinear behavior only when their point of contra-flexure shifts along their span under non-proportional loading. Static analysis of the cable-suspended beam shows that the applied nodal dead loads are resisted mainly by the cable. The cable-suspended beam is subjected to quite small nodal forces and flexural moments. In fact, even the positive static loads are predicted here to introduce negative force and moment at one node of the concrete beam. This counter-intuitive finding justifies the stated objective of the present paper, viz., to investigate the effect of nonlinearity of the cracked concrete deck beam on the static and dynamic response of the structure.

The dead load configuration determined by the BEFS Model is found to be remain unaffected by the nonlinearity of the concrete beam. The variation of the suspender forces, beam forces and moments and beam displacements are predicted to vary nonlinearly with non-proportional loading as well as general proportional loading. The sense of nodal moments and the direction of the support reaction of the beam are unaffected by the bilinearity ratio.

The nodal displacements of the suspension cable are shown to be composed of elastic and configurational components. The linear modal frequencies of the structure are shown to vary nonlinearly with loads. For nonlinear elastic concrete beams, the variation of these frequencies is discontinuous. Beams with lower bilinearity ratios subjected to higher sinusoidal forces are predicted to exhibit higher subharmonic resonance response. Different positive and negative deviations in the nodal moments from the equilibrium state are registered for the nonlinear concrete beams executing steady state vibrations.

Under a particular set of applied nodal loads, the nodal response of the structure gets uncoupled because of the diagonalization of its tangent stiffness matrix. Application of small force at one of the nodes in addition to the above sustained load does not introduce any beam displacements at the other node. Perturbation by some initial velocity or application of additional small sinusoidal force at one node causes only that beam node to vibrate. Displacement response at any beam node equals the total response at the corresponding cable node. Thus, when a beam node does not exhibit any displacement response, the total displacement of the corresponding cable node also vanishes. However, this does not imply that the said cable node does not experience any elastic displacement. It is only that the elastic displacement there is cancelled by the equal and opposite configurational displacement. This mechanical ‘epi-phenomenon’ at the cable node is not accessible to measurement in the laboratory. Being imperceptible, it cannot even be experienced in the field. But, it is amenable to analysis. To recapitulate further, equality of linear modal frequencies of shallow sagging elastic cables has earlier been considered to constitute cross-over phenomenon (Irvine and Caughey 1974). In this paper, a more general criterion, i.e., diagonal stiffness matrix, of crossover phenomenon than the equality of modal frequencies is proposed for the 2-DOF cable-suspended beam structures. Using this newly proposed criterion, a cross-over curve is shown to partition the load space into two regions.

It is generally believed that the force in a suspender attached to the beam at a node cannot exceed the external force applied at that node. This belief is based upon the expectation that both the suspension cable and the suspended beam share the nodal external loads. Such a belief has turned out to be wrong in the present case. Under some applied nodal loads (refer Fig. 2), the beam contribution is negative and consequently the suspender has to resist an axial force greater than the applied nodal load. This has important implications for the design of the suspenders as well as the suspension cable. Further, the bilinear reinforced concrete beam is expected to possess different ultimate resistance to positive and negative flexural moments. As shown in Fig. 10(b), the beam should be designed for resisting both the positive and negative design nodal moments introduced by the loads.

In this paper, the concrete beams are assumed to be cracked *a priori* at all sections. However, this assumption is not valid for real concrete structures wherein the crack formation occurs at discrete locations gradually upon loading. The process of crack formation in concrete structures requires quite complex modelling. A new method of numerical simulation of cohesive crack growth in concrete structures has recently been proposed (Zhang and Bui 2015). For making the theory proposed in this paper more realistic, it should be extended to incorporate the process of crack formation in concrete beams during vibrations.

6. Conclusions

In this paper, authors’ earlier theory of cable-suspended linear elastic beams is generalized to cable-suspended nonlinear elastic beams. The same constitutive equations and equations of motion are valid. This is because, for the chosen first order homogeneous beams, the tangent stiffness matrix plays the role of secant stiffness matrix as well. Some load sets are shown to introduce negative flexural moments, negative beam support reactions and nonlinearity in the response of bilinear beams with different positive and negative flexural rigidity values. Presence of subharmonic resonances is predicted. A more general criterion for crossover phenomenon is proposed in terms of uncoupled nodal response. In such cases, irrespective of the bilinearity ratio of the beam, the suspension cable is predicted to exhibit configurational and elastic response inaccessible to measurement and perception. The proposed theory is claimed to be potentially applicable for analysis and design of suspension bridges with concrete decks.

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