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A general solution to structural performance of pre-twisted Euler beam subject to static load

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Abstract. Based on the coupled elastic bending deformation features and relationships between the internal force and deformation of pre-twisted Euler beam, the generalized strain, the equivalent constitutive equation and the equilibrium equation of pre-twisted Euler beam are developed. Based on the properties of the dual-antisymmetric matrix, the general solution of pre-twisted Euler beam is obtained. By comparison with ANSYS solution by using straight Beam-188 element based on infinite approach strategy, the results show that the developed method is available for pre-twisted Euler beam and also provide an accuracy displacement interpolation function for the subsequent finite element analysis. The effect of pre-twisted angle on the mechanical property has been investigated.

Keywords: pre-twisted beam; coupled displacement; generalized strain; general solution

1. Introduction

The pre-twisted beam, also known as a naturally twisted beam, presents an initially twisted shape in the natural state. Some researchers have called it naturally Twisted Beam (Berdichevskii and Starosel'skii 1985, Polyakov and Yu 1996, Zubov 2006). Static and dynamic analysis of naturally twisted beams have many important applications in mechanical and civil engineering, such as turbine blades, helicopter rotor blades, aircraft propeller blades, wind turbine blades, and others (Subrahmanyam *et al*. 1981, Nabi and Ganesan 1996, Yoo *et al*. 2001, Sinha *et al*. 2011). Berdichevskii and Starosel'skii (1985) investigated the stress state of pre-twisted rod and showed that the spatial problem can be successfully reduced to a Neumann-type problem for a certain system of second-order elliptic equations in the cross-section, and the pre-twisted rod is decomposed into two independent problems, one bending and one extension-torsion. Polyakov *et al*. (1996) developed an applied theory of naturally twisted rods without involving any hypotheses, using rigorous mathematical methods. Problems on large stretching, torsional and bending deformations of a naturally twisted rod, loaded with end forces and moments, are considered from the point of view of the non-linear three-dimensional theory of elasticity, and particular solutions of the equations of elastostatics are found (Zubov 2006). Many research studies are focused on the vibration performance of pre-twisted blades (Subrahmanyam *et al*. 1981, Nabi and Ganesan 1996, Yoo *et al*. 2001, Sinha *et al*. 2011) and beams (Chen and Keer 1993, Banerjee 2001, 2004, Choi *et al*. 2007). Subrahmanyam *et al*. (1981) applied the Reissner method and the total potential energy approach to calculate the natural frequencies and mode shapes of pre-twisted cantilever blades including shear deformation and rotary inertia. Nabi and Ganesan (1996) analyzed the vibration characteristics of pre-twisted metal matrix composite blades by using beam and plate theories. A beam element with eight degrees of freedom per node has been developed with torsion-flexure, flexure-flexure and shear-flexure couplings, which are encountered in twisted composite beams. A triangular plate element was used for the composite material to model the beam as a plate structure. Yoo *et al*. (2001) used a modeling method for the vibration analysis of rotating pre-twisted blades with a concentrated mass. Sinha *et al*. (2011) derived the governing partial differential equation of motion for the transverse deflection of a rotating pre-twisted plate by using the thin shell theory. Chen and Keer (1993) studied the transverse vibration problems of a rotating twisted Timoshenko beam under axial loading, and investigated the effects of the twist angle, rotational speed, and axial force on natural frequencies by the finite element method. Banerjee (2010, 2004) developed an exact dynamic stiffness method to predict the natural frequencies of a pre-twisted beam. Choi *et al*. (2007) studied bending vibration control of the pre-twisted rotating composite thin-walled beam based on a single cell composite beam, including a warping function, centrifugal force, Coriolis acceleration, pre-twist angle and

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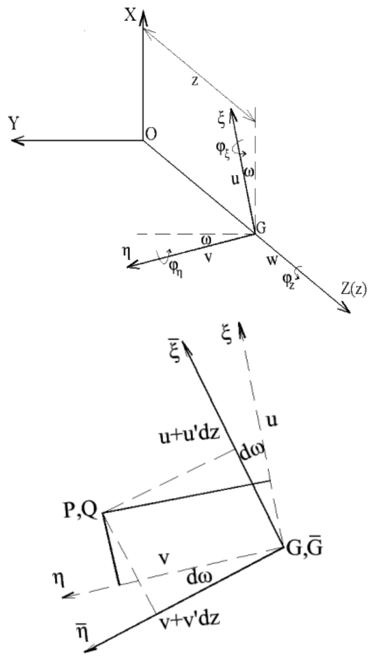

Fig. 1 Global and local coordinate of pre-twisted Euler beam

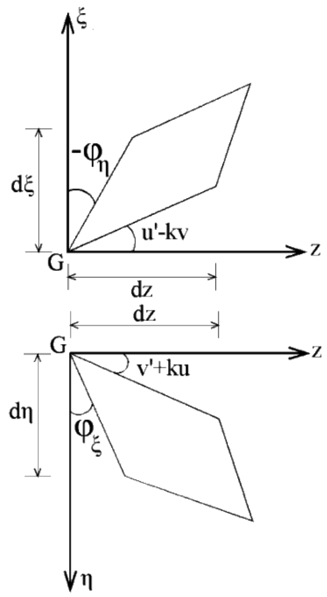

Fig. 2 Infinitesimal element shear strains of pre-twisted Euler beam

piezoelectric effect.

Dynamic stability of a pre-twisted beam is a very important research direction in many fields of aerospace and mechanical engineering. Kar and Ray (1995) dealt with the parametric instability of a pre-twisted, cantilevered, three-layered symmetric sandwich beam subjected to a periodic axial load at the free end by Hamilton's principle and the generalized Galerkin method. Lee (1995) studied the equations of motion of a spinning pre-twisted beam subject to axial loads, by using Euler beam theory. The equations of motion are transformed to the standard form of an eigenvalue problem for determining the perturbation frequencies defining the boundaries of the regions of instability of the spinning beam. Young and Gau (2003), Sabuncu *et al*. (2006), Chen (2010) discussed the influence of thickness-to-width ratio, twist angle, spinning speed and axial load on the natural frequency and buckling load of Timoshenko beams.

Currently, only a few reports have been reported in the existing literature on the general solution of pre-twisted beam (Yu *et al*. 2011). The finite element method provides an effective method and has a wide application (Zupan 2004, Yardimoglu and Yildirim 2004, Petrov and Géradin 1998, Long *et al*. 2013, Chen 2016). The common finite element method to handle static and dynamic problem about pre-twisted beam is based on infinite approach strategy. However, the polynomial displacement functions based on traditional straight beam do not correctly reflect the fact that the strain is zero when rigid motion occurs. Meanwhile, the fact that bending displacements are coupled with each other due to the naturally twisted angle ω will further cause a new error (Zupan 2004, Chen 2014, 2016). As a result, research on the general solution and mechanics performance of pre-twisted beam will have important theoretical and practical meaning.

2. The generalized strain of pre-twisted Euler beam

2.1 The coupled elastic bending displacement

In the global coordinate system *O_XYZ*, the coupled elastic bending displacement behaviour of the pre-twisted thin-walled beam is studied. At any position $Z=z$, we introduce the local coordinate system $G\_\xi\eta z$, where the Gz axis and OZ axis are coincident, $G\xi$ and $G\eta$ are the main bending axis of the section, and the twisted angle of $G\_\xi\eta z$ relative to the *O_XYZ* is $\omega=kz$. The linear displacements are u and v along the axes $G\xi$ and $G\eta$, the rotational displacements are φ_ξ and φ_η. At position $Z=z+dz$, the linear displacements are $u+u'dz$ and $v+v'dz$ along the axis $\overline{G\xi}$ and $\overline{G\eta}$, as shown in Fig. 1.

The incremental displacements of adjacent section in the local coordinate system are given by

$$\Delta u=(u+u'\mathrm{d}z)\cos\mathrm{d}\omega-(v+v'\mathrm{d}z)\sin\mathrm{d}\omega-u \tag{1}$$

$$\Delta v=(v+v'\mathrm{d}z)\cos\mathrm{d}\omega+(u+u'\mathrm{d}z)\sin\mathrm{d}\omega-v \tag{2}$$

To the first order, $\cos d\omega=1$, $\sin d\omega=d\omega$ and $d\omega=kdz$; these are

$$\Delta u=(u'-kv)\mathrm{d}z \tag{3}$$

$$\Delta v=(v'+ku)\mathrm{d}z \tag{4}$$

From the definition of shear strain (Fig. 2), the following equations can be obtained

$$\gamma_{z\xi}=-\varphi_\eta+u'-kv,\ \gamma_{z\eta}=\varphi_\xi+v'+ku \tag{5}$$

Without considering shear deformation of the Euler beam

$$\varphi_\xi=-v'-ku\ ,\quad \varphi_\eta=u'-kv \tag{6}$$

Due to the bending effect, the axial displacement W of any point in $G\_\xi\eta$ plane along the GZ axis is given by

$$W = \eta\varphi_\xi - \xi\varphi_\eta \tag{7}$$

By introducing the global and local coordinate system conversion relation

$$\xi = X\cos\omega + Y\sin\omega,\ \ \eta = -X\sin\omega + Y\cos\omega \tag{8}$$

substituting Eq. (6) and (8) into (7), we can get

$$\begin{aligned} W &= (-X\sin\omega + Y\cos\omega)(-v' - ku) \\ &\quad - (X\cos\omega + Y\sin\omega)(u' - kv) \end{aligned} \tag{9}$$

The normal strain is

$$\begin{aligned} \varepsilon_z = \frac{\partial W}{\partial z} &= (-X\sin\omega + Y\cos\omega)(-v'' - ku') \\ &\quad - (X\cos\omega + Y\sin\omega)(u'' - kv') \\ &\quad (-X\cos\omega - Y\sin\omega)k(-v' - ku) \\ &\quad - (-X\sin\omega + Y\cos\omega)k(u' - kv) \end{aligned} \tag{10}$$

Substituting Eqs. (8) into (10) gives

$$\varepsilon_z = -\xi(u'' - 2kv' - k^2u) - \eta(v'' + 2ku' - k^2v) \tag{11}$$

2.2 Generalized strain of pre-twisted Euler beam

According to the relation of bending moment and section stress, we can get the bending moment at the location $Z=z$ of pre-twisted beam, as following

$$M_\xi = \iint_A \eta E\varepsilon_z \mathrm{d}\xi\mathrm{d}\eta,\ M_\eta = \iint_A -\xi E\varepsilon_z \mathrm{d}\xi\mathrm{d}\eta \tag{12}$$

Substituting Eqs. (6) and (11) into (12), and considering the biaxial symmetry section, namely product of inertia $I_{\xi\eta} = \int_A \xi\eta\mathrm{d}\xi\mathrm{d}\eta = 0$, we can get

$$M_\xi = EI_\xi(\varphi_\xi^{'} - k\varphi_\eta) \tag{13}$$

$$M_\eta = EI_\eta(\varphi_\eta^{'} + k\varphi_\xi) \tag{14}$$

where $I_\xi = \int_A \eta^2\mathrm{d}\xi\mathrm{d}\eta$, $I_\eta = \int_A \xi^2\mathrm{d}\xi\mathrm{d}\eta$ .Considering two-way bending deformation coupling effects on axial deformation, we can get the normal strain

$$\varepsilon_z = w' - \xi(\varphi_\eta^{'} + k\varphi_\xi) + \eta(\varphi_\xi^{'} - k\varphi_\eta) \tag{15}$$

where $w' = \dfrac{\mathrm{d}w}{\mathrm{d}z}$. To solve subsequent equivalent Eqution, we introduce the following generalized strains:

$$\begin{cases} \varepsilon = w' \\ \gamma_\xi = -\varphi_\eta + u' - kv \\ \gamma_\eta = \varphi_\xi + v' + ku \\ k_\xi = \varphi_\xi^{'} - k\varphi_\eta \\ k_\eta = \varphi_\eta^{'} + k\varphi_\xi \\ k_z = \varphi_z^{'} \end{cases} \tag{16}$$

Introducing the assumption of ignoring shear strain, we can get the generalized strain of pre-twisted Euler beam

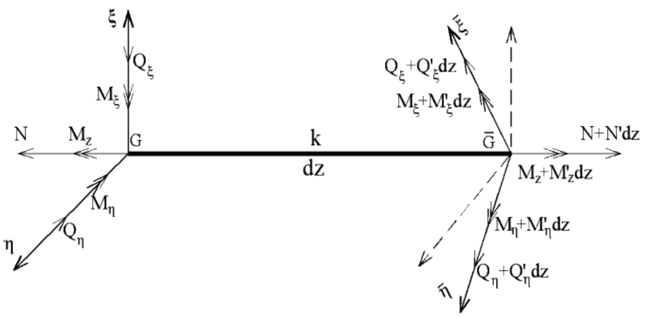

Fig. 3 the internal force relation of pre-twisted Euler beam

$$\begin{cases} \varepsilon = w' \\ k_\xi = -(v'' + 2ku' - k^2v) \\ k_\eta = u'' - 2kv' - k^2u \\ k_z = \varphi_z^{'} \end{cases} \tag{17}$$

3. The equivalent equilibrium equation

3.1 The internal force equilibrium equation

In the local coordinate system $G\_\xi\eta z$, the section internal forces are as follows

$$\begin{bmatrix} N \\ Q_\xi \\ Q_\eta \end{bmatrix} = \iint_A \begin{bmatrix} \sigma_z \\ \tau_{z\xi} \\ \tau_{z\eta} \end{bmatrix} \mathrm{d}\xi\mathrm{d}\eta\ ,\ \begin{bmatrix} M_z \\ M_\xi \\ M_\eta \end{bmatrix} = \iint_A \begin{bmatrix} \xi\tau_{z\eta} - \eta\tau_{z\xi} \\ \eta\sigma_z \\ -\xi\sigma_z \end{bmatrix} \mathrm{d}\xi\mathrm{d}\eta \tag{18}$$

where, N, Q_ξ, Q_η represent the axial force, shear force in the ξ and η direction, respectively. M_z, M_ξ, M_η represent torque, bending moment in the ξ and η direction, respectively. A is the section area.

The equilibrium relations of internal force between z and $z+dz$ sections are as shown in Fig. 3. According to the force and moment equilibrium conditions, we can get

$$\begin{cases} N + N^{'}\mathrm{d}z - N + p_z\mathrm{d}z = 0 \\ (Q_\xi + Q_\xi^{'}\mathrm{d}z)\cos(\mathrm{d}\omega) - Q_\xi - (Q_\eta + Q_\eta^{'}\mathrm{d}z)\sin(\mathrm{d}\omega) + p_\xi\mathrm{d}z = 0 \\ (Q_\eta + Q_\eta^{'}\mathrm{d}z)\cos(\mathrm{d}\omega) - Q_\eta + (Q_\xi + Q_\xi^{'}\mathrm{d}z)\sin(\mathrm{d}\omega) + p_\eta\mathrm{d}z = 0 \end{cases} \tag{19}$$

$$\begin{cases} M_z + M_z^{'}\mathrm{d}z - M_z + m_z\mathrm{d}z = 0 \\ (M_\xi + M_\xi^{'}\mathrm{d}z)\cos(\mathrm{d}\omega) - (M_\eta + M_\eta^{'}\mathrm{d}z)\sin(\mathrm{d}\omega) \\ -M_\xi - Q_\eta\mathrm{d}z + m_\xi\mathrm{d}z = 0 \\ (M_\eta + M_\eta^{'}\mathrm{d}z)\cos(\mathrm{d}\omega) + (M_\xi + M_\xi^{'}\mathrm{d}z)\sin(\mathrm{d}\omega) \\ -M_\eta + Q_\xi\mathrm{d}z + m_\eta\mathrm{d}z = 0 \end{cases} \tag{20}$$

where, the load vectors are $p=[p_z,p_\xi,p_\eta]'$ $m=[m_z,m_\xi,m_\eta]'$. Only considering the first order, the Eqs. (19) and (20) will be simplified, as following

$$\begin{cases} N' + p_z = 0 \\ Q_\xi^{'} - kQ_\eta + p_\xi = 0 \\ Q_\eta^{'} + kQ_\xi + p_\eta = 0 \end{cases} \tag{21}$$

$$\begin{cases} M_z^{'} + m_z = 0 \\ M_\xi^{'} - kM_\eta - Q_\eta + m_\xi = 0 \\ M_\eta^{'} + kM_\xi + Q_\xi + m_\eta = 0 \end{cases} \tag{22}$$

According to the Eq. (22), we can get the relation of bending moment and shear force, as following

$$\begin{cases} Q_\xi = -M_\eta^{'} - kM_\xi - m_\eta \\ Q_\eta = M_\xi^{'} - kM_\eta + m_\xi \end{cases} \tag{23}$$

Substituting the Eqs. (23) into (21) gives

$$\begin{cases} N' + p_z = 0 \\ M_z^{''} + m_z = 0 \\ M_\eta^{''} + k^2 M_\eta + 2kM_\xi^{'} + m_\eta^{'} + km_\xi + p_\xi = 0 \\ M_\xi^{''} - k^2 M_\xi - 2kM_\eta^{'} + m_\xi^{'} - km_\eta + p_\eta = 0 \end{cases} \tag{24}$$

3.2 The equivalent equilibrium equation

According to the Eq. (18), we can get the equivalent constitutive equation, as following

$$\begin{aligned} &N = EA\varepsilon,\ Q_\xi = GA\gamma_\xi,\ Q_\eta = GA\gamma_\eta \\ &M_z = GJk_z,\ M_\xi = EI_\xi k_\xi,\ M_\eta = EI_\eta k_\eta \end{aligned} \tag{25}$$

According to the Eqs. (17), (24) and (25), we can get the equilibrium equation using the parameter of the linear displacement (u, v, w) and rotation displacement (φ_ξ, φ_η, φ_z), as following

$$\begin{cases} EAw'' + p_z = 0 \\ GA(u'' - 2kv' - \varphi_\eta^{'} - k\varphi_\xi - k^2 u) + p_\xi = 0 \\ GA(v'' + 2ku' + \varphi_\xi^{'} - k\varphi_\eta - k^2 v) + p_\eta = 0 \\ GJ\varphi_z^{''} + m_z = 0 \\ EI_\xi \varphi_\xi^{''} - k(EI_\xi + EI_\eta)\varphi_\eta^{'} - (k^2 EI_\eta + GA)\varphi_\xi \\ -GAv' - GAku + m_\xi = 0 \\ EI_\eta \varphi_\eta^{''} + k(EI_\eta + EI_\xi)\varphi_\xi^{'} - (k^2 EI_\xi + GA)\varphi_\eta \\ +GAu' - GAkv + m_\eta = 0 \end{cases} \tag{26}$$

Ignoring the effect on shear deformation, the equilibrium equation is simplified, as following

$$\begin{cases} EAw'' + p_z = 0 \\ GJ\varphi_z^{''} + m_z = 0 \\ EI_\xi \varphi_\xi^{''} - k(EI_\xi + EI_\eta)\varphi_\eta^{'} - k^2 EI_\eta \varphi_\xi + m_\xi = 0 \\ EI_\eta \varphi_\eta^{''} + k(EI_\eta + EI_\xi)\varphi_\xi^{'} - k^2 EI_\xi \varphi_\eta + m_\eta = 0 \end{cases} \tag{27}$$

To solve the above equilibrium equations, we change the above Eqs. (16), (25) and (26) to the following matrixes

$$\begin{cases} [U'] - [\tau][U] - [H][\varphi] = [\varepsilon] \\ [\varphi'] - [\tau][\varphi] = [K] \end{cases} \tag{28}$$

$$\begin{cases} [N] = [B][\varepsilon] \\ [M] = [D][K] \end{cases} \tag{29}$$

$$\begin{cases} [N'] - [\tau][N] + [p] = 0 \\ [M'] - [\tau][M] - [H][N] + [m] = 0 \end{cases} \tag{30}$$

where:

$$[N] = \begin{bmatrix} N \\ Q_\xi \\ Q_\eta \end{bmatrix},\ [M] = \begin{bmatrix} M_z \\ M_\xi \\ M_\eta \end{bmatrix},\ [p] = \begin{bmatrix} p_z \\ p_\xi \\ p_\eta \end{bmatrix},\ [m] = \begin{bmatrix} m_z \\ m_\xi \\ m_\eta \end{bmatrix},$$

$$[U] = \begin{bmatrix} w \\ u \\ v \end{bmatrix},\ [\varphi] = \begin{bmatrix} \varphi_z \\ \varphi_\xi \\ \varphi_\eta \end{bmatrix}$$

$$[\varepsilon] = \begin{bmatrix} \varepsilon \\ \gamma_\xi \\ \gamma_\eta \end{bmatrix},\ [K] = \begin{bmatrix} k_z \\ k_\xi \\ k_\eta \end{bmatrix},\ [\tau] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & k \\ 0 & -k & 0 \end{bmatrix},\ [H] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$$

$$[B] = \begin{bmatrix} EA & 0 & 0 \\ 0 & GA & 0 \\ 0 & 0 & GA \end{bmatrix},\ [\mathrm{D}] = \begin{bmatrix} GJ & 0 & 0 \\ 0 & EI_\xi & 0 \\ 0 & 0 & EI_\eta \end{bmatrix}$$

4. The general solution of pre-twist Euler beam

Introducing the coordinate transformation matrix $[A]$ between the local coordinate system $G\_\xi\eta z$ and the global coordinate system O_XYZ, as following

$$[A] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos kz & \sin kz \\ 0 & -\sin kz & \cos kz \end{bmatrix} \tag{31}$$

By using the solve method of constant variation of differential equations, we can get the general solution of the equilibrium Eq. (30), as following

$$\begin{cases} [N] = [A][N_0] - [A]\int_0^z [A]^T [p]\mathrm{d}z \\ [M] = [A][M_0] \\ \quad + [A]\int_0^z [A]^T [H][A][N_0]\mathrm{d}z - \int_0^z [A]^T [m]\mathrm{d}z \\ \quad - [A]\int_0^z [A]^T [H][A](\int_0^z [A]^T [p]\mathrm{d}z)\mathrm{d}z \end{cases} \tag{32}$$

where $[N_0]$ and $[M_0]$ are the integral constant .

Similarly, the general solution of the Eq. (28) can be obtained, as following

$$\begin{cases} [U] = [A][U_0] \\ \quad + [A]\left\{\int_0^z [A]^T [H][A][\varphi_0]\mathrm{d}z + \int_0^z [A]^T [\varepsilon]\mathrm{d}z\right\} \\ \quad + [A]\int_0^z [A]^T [H][A](\int_0^z [A]^T [K]\mathrm{d}z)\mathrm{d}z \\ [\varphi] = [A][\varphi_0] + [A]\int_0^z [A]^T [K]\mathrm{d}z \end{cases} \tag{33}$$

where, $[U_0]$ and $[\varphi_0]$ are the integral constant vectors.

Introducing a position vector of pre-twisted Euler beam along the axial direction, as following

$$[Z] = [Z(z), X(z), Y(z)]^T \tag{34}$$

According to the properties of the dual-antisymmetric

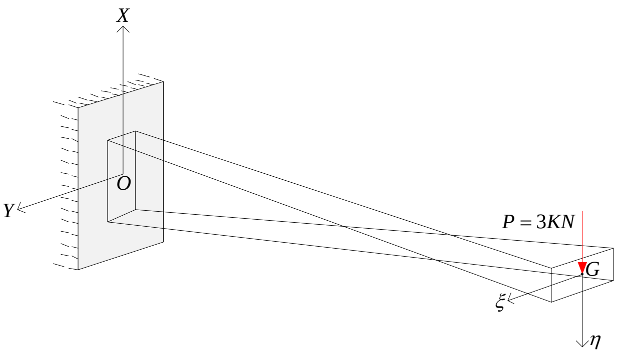

Fig. 4 the geometric parameter with rectangular cross section

matrix $\left[\Omega_{\psi_0}\right]$ of any vector $[\psi_0]=[\psi_{10},\ \psi_{20},\ \psi_{30}]^T$, there is the following conversion formula (Yu 2002)

$$[A]^T[H][A][\psi_0]=\left[\Omega_{\psi_0}\right]([Z]-[Z_0])' \tag{35}$$

So the general solution of the differential Eq. (33) can be further expressed as followings

$$\begin{cases} [U]=[A]\left\{[U_0]-[\Omega_R][\varphi_0]+\int_0^z[A]^T[\varepsilon]\mathrm{d}z\right\} \\ \quad -[A]\left\{[\Omega_R]\int_0^z[A]^T[K]\,\mathrm{d}z-\int_0^z[\Omega_R][A]^T[K]\mathrm{d}z\right\} \\ [\varphi]=[A][\varphi_0]+[A]\int_0^z[A]^T[K]\mathrm{d}z \end{cases} \tag{36}$$

where $[\Omega_R]$ is a dual-antisymmetric matrix generated by the vector $R=[Z]-[Z_0]$.

5. The mechanical performance analysis

5.1 The analysis example

The analysis model in our example is a cantilever beam, whose length l is 6000 mm; the cross-section is a rectangular cross section (Fig. 4): the height h is 600 mm, the width is 200 mm. The pre-twisted angle ω is 0.5 π. The steel material modulus E is 2.1×10^5 Mpa. The concentrated force P is 3 kN.

The load vectors in this example are $[p]=[m]=[0]$; the coordinate transformation matrix is as follows

$$[A]_{z=l}=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos kL & \sin kL\\ 0 & -\sin kL & \cos kL\end{bmatrix} \tag{37}$$

The position vector is $[R]=([Z]-[Z_0])=(z,0,0)^T$, whose the dual-antisymmetric matrix is $[\Omega_R]=\begin{bmatrix}0 & 0 & 0\\ 0 & 0 & -z\\ 0 & z & 0\end{bmatrix}$.

The boundary conditions are as follows

$$\begin{cases} [U]_{z=0}=[\varphi]_{z=0}=(0,0,0)^T \\ [N]_{z=L}=(0,0,p)^T=(0,0,3000)^T \\ [M]_{z=L}=(0,0,0)^T \end{cases} \tag{38}$$

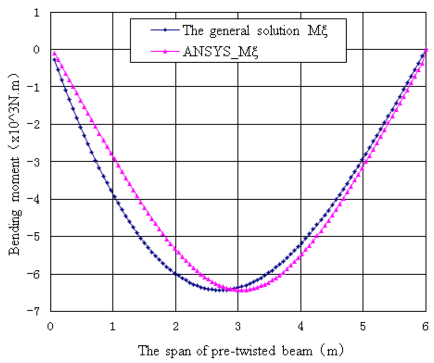

Fig. 5 (a) the bending moment M_ξ

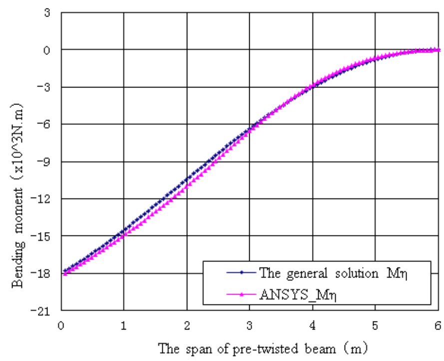

Fig. 5 (b) the bending moment M_η

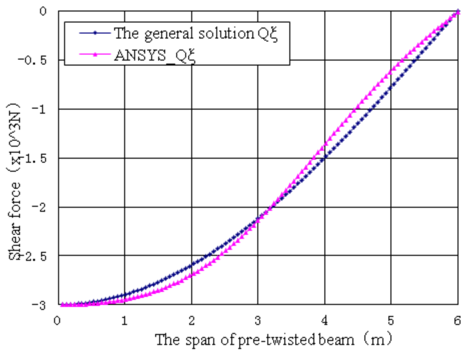

Fig. 5 (c) the shear force Q_ξ

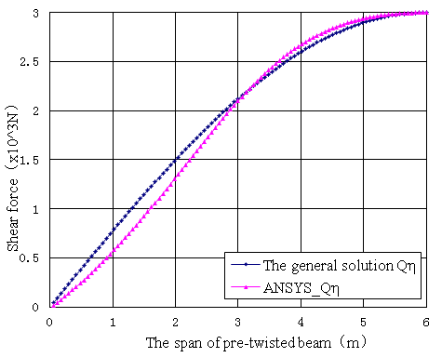

Fig. 5 (d) the shear force Q_ξ

5.2 The comparison results

According to the comparisons between the general solution and ANSYS by using the straight Beam-188 element based on infinite approach method (the numbers of

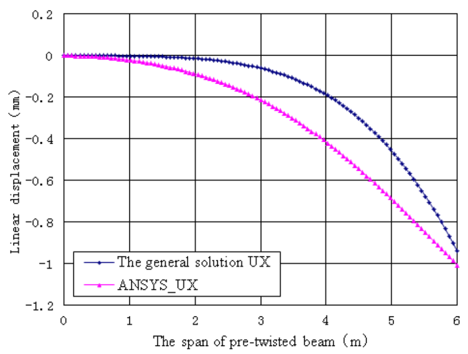

Fig. 6 (a) the deformation UX

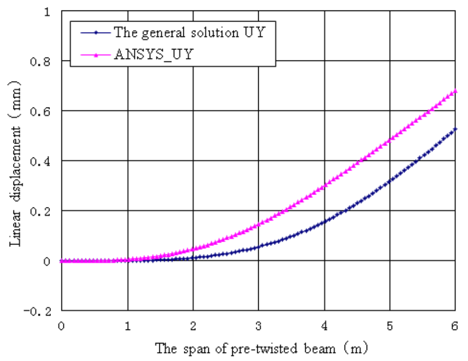

Fig. 6 (b) the deformation UY

beam elements are 100), we can get the following results (Figs. 5-6):

(1) The change trends of internal force and displacement are almost same; the maximal difference values of bending moment and shear force are less than 1%.

(2) The linear displacements using the general solution are smaller than the ANSYS results, the UX is -0.9372 mm at the end of the cantilever beam, the ANSYS result is -1.0053 mm, and the difference value is 6.77%. The UY is 0.5267 mm, the ANSYS result is 0.6805 mm, and the difference value is 22.6%.

(3) The possible reason of the linear smaller displacement by using the general solution could be not to consider the influences of shear deformation and discrete error by using infinite approach method in the ANSYS program.

6. Parametric analysis

6.1 The effect of pre-twisted angle on deflection

The effect of pre-twisted angle ω on the deflections has been investigated (Fig. 7).

(1) When $\omega=kz\in[0,\ \pi/2]$, the displacement Ux corresponding to the main axis $G\eta$ increased gradually with increasing of the pre-twisted angle, and the displacement Uy corresponding to the secondary axis $G\xi$ is also increased. The equivalent stiffness is also shown to be decreased along main axis direction as the increment of pre-twisted angle, and then the displacement is increased when the pre-twisted angle changes in the range of $[0,0.5\pi]$. The

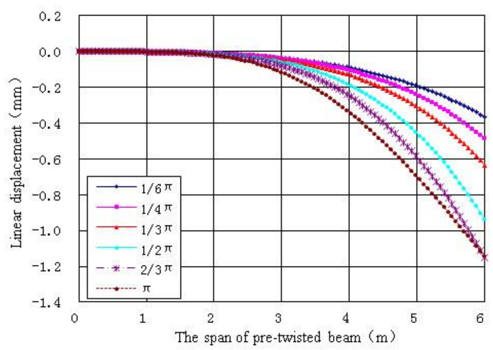

Fig. 7 (a) the deformation UX

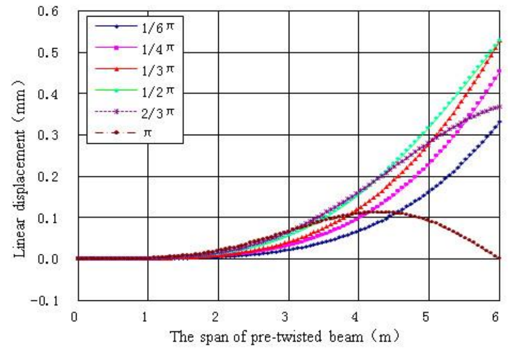

Fig. 7 (b) the deformation UY

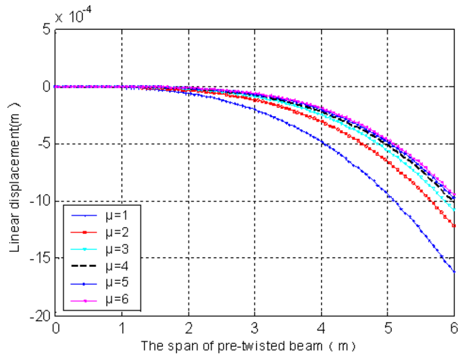

Fig. 8 (a) the deformation UX

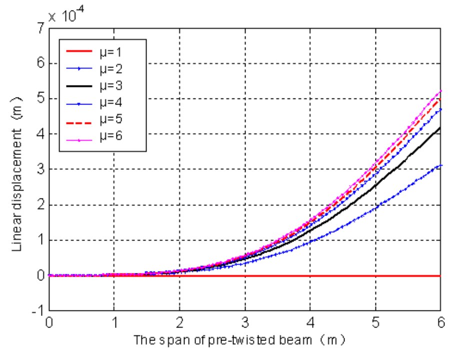

Fig. 8 (b) the deformation UY

displacements along main axis and secondary axis direction are coupled to each other as the existence of pre-twisted angle. The coupling effect will become stronger, and the lateral displacement Uy will also be increased with the increasing of pre-twisted angle.

(2) When $\omega=kz\in[\pi/2,\pi]$, the displacement UX corresponding to the flexural main axis $G\eta$ increased gradually with the increasing of pre-twisted angle, while the displacement UY corresponding to the flexural secondary axis $G\xi$ decreased. It is indicated that the coupling effect became smaller, and lateral displacement decreased with the increment of pre-twisted angle when the pre-twisted angle changes in the range of $[\pi/2,\pi]$.

6.2 The effect of flexural stiffness ratio on deflection

Introducing the flexural stiffness ratio of pre-twisted beam

$$\mu=\frac{I_\eta}{I_\xi}\geq 1 \tag{39}$$

According to the above example, and assuming the flexural stiffness EI_ξ along secondary axis $G\xi$ direction remain the same, the effect of parameter μ on the deflections of pre-twisted rectangular beam has been investigated by using section flexural stiffness ratio μ from 1 to 6.

From the figure 10, it is indicated that the displacement UX decreased corresponding to the flexural main axis $G\eta$, while the displacement UY increased corresponding to flexural secondary axis $G\xi$ with the increasing of flexural stiffness ratio μ. The coupled effect of pre-twisted beam became stronger between the flexural main axis and secondary axis with the increasing of flexural stiffness ratio μ.

7. Conclusions

- Based on the coupled elastic bending deformation features and relationship between the internal force and deformation of pre-twisted Euler beam, the generalized strain, the equivalent constitutive equation and the internal force equilibrium equation of pre-twisted Euler beam are developed.
- Based on the properties of the dual-antisymmetric matrix, the general solution of pre-twisted Euler beam is obtained. By comparison with ANSYS solution by using straight Beam-188 element based on infinite approach method, the results show that the developed method is available for pre-twisted Euler beam and provides an accuracy displacement interpolation function for the subsequent finite element analysis.
- The effect of pre-twisted angle on the deflections has been investigated. The displacements along section flexural main axis and secondary axis direction are coupled to each other because of the existence of pre-twisted angle.
- The equivalent stiffness decreased along section flexural main axis direction as the increment of pre-twisted angle when the pre-twisted angle changes in the range of $[0,\pi/2]$. While the coupling effect became smaller, and lateral displacement decreased with the increment of pre-twisted angle when the pre-twisted angle changes in the range of $[\pi/2,\pi]$.

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