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Probabilistic damage detection of structures with uncertainties under unknown excitations based on Parametric Kalman filter with unknown Input

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Abstract. System identification and damage detection for structural health monitoring have received considerable attention. Various time domain analysis methodologies based on measured vibration data of structures have been proposed. Among them, recursive least-squares estimation of structural parameters which is also known as parametric Kalman filter (PKF) approach has been studied. However, the conventional PKF requires that all the external excitations (inputs) be available. On the other hand, structural uncertainties are inevitable for civil infrastructures, it is necessary to develop approaches for probabilistic damage detection of structures. In this paper, a parametric Kalman filter with unknown inputs (PKF-UI) is proposed for the simultaneous identification of structural parameters and the unmeasured external inputs. Analytical recursive formulations of the proposed PKF-UI are derived based on the conventional PKF. Two scenarios of linear observation equations and nonlinear observation equations are discussed, respectively. Such a straightforward derivation of PKF-UI is not available in the literature. Then, the proposed PKF-UI is utilized for probabilistic damage detection of structures by considering the uncertainties of structural parameters. Structural damage index and the damage probability are derived from the statistical values of the identified structural parameters of intact and damaged structure. Some numerical examples are used to validate the proposed method.

Keywords: parametric Kalman filter; unknown Input; structural identification; probabilistic damage detection; uncertainties

1. Introduction

One of the important tasks of structural health monitoring (SHM) is to identify the state of the structures and detect structural damage for the reliability and safety of structures. When a structure is damaged, such as cracking in a certain structural element, the stiffness of the damaged component is usually reduced. So, the variations of structural parameters could indicate the structural damage (Zhong *et al*. 2003, Yang *et al*. 2007, Jiang *et al*. 2011, Li and Chen 2013, Zhang *et al*. 2015, Lin and Liang 2015, Yu and Zhu 2015). In the past decades, various approaches in time domain analyses have been developed such as the methods of least-squares estimation (Yang *et al*. 2004, 2005), sequential nonlinear least-squares estimation (Yang *et al*. 2006), the finite element model updating and structural damage identification based on OMAX data (operational modal analysis with eXogenous forces) (Reynders *et al*. 2010), response surface metamodels for structural damage identification (Rutherforda *et al*. 2005, Fang *et al*. 2011), the two-stage Kalman estimation approach for the identification of nonlinear structural parameters (Lei *et al*. 2015). However, these approaches are only applicable when the information of external inputs to structures is available. In practice, it is difficult or even impossible to directly measure all external inputs on the structures. The information of external inputs is important in SHM. So, it is necessary to develop algorithms for the structural damage detection with unknown external inputs.

There have been some approaches for simultaneous identification of structural damage and unknown external inputs, e.g., numerical iterative procedures based on the classical least squares estimation or extended Kalman filter for identification of the constant structural parameters (Wang *et al.* 1994, 1997, Ling *et al.* 2004), the recursive least squares estimation with unknown inputs (RLSE-UI) approach for damage identification of structures (Yang *et al*. 2007), the adaptive quadratic sum-squares error with unknown inputs (AQSSE-UI) for the detection of structural damage (Huang *et al.* 2010). However, the derivations of these approaches are quite involved, e.g., the mathematical derivations of the refereed RLSE-UI by Yang *et al.* (2007) were presented in both the paper txt and the Appendix with many page spaces，but the final recursive estimation expressions are analogous to those of the parameter Kalman filter (PKF) (Li *et al.* 2002), which implies the direct extension of the conventional PKF for simultaneous identification of structural damage and unknown external inputs.

Detection of structural damage in civil engineering always involves uncertainties from environment measurement noise, modeling error, and uncertainties in structures. These errors and uncertainties can result in mistake or low accuracy in damage detection. So, the uncertainties in structures limit the successful use of those

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deterministic damage detection methods. Some approaches with consideration of the uncertainties have been developed for structural damage detection. There are two main methods: Bayesian model updating damage detection methods and stochastic finite element model updating methods.

The main idea of Bayesian method is that the probability distribution of model parameters are assumed given, uncertain parameters are updated by the measurement signals, and the posterior probability distribution of structural parameters can be obtained. Then, the probability and extent of damage could be detected by the comparisons of the probability distribution of undamaged model and damaged model. Comprehensive Bayesian model updating was presented by Beck *et al*. (1998) and some recent works have been conducted by Hao *et al.* (2015), Mustafa *et al.* (2015), Sun *et al.* (2015), etc. Stochastic finite element model updating methods is another very popular technique for uncertainty propagation. The probability distribution of structural parameters is obtained by the stochastic simulation of test data and model parameter perturbation, and the prior probability distribution of structural parameters is updated by measurement data. A statistical method for the structural damage detection based on the measured acceleration response considering the uncertainties in measurement noise was presented by Li and Law (2008). A new stochastic damage detection method was proposed by Xu and Zhang (2011) for building structures with parametric uncertainties.

However, these approaches above are applicable when the information of external inputs to structures is available. Therefore, it is necessary to develop algorithms with consideration of the uncertainties in structures for the structural damage detection with unknown external inputs.

In this paper, a parametric Kalman filter with unknown inputs (PKF-UI) is proposed for the simultaneous identification of structural parameters and the unmeasured external inputs. Analytical recursive formulations of the proposed PKF-UI are derived based on the conventional PKF. Such a straightforward derivation and formulation of PKF-UI is not available in the literature. Then, the proposed PKF-UI is utilized for probabilistic damage detection of structures by considering the uncertainties of structural parameters. Structural damage index and the damage probability are derived from the statistical values of the identified structural parameters of intact and damaged structure. Some numerical examples are used to validate the proposed method.

2. A brief review of the conventional PKF

The equation of motion of an n-DOF linear time-invariant structure can be expressed as

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(\mathrm{t})+\boldsymbol{F}\left[\boldsymbol{x}(\mathrm{t}),\dot{\boldsymbol{x}}(\mathrm{t}),\boldsymbol{\theta}\right]=\boldsymbol{\eta}\boldsymbol{f}(\mathrm{t}) \quad (1)$$

where $\ddot{\boldsymbol{x}}(\mathrm{t})$, $\dot{\boldsymbol{x}}(\mathrm{t})$ and $\boldsymbol{x}(\mathrm{t})$ are n-dimensional vectors of structural acceleration, velocity and displacement, respectively, $\boldsymbol{\theta}$ is a m-dimensional time-invariant parametric vector involving unknown parameters to be estimated, $\boldsymbol{F}\left[\boldsymbol{x}(\mathrm{t}),\dot{\boldsymbol{x}}(\mathrm{t}),\boldsymbol{\theta}\right]$ is a force vector which can be linear or nonlinear function of the displacements, velocities and the structural parameters, $\boldsymbol{f}(t)$ is a p-dimensional external inputs vector, and $\boldsymbol{\eta}$ is the corresponding influence matrix associated with the external inputs $\boldsymbol{f}(t)$.

The observation equation can be expressed as

$$\boldsymbol{y}(\mathrm{t})=\boldsymbol{\varphi}\left[\boldsymbol{x}(\mathrm{t}),\dot{\boldsymbol{x}}(\mathrm{t})\right]\boldsymbol{\theta}+\boldsymbol{\upsilon}(\mathrm{t}) \quad (2)$$

in which $\boldsymbol{y}(\mathrm{t})=\boldsymbol{\eta}\boldsymbol{f}(\mathrm{t})-\boldsymbol{M}\ddot{\boldsymbol{x}}(\mathrm{t})$, $\boldsymbol{\varphi}$ [ ] is the observation matrix composed of the system response vectors, and $\boldsymbol{\upsilon}(t)$ is a measurement noise vector, which is assumed a Gaussian white noise vector with zero mean and a covariance matrix $\boldsymbol{R}(t)$.

The discrete equation for the observation equation in Eq. (2) is expressed by

$$\boldsymbol{y}_{k+1}=\boldsymbol{\varphi}_{k+1}\boldsymbol{\theta}_{k+1}+\boldsymbol{\upsilon}_{k+1} \quad (3)$$

in which $\boldsymbol{y}_{k+1}$ is the measured response vector at time $t=(k+1)\Delta t$ and Δt is the sampling time step.

Based on the conventional PKF approach, the estimation value of the unknown parametric vector at time $t=(k+1)\Delta t$ is given by

$$\hat{\boldsymbol{\theta}}_{k+1|k+1}=\hat{\boldsymbol{\theta}}_{k|k}+\boldsymbol{K}_{k+1}(\boldsymbol{y}_{k+1}-\boldsymbol{\varphi}_{k+1}\hat{\boldsymbol{\theta}}_{k|k}) \quad (4)$$

where $\hat{\boldsymbol{\theta}}_{k+1|k+1}$ and $\hat{\boldsymbol{\theta}}_{k|k}$ denote the estimated values of $\boldsymbol{\theta}$ at time $t=(k+1)\Delta t$ and $t=k\Delta t$, respectively, and $\boldsymbol{K}$ is the Kalman gain matrices given by

$$\boldsymbol{K}_{k+1}=\tilde{\mathbf{P}}_{k+1|k}\boldsymbol{\varphi}_{k+1}^{\mathrm{T}}(\boldsymbol{\varphi}_{k+1}\tilde{\mathbf{P}}_{k+1|k}\boldsymbol{\varphi}_{k+1}^{\mathrm{T}}+\boldsymbol{R}_{k+1})^{-1} \quad (5)$$

in which $\tilde{\mathbf{P}}_{k+1|k}$ is the covariance matrix of estimation error of the estimated $\tilde{\boldsymbol{\theta}}_{k+1|k}$. The recursive estimation of covariance matrix of error is given by

$$\hat{\boldsymbol{P}}_{k+1|k}=\left(\boldsymbol{I}-\boldsymbol{K}_{k+1}\boldsymbol{\varphi}_{k+1}\right)\tilde{\boldsymbol{P}}_{k+1|k} \quad (6)$$

In the above procedure, it is assumed that external input vector $\boldsymbol{f}(t)$ is known, which is the limitation of the conventional PKF.

3. The direct extension of PKF to PKF-UI

When external inputs to the above n-DOF structure are unknown, the equation of motion is rewritten by

$$\boldsymbol{M}\ddot{\boldsymbol{x}}(\mathrm{t})+\boldsymbol{F}\left[\boldsymbol{x}(\mathrm{t}),\dot{\boldsymbol{x}}(\mathrm{t}),\boldsymbol{\theta}\right]=\boldsymbol{\eta}^{u}\boldsymbol{f}^{u}(\mathrm{t}) \quad (7)$$

where $\boldsymbol{f}^{u}(t)$ is an unmeasured p-dimensional external inputs vector, and $\boldsymbol{\eta}^{u}$ is the corresponding influence matrix associated with the unknown external input vector $\boldsymbol{f}^{u}(t)$.

3.1 PKF-UI with linear observation equation

If the observation equation associated with the equation of motion in Eq .(7) is expressed as a linear equation by

$$\boldsymbol{y}(\mathrm{t})=\boldsymbol{\varphi}\left[\boldsymbol{x}(\mathrm{t}),\dot{\boldsymbol{x}}(\mathrm{t})\right]\boldsymbol{\theta}+\boldsymbol{\eta}^{u}\boldsymbol{f}^{u}(\mathrm{t})+\boldsymbol{\upsilon}(\mathrm{t}) \quad (8)$$

in which $\boldsymbol{y}(\mathrm{t}) = \boldsymbol{M}\ddot{\boldsymbol{x}}(\mathrm{t})$

The discrete equation for the above linear observation equation can be rewritten as

$$\boldsymbol{y}_{k+1} = \boldsymbol{\varphi}_{k+1}\boldsymbol{\theta}_{k+1} + \boldsymbol{\eta}^u \boldsymbol{f}^u_{k+1} + \boldsymbol{\upsilon}_{k+1} \tag{9}$$

Analogous to the procedure of the conventional PKF, the estimation value of the unknown parametric vector at time $t=(k+1)\Delta t$ is given by

$$\hat{\boldsymbol{\theta}}_{k+1|k+1} = \hat{\boldsymbol{\theta}}_{k|k} + \boldsymbol{K}_{k+1}(\boldsymbol{y}_{k+1} - \boldsymbol{\varphi}_{k+1}\hat{\boldsymbol{\theta}}_{k|k} - \boldsymbol{\eta}^u \hat{\boldsymbol{f}}^u_{k+1|k+1}) \tag{10}$$

where $\hat{\boldsymbol{f}}^u_{k+1|k+1}$ denotes the estimated values of unknown external input vector $\boldsymbol{f}^u(t)$at time $t=(k+1)\Delta t$. $\boldsymbol{K}_{k+1}$ is the Kalman gain matrix at time $t=(k+1)\Delta t$.

Under the condition that the number of response measurements is not less than that of unknown external inputs, $\hat{\boldsymbol{f}}^u_{k+1|k+1}$ can be estimated by minimizing the following error vector as

$$\boldsymbol{\Delta}_{k+1} = \boldsymbol{y}_{k+1} - \boldsymbol{\varphi}_{k+1}\hat{\boldsymbol{\theta}}_{k+1|k+1} - \boldsymbol{\eta}^u \hat{\boldsymbol{f}}^u_{k+1|k+1} \tag{11}$$

By inserting the expression of $\hat{\boldsymbol{\theta}}_{k+1|k+1}$ in Eq. (10) into the above error vector in Eq. (11), $\boldsymbol{\Delta}_{k+1}$ can be rewritten as

$$\begin{aligned}\boldsymbol{\Delta}_{k+1} &= \left(\mathbf{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{y}_{k+1} - \boldsymbol{\varphi}_{k+1}\hat{\boldsymbol{\theta}}_{k|k}\right) \\ &\quad - \left(\mathbf{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\boldsymbol{\eta}^u \hat{\boldsymbol{f}}^u_{k+1|k+1}\end{aligned} \tag{12}$$

Then, $\hat{\boldsymbol{f}}^u_{k+1|k+1}$ can be estimated from above Eq. (12) by the least-squares estimation as

$$\hat{\boldsymbol{f}}^u_{k+1|k+1} = \boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{y}_{k+1} - \boldsymbol{\varphi}_{k+1}\hat{\boldsymbol{\theta}}_{k|k}\right) \tag{13}$$

where $\boldsymbol{S}_{k+1} = \left[\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\boldsymbol{\eta}^u\right]^{-1}$

The error of the estimated $\hat{\boldsymbol{f}}^u_{k+1|k+1}$ can be evaluated by

$$\begin{aligned}\hat{\boldsymbol{e}}^f_{k+1|k+1} &= \boldsymbol{f}^u_{k+1} - \hat{\boldsymbol{f}}^u_{k+1|k+1} \\ &= \boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\left(\boldsymbol{\varphi}_{k+1}\hat{\boldsymbol{e}}^{\theta}_{k|k} + \boldsymbol{v}_{k+1}\right)\end{aligned} \tag{14}$$

in which $\hat{\boldsymbol{e}}^{\theta}_{k|k} = \boldsymbol{\theta}_k - \hat{\boldsymbol{\theta}}_{k|k}$

Based on Eq. (10) and Eq. (13), it is known that the error $\hat{\boldsymbol{e}}^{\theta}_{k+1|k+1}$ can be estimated by

$$\begin{aligned}\hat{\boldsymbol{e}}^{\theta}_{k+1|k+1} &= \boldsymbol{\theta}_{k+1} - \hat{\boldsymbol{\theta}}_{k+1|k+1} \\ &= \hat{\boldsymbol{e}}^{\theta}_{k|k} - \boldsymbol{K}_{k+1}(\boldsymbol{\varphi}_{k+1}\hat{\boldsymbol{e}}^{\theta}_{k|k} + \boldsymbol{\eta}^u\hat{\boldsymbol{e}}^f_{k+1|k+1} + \boldsymbol{v}_{k+1})\end{aligned} \tag{15}$$

By inserting $\hat{\boldsymbol{e}}^f_{k+1|k+1}$ in Eq. (14) into Eq. (15), $\hat{\boldsymbol{e}}^{\theta}_{k+1|k+1}$ can be expressed by

$$\begin{aligned}\hat{\boldsymbol{e}}^{\theta}_{k+1|k+1} &= \left(\boldsymbol{I}_l + \boldsymbol{K}_{k+1}\boldsymbol{\eta}^u\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{\varphi}_{k+1}\right)\left(\boldsymbol{I}_l - \boldsymbol{K}_{k+1}\boldsymbol{\varphi}_{k+1}\right)\hat{\boldsymbol{e}}^{\theta}_{k|k} \\ &\quad - \boldsymbol{K}_{k+1}\left[\boldsymbol{I}_l - \boldsymbol{\eta}^u\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\right]\boldsymbol{v}_{k+1}\end{aligned} \tag{16}$$

and the error covariance matrix $\hat{\boldsymbol{P}}^{\theta}_{k+1|k+1}$ can be estimated by

$$\begin{aligned}\hat{\boldsymbol{P}}^{\theta}_{k+1|k+1} &= \left(\boldsymbol{I}_l + \boldsymbol{K}_{k+1}\boldsymbol{\eta}^u\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{\varphi}_{k+1}\right)\left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{\varphi}_{k+1}\right)\hat{\boldsymbol{P}}^{\theta}_{k|k} \\ &\quad \left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{\varphi}_{k+1}\right)^T\left(\boldsymbol{I}_l + \boldsymbol{K}_{k+1}\boldsymbol{\eta}^u\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{\varphi}_{k+1}\right)^T \\ &\quad + \boldsymbol{K}_{k+1}\left[\boldsymbol{I}_l - \boldsymbol{\eta}^u\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\right]\boldsymbol{R}_{k+1} \\ &\quad \left[\boldsymbol{I}_l - \boldsymbol{\eta}^u\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\left(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}\right)\right]^{\mathrm{T}}\end{aligned} \tag{17}$$

To minimize the error covariance matrix $\hat{\boldsymbol{P}}^{\theta}_{k+1|k+1}$, Kalman gain matrix $\boldsymbol{K}_{k+1}$ should be selected as

$$\boldsymbol{K}_{k+1} = \hat{\mathbf{P}}^{\theta}_{k|k}\boldsymbol{\varphi}^{\mathrm{T}}_{k+1}(\boldsymbol{\varphi}_{k+1}\hat{\mathbf{P}}^{\theta}_{k|k}\boldsymbol{\varphi}^{\mathrm{T}}_{k+1} + \boldsymbol{R}_{k+1})^{-1} \tag{18}$$

Then, the estimation of $\hat{\boldsymbol{P}}^{\theta}_{k+1|k+1}$ in Eq. (17) can be simplified as

$$\hat{\boldsymbol{P}}^{\theta}_{k+1|k+1} = \left(\boldsymbol{I}_l + \boldsymbol{K}_{k+1}\boldsymbol{\eta}^u\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}\boldsymbol{\varphi}_{k+1}\right)\left(\boldsymbol{I} - \boldsymbol{K}_{k+1}\boldsymbol{\varphi}_{k+1}\right)\hat{\boldsymbol{P}}^{\theta}_{k|k} \tag{19}$$

From the above derivation of proposed PKF-UI, it is noted that the analytical recursive formulations of the proposed PKF-UI is the direct extension of the conventional PKF.

3.2 PKF-UI with nonlinear observation equation

When the observation equation is nonlinear, it can be expressed as

$$\boldsymbol{y}(\mathrm{t}) = \boldsymbol{h}\left(\boldsymbol{\theta}, \boldsymbol{x}(\mathrm{t}), \dot{\boldsymbol{x}}(\mathrm{t}), \boldsymbol{f}^u(\mathrm{t})\right) + \boldsymbol{\upsilon}(\mathrm{t}) \tag{20}$$

in which $\boldsymbol{y}(\mathrm{t}) = \boldsymbol{M}\ddot{\boldsymbol{x}}(\mathrm{t})$, $\boldsymbol{h}$ ( ) is a nonlinear function of structural response vectors, structural parameters and the unknown external inputs. The discrete form of the above observation equation is expressed as

$$\boldsymbol{y}_{k+1} = \boldsymbol{h}\left(\boldsymbol{\theta}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \boldsymbol{f}^u_{k+1}\right) + \boldsymbol{\upsilon}_{k+1} \tag{21}$$

The nonlinear function $\boldsymbol{h}\left(\boldsymbol{\theta}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \boldsymbol{f}^u_{k+1}\right)$ can be expanded at $\hat{\boldsymbol{\theta}}_{k|k}$ and $\hat{\boldsymbol{f}}^u_{k|k}$ by the Taylor series expansion to the first order as

$$\begin{aligned}&\boldsymbol{h}\left(\boldsymbol{\theta}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \boldsymbol{f}^u_{k+1}\right) \\ &= \boldsymbol{h}(\hat{\boldsymbol{\theta}}_{k|k}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \hat{\boldsymbol{f}}^u_{k|k}) + \boldsymbol{\varphi}_{k+1}(\boldsymbol{\theta} - \hat{\boldsymbol{\theta}}_{k|k}) + \boldsymbol{\eta}^u(\boldsymbol{f}^u_{k+1} - \hat{\boldsymbol{f}}^u_{k|k})\end{aligned} \tag{22}$$

in which

$$\boldsymbol{\varphi}_{k+1} = \left.\frac{\partial \boldsymbol{h}(\boldsymbol{\theta}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \boldsymbol{f}^u_{k+1})}{\partial \boldsymbol{\theta}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{k|k};\ \boldsymbol{f}^u_{k+1}=\hat{\boldsymbol{f}}^u_{k|k}} \tag{23a}$$

$$\boldsymbol{\eta}^u = \left.\frac{\partial \boldsymbol{h}(\boldsymbol{\theta}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \boldsymbol{f}^u_{k+1})}{\partial \boldsymbol{f}^u_{k+1}}\right|_{\boldsymbol{\theta}=\hat{\boldsymbol{\theta}}_{k|k};\ \boldsymbol{f}^u_{k+1}=\hat{\boldsymbol{f}}^u_{k|k};} \tag{23b}$$

Based on linearized observation equation in Eq. (22), structural unknown parametric vector can be recursively estimated by the conventional PKF as follows

$$\begin{aligned}\hat{\boldsymbol{\theta}}_{k+1|k+1} = \hat{\boldsymbol{\theta}}_{k|k} + \boldsymbol{K}_{k+1}\Big[&\boldsymbol{y}_{k+1} - \boldsymbol{h}(\hat{\boldsymbol{\theta}}_{k|k}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \hat{\boldsymbol{f}}^u_{k|k}) \\ &- \boldsymbol{\eta}^u(\hat{\boldsymbol{f}}^u_{k+1|k+1} - \hat{\boldsymbol{f}}^u_{k|k})\Big]\end{aligned} \tag{24}$$

Under the condition that the number of response measurements is not less than the number of unknown external inputs, $\hat{\boldsymbol{f}}^{u}_{k+1|k+1}$ can be estimated by minimizing the following error vector as

$$\begin{aligned}\Delta_{k+1} = &\boldsymbol{y}_{k+1} - \boldsymbol{h}(\hat{\boldsymbol{\theta}}_{k|k}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \hat{\boldsymbol{f}}^{u}_{k|k}) \\ &-\boldsymbol{\varphi}_{k+1}(\hat{\boldsymbol{\theta}}_{k+1|k+1} - \hat{\boldsymbol{\theta}}_{k|k}) - \boldsymbol{\eta}^{u}(\hat{\boldsymbol{f}}^{u}_{k+1|k+1} - \hat{\boldsymbol{f}}^{u}_{k|k})\end{aligned} \tag{25}$$

By inserting the expression of $\hat{\boldsymbol{\theta}}_{k+1|k+1}$ in Eq. (24) into the above error vector, $\boldsymbol{\Delta}_{k+1}$ can be expressed as

$$\begin{aligned}\Delta_{k+1} = &(\mathbf{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1})\left(\boldsymbol{y}_{k+1} - \boldsymbol{h}(\hat{\boldsymbol{\theta}}_{k|k}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \hat{\boldsymbol{f}}^{u}_{k|k}) + \boldsymbol{\eta}^{u}\hat{\boldsymbol{f}}^{u}_{k|k}\right) \\ &-(\mathbf{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1})\boldsymbol{\eta}^{u}\hat{\boldsymbol{f}}^{u}_{k+1|k+1}\end{aligned} \tag{26}$$

Then, $\hat{\boldsymbol{f}}^{u}_{k+1|k+1}$ can be estimated from Eq. (26) by the least-squares estimation as

$$\begin{aligned}\hat{\boldsymbol{f}}^{u}_{k+1|k+1} = &\boldsymbol{S}_{k+1}\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1}) \\ &\left(\boldsymbol{y}_{k+1} - \boldsymbol{h}(\hat{\boldsymbol{\theta}}_{k|k}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \hat{\boldsymbol{f}}^{u}_{k|k}) + \boldsymbol{\eta}^{u}\hat{\boldsymbol{f}}^{u}_{k|k}\right)\end{aligned} \tag{27}$$

in which $\boldsymbol{S}_{k+1} = \left[\boldsymbol{\eta}^{uT}\boldsymbol{R}^{-1}_{k+1}(\boldsymbol{I}_l - \boldsymbol{\varphi}_{k+1}\boldsymbol{K}_{k+1})\boldsymbol{\eta}^{u}\right]^{-1}$

Based on Eqs. (24)-(27), the error vectors $\hat{\boldsymbol{e}}^{\boldsymbol{f}}_{k+1|k+1}$ and $\hat{\boldsymbol{e}}^{\boldsymbol{\theta}}_{k+1|k+1}$, the error covariance matrices $\hat{\boldsymbol{P}}^{\boldsymbol{f}}_{k+1|k+1}$ and $\hat{\boldsymbol{P}}^{\boldsymbol{\theta}}_{k+1|k+1}$, and the Kalman gain matrix $\boldsymbol{K}_{k+1}$ can be derived as the same as those in Eqs. (14)-(19).

In summary, the derivation of the proposed PKF-UI is completely based on the conventional PKF. The recursive procedures of the proposed PKF-UI are analogous to those of the conventional PKF. The flowchart of the proposed PKF-UI is shown in Fig. 1. Therefore, the proposed PKF-UI is a direct extension of the conventional PKF, which simplifies the complex derivations in previous last-squares estimation with unknown inputs (Yang *et al*. 2007).

4. Probabilistic damage detection of structures with uncertainties based on PKF-UI

Structural uncertainties are inevitable for civil infrastructures, it is necessary to develop approaches for probabilistic damage detection of structures.

It is assumed that the probability distribution of uncertain parametric vector $\boldsymbol{\Theta}$ is obtained when the information of the uncertain parameters is sufficient. Then, the estimation of $\hat{\boldsymbol{\theta}}_{k|k}$ and $\hat{\boldsymbol{f}}^{u}_{k|k}$ can be expanded at the corresponding mean value $\bar{\boldsymbol{\Theta}}$ of uncertain parameters by Taylor series expansion to the first order as (Li and Law 2008, Law and Li 2010)

$$\hat{\boldsymbol{\theta}}_{k}(\boldsymbol{\Theta}) = \hat{\boldsymbol{\theta}}_{k}(\bar{\boldsymbol{\Theta}}) + \frac{\partial\hat{\boldsymbol{\theta}}_{k}(\boldsymbol{\Theta})}{\partial\boldsymbol{\Theta}}\Big|_{\boldsymbol{\Theta}=\bar{\boldsymbol{\Theta}}}(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) \tag{28a}$$

$$\hat{\boldsymbol{f}}^{u}_{k}(\boldsymbol{\Theta}) = \hat{\boldsymbol{f}}^{u}_{k}(\bar{\boldsymbol{\Theta}}) + \frac{\partial\hat{\boldsymbol{f}}^{u}_{k}(\boldsymbol{\Theta})}{\partial\boldsymbol{\Theta}}\Big|_{\boldsymbol{\Theta}=\bar{\boldsymbol{\Theta}}}(\boldsymbol{\Theta} - \bar{\boldsymbol{\Theta}}) \tag{28b}$$

where $\hat{\boldsymbol{\theta}}_{k}(\bar{\boldsymbol{\Theta}})$ and $\hat{\boldsymbol{f}}^{u}_{k}(\bar{\boldsymbol{\Theta}})$ denote the identified structural identified parametric vector and unknown external inputs with the mean value $\bar{\boldsymbol{\Theta}}$ of the uncertain parameters, respectively, which can be obtained by the proposed PKF-UI. $\frac{\partial\hat{\boldsymbol{\theta}}_{k}(\boldsymbol{\Theta})}{\partial\boldsymbol{\Theta}}\Big|_{\boldsymbol{\Theta}=\bar{\boldsymbol{\Theta}}}$ and $\frac{\partial\hat{\boldsymbol{f}}^{u}_{k}(\boldsymbol{\Theta})}{\partial\boldsymbol{\Theta}}\Big|_{\boldsymbol{\Theta}=\bar{\boldsymbol{\Theta}}}$ are the sensitivity matrix of the estimated $\hat{\boldsymbol{\theta}}_{k|k}$ and $\hat{\boldsymbol{f}}^{u}_{k|k}$ with respect to the uncertain structural parametric vector Θ, which can be estimated by central difference approach.

Based on Eq.(28a), the probability density functions of unknown structural parametric vector $\boldsymbol{\theta}$ can be achieved when probability distribution of the uncertain parametric vector Θ is known.

A confidence level of the i^{th} identified structural parameter $\theta^{u}_{i}(\boldsymbol{\Theta})$ in undamaged structural model is $1-\alpha_i$ (i=1,2,…,m), i.e.

$$\text{Prob}(L_{\text{i}} \le \theta^{u}_{i}(\boldsymbol{\Theta}) < \infty) = 1 - \alpha_{\text{i}} \tag{29}$$

where L_{i} is lower bound of confidence interval for the i^{th} identified structural parameter $\theta^{u}_{i}(\boldsymbol{\Theta})$.

Then, the probabilities of damage existence (PDE) for the i^{th} identified structural parameter $\theta^{D}_{i}(\boldsymbol{\Theta})$ is defined by (Wang *et al*. 2014)

$$\text{PDE}_{i} = \text{Prob}(-\infty < \theta^{D}_{i}(\boldsymbol{\Theta}) \le L_{\text{i}}) \tag{30}$$

where $\theta^{D}_{i}(\boldsymbol{\Theta})$ denotes the i^{th} identified structural parameter in the damaged structural model.

Finally, structural damage extent (DE) for the i^{th} identified structural parameter is defined by (Zhang *et al*. 2011)

$$\text{DE}_{i} = \frac{\theta^{\text{u}}_{i}(\bar{\boldsymbol{\Theta}}) - \theta^{D}_{i}(\bar{\boldsymbol{\Theta}})}{\theta^{\text{u}}_{i}(\bar{\boldsymbol{\Theta}})} \times 100\% \tag{31}$$

5. Numerical example validations

Some numerical examples are used to validate the proposed method. Two scenarios of linear observation equations and nonlinear observation equations are discussed in the following two examples, respectively.

5.1 Example 1: Probabilistic damage detection of a shear frame with uncertainty of mass density

A 10-story shear frame building shown in Fig. 2 is used as a numerical example to evaluate the proposed method. The deterministic parameters of the building are story stiffness as: $\boldsymbol{k}_i$=[2.713, 2.685, 2.657, 2.648, 2.639, 2.629, 2.604, 2.589, 2.576, 2.558]×10^5 N/m, (i=1,2,…,10).

The density of structural material is considered as a random variable with a normal distribution with mean value ρ_0=7850 kg/m^3 and the standard deviation σ of 7% of the mean value. So, the mass of floor calculated by mean value ρ_0 are $\boldsymbol{m}_i(\rho_0)$=[67.96, 65.49, 63.08, 62.59, 61.92, 60.49, 59.92, 58.42, 57.48, 56.84] *kg* (i=1,2,…,10).

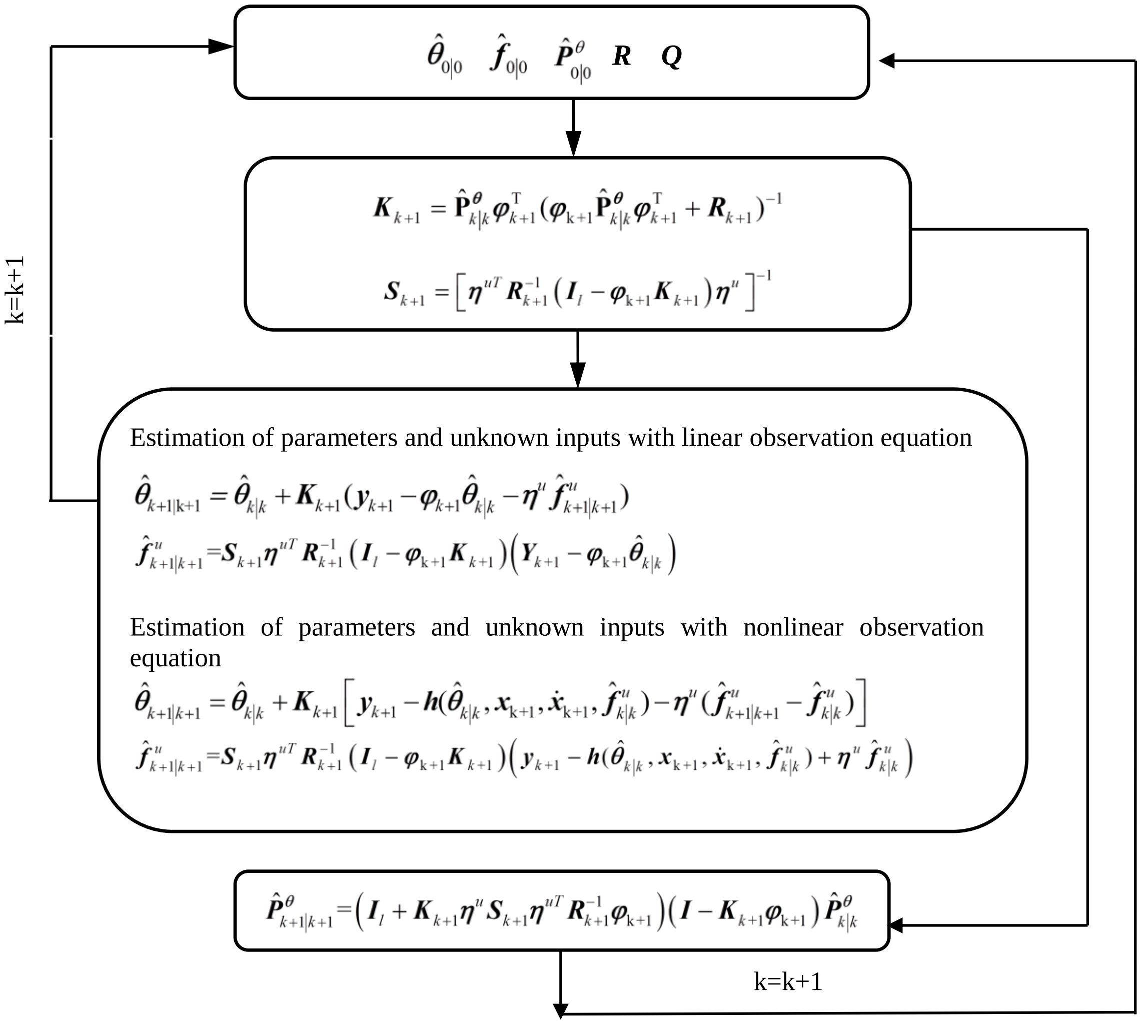


Fig. 1 The flowchart of the proposed PKF-UI

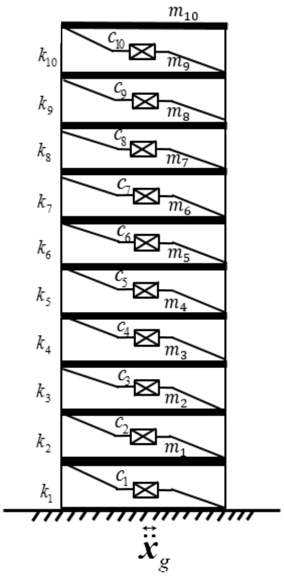


Fig. 2 A 10-storey shear building under unknown input of ground excitation

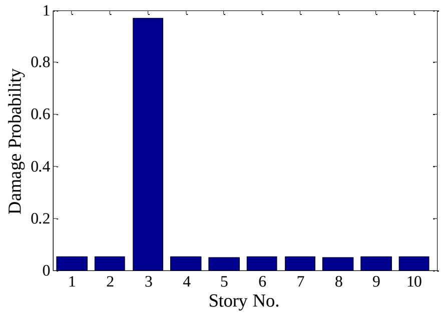
The frame is subjected to an unknown input of ground excitation in the K-T spectrum with the spectral density function in the form as

$$S_g(\omega) = \frac{1 + 4\xi_g^2(\frac{\omega}{\omega_g})^2}{\left[1 - (\frac{\omega}{\omega_g})^2\right]^2 + 4\xi_g^2(\frac{\omega}{\omega})^2} S_0 \tag{32}$$

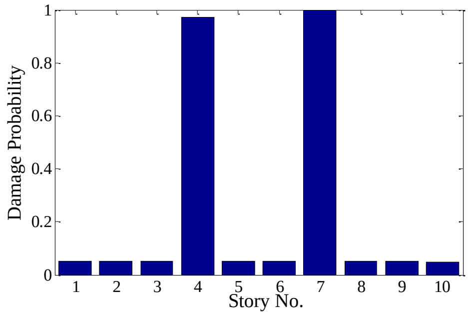
in which ω_g, ξ_g and S_0 are the characteristic parameters of the ground motion. These parameters are selected as ω_g =15.0 rad/s, ξ_g=0.6, S_0=4.64×10^{-4} m^2/rads^3. The time duration of the simulated acceleration is 15s and the sampling frequency is 1000Hz. All the velocity and displacement and acceleration measurements at each floor are polluted by white noises with 5% noise-to-signal ratio in root mean square (rms).

Rayleigh damping is adopted with $\boldsymbol{C}=\alpha\boldsymbol{M}+\beta\boldsymbol{K}$, where α and β are two unknown coefficients of Rayleigh damping. The unknown parametric vector to be identified in this example is $\boldsymbol{\theta}=\{k_1, k_2, \ldots, k_n, \beta k_1, \beta k_2, \ldots, \beta k_n, \alpha\}^T$ ($\boldsymbol{n}$=10). The unknown input of ground excitation in K-T spectrum will also be identified.

Two structural damage patterns are considered herein. In the damage pattern A, the stiffness of the 3$^{\text{th}}$ story is reduced by 10%. In the damage pattern B, both the stiffness

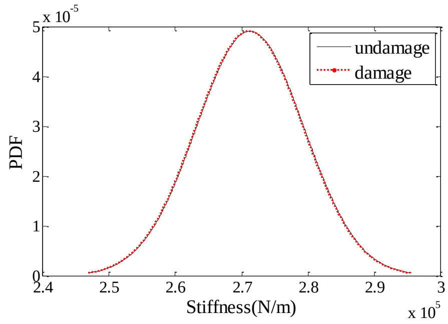


(a) Damage probability of each story for pattern A

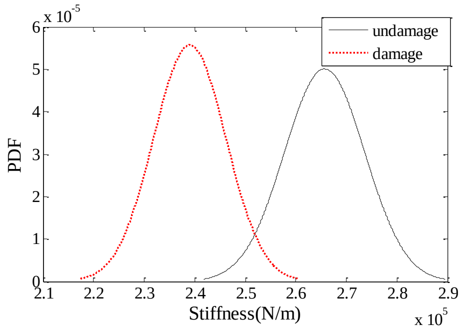


(b) Damage probability of each story for pattern B

Fig. 3 Damage probability of each story for pattern A and pattern B



(a) PDF of the 1st story



(b) PDF of the 3th story

Fig. 4 Probability density functions of some story for damage pattern A

of the 4th story and the 7th story are reduced by 10% and 15%, respectively.

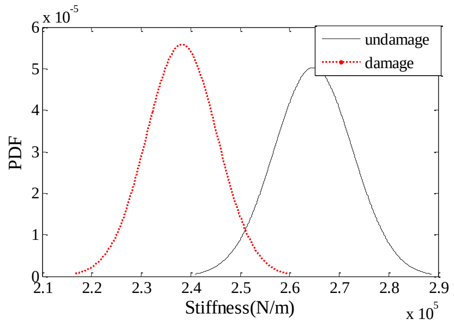
In this numerical example, the observation equation is a linear one expressed as

$$\begin{aligned} \boldsymbol{y}_{k+1} &= \boldsymbol{\varphi}_{k+1}\boldsymbol{\theta}_{k+1} + \boldsymbol{\eta}^u \boldsymbol{f}^u_{k+1} + \boldsymbol{\upsilon}_{k+1} \\ &= -(\alpha \boldsymbol{M} + \beta \boldsymbol{K})\dot{\boldsymbol{x}}_{k+1} - \boldsymbol{K}\boldsymbol{x}_{k+1} + \boldsymbol{\eta}^u \boldsymbol{f}^u_{k+1} + \boldsymbol{\upsilon}_{k+1} \end{aligned} \tag{33}$$

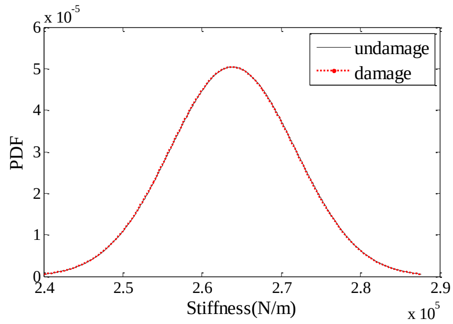
in which

$$\begin{aligned} \boldsymbol{\varphi}_{k+1} &= \frac{\partial[(-\alpha \boldsymbol{M} - \beta \boldsymbol{K})\dot{\boldsymbol{x}}_{k+1} - \boldsymbol{K}\boldsymbol{x}_{k+1}]}{\partial \boldsymbol{\theta}} \\ &= -\frac{\partial \alpha}{\partial \boldsymbol{\theta}}\boldsymbol{M}\dot{\boldsymbol{x}}_{k+1} - \frac{\partial(\beta \boldsymbol{K})}{\partial \boldsymbol{\theta}}\dot{\boldsymbol{x}}_{k+1} - \frac{\partial \boldsymbol{K}}{\partial \boldsymbol{\theta}}\boldsymbol{x}_{k+1} \end{aligned} \tag{34}$$

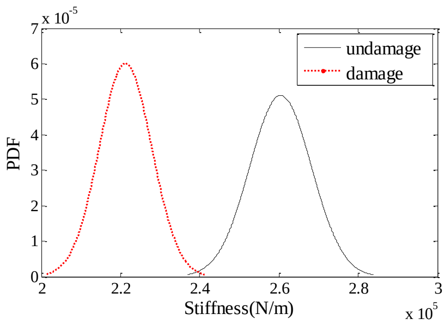
The identified story stiffness $k_i(\rho)$ is also a random



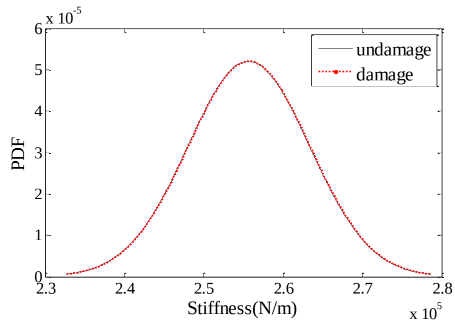
(a) PDF of the 4th story



(b) PDF of the 5th story



(c) PDF of the 7th story



(d) PDF of the 10th story

Fig. 5 Probability density functions of some story for damage pattern B

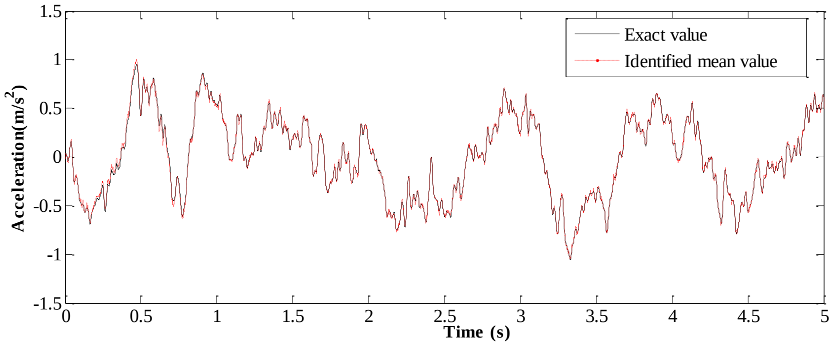


Fig. 6 Comparison of identified mean value of ground acceleration and exact value

Table 1 Undamaged and damaged structural parameters, DE and PDE for pattern A

Story No.	Undamaged Stiffness (N/m)	Identified mean value of undamaged stiffness (N/m)	Error (%)	Damaged Stiffness (N/m)	Identified mean value of damaged stiffness (N/m)	Error (%)	DE (%)	PDE (%)
1	271300	271307	0.003	271300	271191	-0.04	0.04	5.14
2	268500	268502	0.001	268500	268604	0.039	-0.04	5.13
3	265700	265722	0.008	239130	239043	-0.036	**10.04**	**97.09**
4	264800	264806	0.002	264800	264905	0.040	-0.04	5.14
5	263900	263897	-0.001	263900	264021	0.046	-0.05	5.09
6	262900	262930	0.011	262900	262812	-0.033	0.04	5.15
7	260400	260434	0.013	260400	260307	-0.036	0.05	5.16
8	258900	258810	-0.035	258900	258803	-0.038	0.00	5.01
9	257600	257508	-0.036	257600	257789	0.073	-0.11	5.12
10	255800	255665	-0.053	255800	255525	-0.107	0.05	5.18

Table 2 Undamaged and damaged structural parameters, DE and PDE for pattern B

Story No.	Undamaged Stiffness (N/m)	Identified mean value of undamaged stiffness (N/m)	Error (%)	Damaged Stiffness (N/m)	Identified mean value of damaged stiffness (N/m)	Error (%)	DE (%)	PDE (%)
1	271300	271307	0.003	271300	271386	0.032	-0.03	5.12
2	268500	268502	0.001	268500	268417	-0.031	0.03	5.11
3	265700	265722	0.008	265700	265778	0.030	-0.02	5.13
4	264800	264806	0.002	238320	238240	-0.034	**10.03**	**97.07**
5	263900	263897	-0.001	263900	263957	0.021	-0.02	5.07
6	262900	262930	0.011	262900	262973	0.027	-0.02	5.13
7	260400	260434	0.013	221340	221275	-0.029	**15.04**	**99.00**
8	258900	258810	-0.035	258900	258829	-0.027	-0.01	4.98
9	257600	257508	-0.036	257600	257450	-0.058	0.02	5.08
10	255800	255665	-0.053	255800	255587	-0.083	0.03	5.10

variable. Based on Eq. (28a), $k_i(\rho)$ can be expanded at the mean value ρ_0 by the Taylor series expansion to the first order as

$$k_i(\rho) = k_i(\rho_0) + \left.\frac{\partial k_i}{\partial \rho}\right|_{\rho=\rho_0} (\rho - \rho_0) \tag{35}$$

in which $\left.\frac{\partial k_i}{\partial \rho}\right|_{\rho=\rho_0}$ is estimated by method of central differences as:

$$\left.\frac{\partial k_i}{\partial \rho}\right|_{\rho=\rho_0} = \frac{k_i(\rho_0+\Delta\rho) - k_i(\rho_0-\Delta\rho)}{2\Delta\rho}, (i=1,\ldots,\text{n, n}=10) \tag{36}$$

Probability density functions of each story $k_i(\rho)$ can be achieved by Eq. (35) when probability distribution of density ρ is known. The confidence level of all the identified structural parameter is $1-\alpha_i$=95%.

The damage probability of each story for pattern A and pattern B are shown in Fig. 3(a) and 3(b), respectively. As seen from these figures, the damage probability of the damaged story is clearly higher than that of other stories.

The probability density functions of some story $k_i(\rho)$for damage pattern A and pattern B are shown in Fig. 4 and Fig. 5, respectively. In these figures, the probability density functions of the damaged story changes obviously.

The comparison of identified mean value of ground acceleration and exact value are shown in Fig. 6. It is clearly shown that the identified acceleration is closed to the exact value.

The Identified mean value of undamaged and damaged story stiffness, damage extent (DE) and damage probability (PDE) of each story for pattern A and pattern B are listed in Table 1 and Table 2, respectively.

5.2 Example 2: probabilistic damage detection of a shear frame with uncertainty of damping ratio

A 10-story shear building with uncertainty of damping ratio is used as a numerical example to evaluate the effectiveness of the proposed method. The deterministic parameters of the building are the floor mass and story stiffness as: $\boldsymbol{m}_i$=[67.955, 65.485, 63.079, 62.591, 61.918, 60.485, 59.922, 58.418, 57.484, 56.837]×10 kg, $\boldsymbol{k}_i$=[2.713, 2.685, 2.657, 2.648, 2.639, 2.629, 2.604, 2.589, 2.576, 2.558]× 10^4 N/m (i=1,2,…,10).

The first damping ratio ζ_1 and the second damping ratio ζ_2 are selected as independent random variables and approximated as lognormal distributions with mean values and the standard deviations are $\bar{\zeta}_1$ =1.0% σ_1 =20% $\bar{\zeta}_1$ and $\bar{\zeta}_2$ =2.14% σ_2 =20% $\bar{\zeta}_2$ respectively.

The frame is also subjected to the same unknown input of ground excitation in the K-T spectrum as shown in example 1. All the velocity and displacement and acceleration measurements at each story are polluted by white noises with 5% noise-to-signal ratio in root mean square (rms).

Rayleigh damping is adopted, $\boldsymbol{C}=\alpha\boldsymbol{M}+\beta\boldsymbol{K}$ where α and β are two coefficients of Rayleigh damping, which are the implicit functions of the identified parametric vector $\boldsymbol{\theta}=\{k_1,k_2,\ldots,k_n\}^T$. The unknown input of ground excitation in K-T spectrum will also be estimated, structural damage is assumed as the reduction of the 3th story stiffness by 10%.

In this numerical example, the observation equation is a nonlinear one expressed as

$$\begin{aligned}\boldsymbol{y}_{k+1} &= \boldsymbol{h}\left(\boldsymbol{\theta}, \boldsymbol{x}_{k+1}, \dot{\boldsymbol{x}}_{k+1}, \boldsymbol{f}^u_{k+1}\right)+\boldsymbol{\upsilon}_{k+1} \\ &= -\left(\alpha\boldsymbol{M}+\beta\boldsymbol{K}\right)\dot{\boldsymbol{x}}_{k+1}-\boldsymbol{K}\boldsymbol{x}_{k+1}+\boldsymbol{\eta}^u\boldsymbol{f}^u_{k+1}+\boldsymbol{v}_{k+1}\end{aligned} \tag{37}$$

where α and β are two coefficients of Rayleigh damping defined by

$$\begin{Bmatrix}\alpha\\ \beta\end{Bmatrix}=\frac{2\omega_1\omega_2}{\omega_2^2-\omega_1^2}\begin{bmatrix}\omega_2 & -\omega_1\\ -\dfrac{1}{\omega_2} & \dfrac{1}{\omega_1}\end{bmatrix}\begin{Bmatrix}\xi_1\\ \xi_2\end{Bmatrix} \tag{38}$$

ω_1 and ω_2 are the first and second natural frequencies of the building, respectively.

Then based on Eq. (23)

$$\boldsymbol{\varphi}_{k+1}=-\boldsymbol{M}\dot{\boldsymbol{x}}_{k+1}\cdot\frac{\partial\alpha}{\partial\boldsymbol{\omega}}\cdot\frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{\theta}}-\boldsymbol{K}\dot{\boldsymbol{x}}_{k+1}\cdot\frac{\partial\beta}{\partial\boldsymbol{\omega}}\cdot\frac{\partial\boldsymbol{\omega}}{\partial\boldsymbol{\theta}}-\beta\dot{\boldsymbol{x}}_{k+1}-\boldsymbol{x}_{k+1} \tag{39}$$

where $\boldsymbol{\omega}=\begin{bmatrix}\omega_1\\ \omega_2\end{bmatrix}$, the value of $\dfrac{\partial\alpha}{\partial\boldsymbol{\omega}}$ and $\dfrac{\partial\beta}{\partial\boldsymbol{\omega}}$ can be worked out as

$$\frac{\partial\alpha}{\partial\boldsymbol{\omega}}=\frac{2}{(\omega_1^2-\omega_2^2)^2}\begin{bmatrix}\omega_2^2\left(\omega_1^2+\omega_2^2\right) & -2\omega_1\omega_2^3\\ -2\omega_1^3\omega_2 & \omega_1^2\left(\omega_2^2+\omega_1^2\right)\end{bmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \tag{40}$$

$$\frac{\partial\beta}{\partial\boldsymbol{\omega}}=\frac{2}{(\omega_1^2-\omega_2^2)^2}\begin{bmatrix}-\omega_1^2-\omega_2^2 & 2\omega_1\omega_2\\ 2\omega_1\omega_2 & -\omega_1^2-\omega_2^2\end{bmatrix}\begin{pmatrix}\zeta_1\\ \zeta_2\end{pmatrix} \tag{41}$$

Because of

$$(\boldsymbol{K}-\omega_j^2\boldsymbol{M})\cdot\boldsymbol{\Phi}_j=\boldsymbol{0} \quad (j=1,2) \tag{42}$$

where $\boldsymbol{\Phi}_j=[\phi_{j,1},\phi_{j,2},\cdots,\phi_{j,10}]^T$, both sides of the Eq.(42) are made partial derivative by θ_i, and left multiplied by $\boldsymbol{\Phi}_j^T$. Then

$$\boldsymbol{\Phi}_j^T(\frac{\partial\boldsymbol{K}}{\partial\theta_i}-\frac{\partial\omega_j^2}{\partial\theta_i}\boldsymbol{M}-\omega_j^2\frac{\partial\boldsymbol{M}}{\partial\theta_i})\cdot\boldsymbol{\Phi}_j=\boldsymbol{0} \tag{43}$$

in which

$$\boldsymbol{\Phi}_j^T(\frac{\partial\boldsymbol{K}}{\partial\theta_i}-\omega_j^2\frac{\partial\boldsymbol{M}}{\partial\theta_i})\cdot\boldsymbol{\Phi}_j=\frac{\partial\omega_j^2}{\partial\theta_i}\boldsymbol{\Phi}_j^T\boldsymbol{M}\boldsymbol{\Phi}_j=\frac{\partial\omega_j^2}{\partial\theta_i} \tag{44}$$

$$\boldsymbol{\Phi}_j^T\frac{\partial\boldsymbol{M}}{\partial\theta_i}\boldsymbol{\Phi}_j=\boldsymbol{0} \tag{45}$$

Based on Eq. (44)

$$\frac{\partial\omega_j^2}{\partial\theta_i}=\boldsymbol{\Phi}_j^T\frac{\partial\boldsymbol{K}}{\partial\theta_i}\boldsymbol{\Phi}_j=\left[\phi_{j,1}^2,(\phi_{j,1}-\phi_{j,2})^2,\cdots,(\phi_{j,(n-1)}-\phi_{j,n})^2\right] \tag{46}$$
$$(j=1,2)\ (n=10)$$

$$\frac{\partial\omega_j}{\partial\theta_i}=\begin{bmatrix}\dfrac{1}{2\omega_1}\left[\phi_{1,1}^2,\ (\phi_{1,1}-\phi_{1,2})^2,\cdots,(\phi_{1,(n-1)}-\phi_{1,n})^2\right]\\ \dfrac{1}{2\omega_2}\left[\phi_{2,1}^2,(\phi_{2,1}-\phi_{2,2})^2,\cdots,(\phi_{2,(n-1)}-\phi_{2,n})^2\right]\end{bmatrix}\ (n=10) \tag{47}$$

Eventually, the value of $\boldsymbol{\varphi}_{k+1}$ can be gain by substituting Eqs. (40), (41) and (47) to Eq. (39).

The identified story stiffness $k_i(\zeta_1, \zeta_2)$ is also a random variable. Based on Eq. (28a), $k_i(\zeta_1, \zeta_2)$ could be expanded at the mean value $\bar{\zeta}_1$ and $\bar{\zeta}_2$ by the Taylor series expansion to the first order as

$$\begin{aligned}k_i(\zeta_1,\zeta_2)&=k_i(\bar{\zeta}_1,\bar{\zeta}_2)+\left.\frac{\partial k_i}{\partial\zeta_1}\right|_{\substack{\zeta_1=\bar{\zeta}_1\\ \zeta_2=\bar{\zeta}_2}}(\zeta_1-\bar{\zeta}_1)\\ &+\left.\frac{\partial k_i}{\partial\zeta_2}\right|_{\substack{\zeta_1=\bar{\zeta}_1\\ \zeta_2=\bar{\zeta}_2}}(\zeta_2-\bar{\zeta}_2)\end{aligned} \tag{48}$$

In which $\left.\dfrac{\partial k_i}{\partial\zeta_1}\right|_{\substack{\zeta_1=\bar{\zeta}_1\\ \zeta_2=\bar{\zeta}_2}}$ and $\left.\dfrac{\partial k_i}{\partial\zeta_2}\right|_{\substack{\zeta_1=\bar{\zeta}_1\\ \zeta_2=\bar{\zeta}_2}}$ could be calculated by method of central differences.

Probability density functions of each story $k_i(\zeta_1, \zeta_2)$ can be achieved by Eq. (48) when probability distribution of ζ_1 and ζ_2 are known . The confidence level in this example is the same as the example 1.

The damage probability of each story is shown in Fig. 7. Obviously, the damage probability of damaged story is clearly higher than that of other story. The comparison of identified mean value of ground acceleration and exact value are shown in Fig. 8. It is clear that the identified mean value of ground acceleration is closed to the exact value.

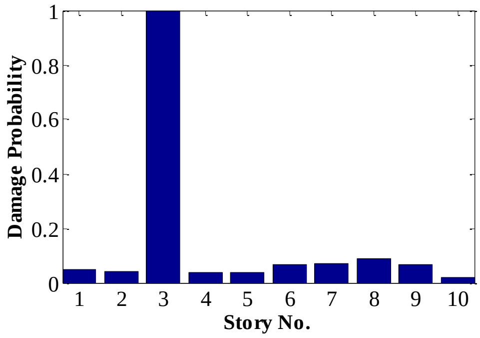


Fig. 7 Damage probability of each story

The identified mean value of undamaged and damaged story stiffness, damage extent (DE) and damage probability (PDE) of each story are listed in Table 3. It is clear that the stiffness of damaged story is significantly reduced and close to the exact value.

6. Conclusions

In this paper, probabilistic damage detection of structures with uncertainties under unknown excitations is investigated based on a proposed parametric Kalman filter with unknown inputs (PKF-UI) for the simultaneous identification of structural parameters and the unmeasured external inputs. The proposed PKF-UI is a direct extension of the conventional PKF with all analytical recursive formulations analogously derived. Two scenarios of linear observation equations and nonlinear observation equations are discussed, respectively. Such a straightforward derivation and formulation of PKF-UI is not available in the literature.

The proposed PKF-UI is utilized for probabilistic damage detection of structures under unknown excitations by considering the uncertainties of structural parameters. Structural damage index and the damage probability are given from the statistical values of the identified structural parameters of intact and damaged structure. Some numerical examples have validated the good performances of proposed method in the probabilistic damage detection of structures with uncertainties under unknown excitations.

Acknowledgments

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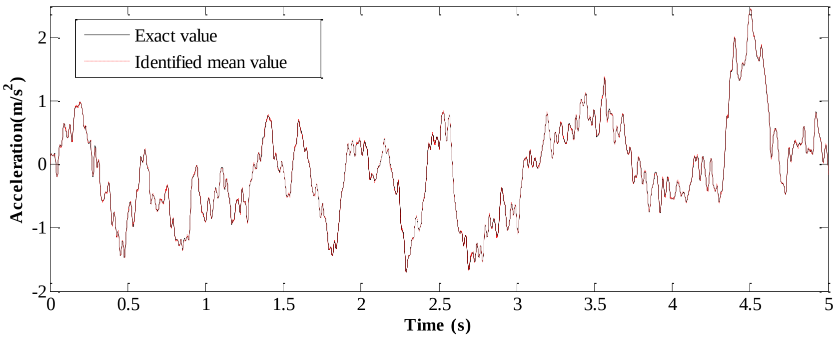

Fig. 8 Comparison of identified mean value of ground acceleration and exact value

Table 3 Undamaged and damaged structural parameters, DE and PDE of each story

Story No.	Undamaged Stiffness (N/m)	Identified mean value of undamaged stiffness (N/m)	Error (%)	Damaged Stiffness (N/m)	Identified mean value of damaged stiffness (N/m)	Error (%)	DE (%)	PDE (%)
1	27130	27126	-0.014	27130	27126	-0.013	0.001	4.78
2	26850	26845	-0.019	26850	26845	-0.017	0.002	4.24
3	26570	26570	0.000	23913	23913	0.002	**9.998**	**100.00**
4	26480	26471	-0.033	26480	26471	-0.034	-0.001	3.94
5	26390	26386	-0.015	26390	26386	-0.016	-0.001	3.98
6	26290	26283	-0.027	26290	26282	-0.030	0.004	6.67
7	26040	26038	-0.007	26040	26037	-0.010	0.004	7.15
8	25890	25881	-0.036	25890	25879	-0.041	0.008	8.76
9	25760	25763	0.011	25760	25762	0.008	0.004	6.67
10	25580	25582	0.009	25580	25584	0.015	-0.008	2.11

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