Structural Engineering and Mechanics, *Vol. 63, No. 4 (2017) 551-565*
DOI: https://doi.org/10.12989/sem.2017.63.4.551

Free vibration analysis of cracked Timoshenko beams carrying spring-mass systems

Guojin Tan[a], Jinghui Shan[b], Chunli Wu[*] and Wensheng Wang[c]

College of Transportation, Jilin University, Renmin Street 5988, Changchun 130022, China

(Received January 16, 2017, Revised May 21, 2017, Accepted June 14, 2017)

Abstract. In this paper, an analytical approach is proposed for determining vibration characteristics of cracked non-uniform continuous Timoshenko beam carrying an arbitrary number of spring-mass systems. This method is based on the Timoshenko beam theory, transfer matrix method and numerical assembly method to obtain natural frequencies and mode shapes. Firstly, the beam is considered to be divided into several segments by spring-mass systems and support points, and four undetermined coefficients of vibration modal function are contained in each sub-segment. The undetermined coefficient matrices at spring-mass systems and pinned supports are obtained by using equilibrium and continuity conditions. Then, the overall matrix of undetermined coefficients for the whole vibration system is obtained by the numerical assembly technique. The natural frequencies and mode shapes of a cracked non-uniform continuous Timoshenko beam carrying an arbitrary number of spring-mass systems are obtained from the overall matrix combined with half-interval method and Runge-Kutta method. Finally, two numerical examples are used to verify the validity and reliability of this method, and the effects of cracks on the transverse vibration mode shapes and the rotational mode shapes are compared. The influences of the crack location, depth, position of spring-mass system and other parameters on natural frequencies of non-uniform continuous Timoshenko beam are discussed.

Keywords: non-uniform Timoshenko beam; free vibration; crack; spring-mass system; numerical assembly technique

1. Introduction

Cracks present a serious threat to proper performance of structures. It is an important means to ensure the safety of structure through taking some remedial measures or evaluating the bearing capacity of the structural residual load as soon as possible whenever cracks appear. The structural dynamic characteristics such as natural frequencies and mode shapes will be changed due to the presence of cracks, which means that the dynamic characteristics have great potential for the diagnosis of cracks. The non-uniform continuous Timoshenko beam carrying various concentrated elements (such as linear springs, point masses and spring-mass systems, etc.) are widely used in the field of mechanical, civil engineering and so on. Thus, it is of great significance to study the free vibration analysis of a non-uniform continuous Timoshenko beam carrying an arbitrary number of spring-mass systems.

Some studies have been performed to analyze the vibration characteristic of Timoshenko beams carrying various concentrated elements by several researchers and some corresponding research papers have been published. Rossi *et al*. (1993) studied the free vibration of a single span uniform Timoshenko beam carrying a spring-mass system which acts as a dynamic absorber cancelling out the motion at the point of attach. Wu and Chen (2001) obtained the exact natural frequencies and mode shapes of a uniform Timoshenko beam with various boundary conditions and carrying multiple spring-mass systems with the presented numerical assembly technique. Wang *et al*. (2007) studied the natural frequencies and mode shapes of a uniform Timoshenko beam carrying multiple intermediate spring-mass systems with the effects of shear deformation and rotatory inertia. Yesilce *et al*. (2008) studied the free vibration of a uniform multi-span Timoshenko beam carrying multiple spring mass systems by using scant method. Lin (2009) utilized the numerical assembly method to determine the exact natural frequencies and mode shapes of the uniform multi-span Timoshenko beam carrying a number of various concentrated elements including spring-mass systems etc. Yesilce and Demirdag (2008) obtained the frequency values and mode shapes of the multi-span uniform Timoshenko beam carrying multiple spring-mass systems with the axial force effect. Wu and Chang (2013) studied the exact natural frequencies and associated mode shapes for an axial-loaded multi-step Timoshenko beam carrying various concentrated elements by using continuous mass transfer matrix method. EI-Sayed *et al*. (2016) dealt with the analysis of the vibration of an axially loaded beam system carrying ends consisting of non-concentrated tip masses and three spring-two mass sub-systems. From above literature review, one sees that most investigations were concentrated on intact uniform Timoshenko beams,

∗Corresponding author, Ph.D.
E-mail: 463069285@qq.com
[a]Professor
E-mail: tgj@jlu.edu.cn
[b]M.S.
E-mail: 313774894@qq.com
[c]M.S.
E-mail: 823883772@qq.com

http://www.techno-press.com/journals/sem&subpage=7
ISSN: 1225-4568 (Print), 1598-6217 (Online)

however, the problems regarding free vibration of cracked non-uniform continuous Timoshenko beam carrying various concentrated elements are very rare.

During the last decades, the vibration behavior of cracked simple beam has been investigated by many researchers. And various kinds of analytical, semi-analytical and numerical methods have been employed to solve the problem of a cracked simple beam (Zheng and Fan 2001, Khiem and Lien 2001, Torabi *et al*. 2014, Mazanoglu *et al*. 2009, Zheng and Ji 2012). Kisa *et al*. (1998) studied the vibrational characteristics of a cracked uniform Timoshenko beam by integrating the finite element method and component mode synthesis. Finite element method was also used by Viola *et al*. (2001) to model the damaged structure. A new finite spectral element of a cracked uniform Timoshenko beam was introduced by Krawczuk *et al*. (2003) for modal and elastic wave propagation analysis. Lin (2004) obtained the characteristic equation of a simply supported uniform Timoshenko beam with an open crack by using the analytical transfer matrix method. By the use of the energy approach, Swamidas *et al*. (2004) predicted the effect of crack size and location on the natural frequencies of cracked uniform simply supported Timoshenko beam. Loya *et al*. (2006) obtained the natural frequencies of uniform Timoshenko beams with a crack by the perturbation method. All the above studies considered only one crack on the Timoshenko beam.

The case that Timoshenko beam has more than one crack was considered in the study of Zheng and Fan (2001). They obtained the natural frequencies of a Timoshenko beam which can have non-uniform cross-sectional areas with an arbitrary number of transverse open cracks and point-spring supports by using modified Fourier series. Li (2003) also established the frequency equation for a uniform Timoshenko beam with any kind of two end supports and an arbitrary number of cracks from a second-order determinant. Aydin (2007) presented an efficient analytical approach to determine the vibrational frequencies and mode shape functions of axially-loaded uniform Timoshenko beams with an arbitrary number of cracks. From the above literature analysis, it can be seen that the study of the free vibration of cracked Timoshenko beam carrying spring-mass systems has not yet been involved.

In this paper, a method is presented for free vibration analysis of cracked continuous Timoshenko beam carrying arbitrary number spring-mass systems, which is available to any form of the variable cross section. In this method, the transfer matrix method and numerical assembly technique are used to construct the characteristic equation of the whole vibration system. According to the characteristic equation, the natural frequencies and mode shapes of cracked non-uniform continuous Timoshenko beam are obtained by using the half-interval method and Runge-Kutta method. By this method the influence of crack on the transverse vibration mode shapes and the rotational mode shapes is discussed, and the effects of the parameters of crack and spring-mass system on the non-uniform continuous Timoshenko beam are discussed. The results of the discussion have certain reference value for the design and crack diagnosis of this type structure.

2. Differential equation of a non-uniform Timoshenko beam

The motion equations of a non-uniform Timoshenko beam can be expressed as

$$\frac{\partial M(x,t)}{\partial x} = Q(x) - \rho I(x)\frac{\partial \varphi^2(x,t)}{\partial t^2} \tag{1}$$

$$\frac{\partial Q(x,t)}{\partial x} = \rho A(x)\frac{\partial Y^2(x,t)}{\partial t^2} \tag{2}$$

where $M(x,t)$, $Q(x,t)$ are the bending moment and shear force at axial coordinate x and time t, respectively; ρ is density of material; $I(x)$, $A(x)$ are the moment of inertia and area of the cross-sectional area at axial coordinate x, respectively; $\varphi(x,t)$, $Y(x,t)$ are the rotation due to pure bending and transverse deflection at axial coordinate x and time t, respectively.

Based on Timoshenko beam theory, it has

$$M(x,t) = -EI(x)\frac{\partial \varphi(x,t)}{\partial x} \tag{3}$$

$$Q(x,t) = -kGA(x)\left[\varphi(x,t) - \frac{\partial Y(x,t)}{\partial x}\right] \tag{4}$$

where E is Young's modulus, G is shear modulus, k is a constant related to the shape of the cross section, for a rectangular cross section, k can be taken as 5/6.

Assuming the whole vibration system performs a harmonic free vibration, it has

$$Y(x,t) = \bar{Y}(x)e^{j\omega t} \tag{5a}$$

$$\varphi(x,t) = \bar{\varphi}(x)e^{j\omega t} \tag{5b}$$

$$M(x,t) = \bar{M}(x)e^{j\omega t} \tag{5c}$$

$$Q(x,t) = \bar{Q}(x)e^{j\omega t} \tag{5d}$$

where ω is the natural circular frequency of the Timoshenko beam and $j = \sqrt{-1}$.

The substitution of Eqs. (5a)-(5d) into Eqs. (1)-(4) gives

$$\frac{d\bar{M}(x)}{dx} = \bar{Q}(x) + \omega^2 \rho I(x)\bar{\varphi}(x) \tag{6}$$

$$\frac{d\bar{Q}(x)}{dx} = -\omega^2 \rho A(x)\bar{Y}(x) \tag{7}$$

$$\frac{d\bar{\varphi}(x)}{dx} = -\frac{\bar{M}(x)}{EI(x)} \tag{8}$$

$$\frac{d\bar{Y}(x)}{dx} = \bar{\varphi}(x) + \frac{Q(x)}{kGA(x)} \tag{9}$$

Let $\{Z(x)\} = \left[\bar{Y}(x) \quad \bar{\varphi}(x) \quad \bar{M}(x) \quad \bar{Q}(x)\right]^T$, Eqs. (6)-(9) can be written as

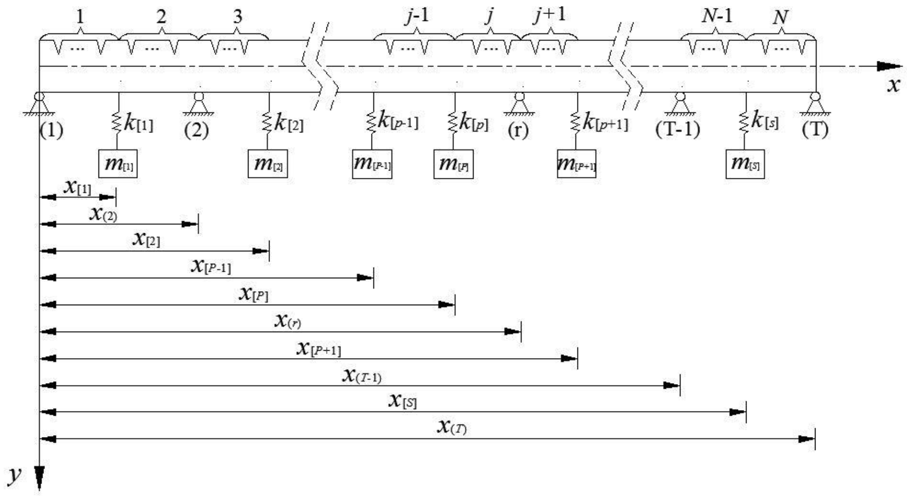


Fig. 1 Sketch of a continuous Timoshenko beam carried by *S* spring-mass systems and *T* pinned supports with any number of cracks

$$\frac{d\{Z(x)\}}{dx} = [U(x)]\{Z(x)\} \tag{10}$$

The specific form of the coefficient matrix $[U(x)]$ is

$$[U(x)] = \begin{bmatrix} 0 & 1 & 0 & 1/kGA(x) \\ 0 & 0 & -1/EI(x) & 0 \\ 0 & \omega^2\rho I(x) & 0 & 1 \\ -\omega^2\rho A(x) & 0 & 0 & 0 \end{bmatrix} \tag{11}$$

By using $\{Z(0)\}$, the vector of undetermined coefficients at the left boundary of beam can be expressed as $\{Z(0)\} = [\overline{Y}(0) \quad \overline{\varphi}(0) \quad \overline{M}(0) \quad \overline{Q}(0)]^T$.

By using the transfer matrix method, it has

$$\{Z(x)\} = [T(x)]\{Z(0)\} \tag{12}$$

where $[T(x)]$ is the transfer matrix, it shows the transfer relationship of undetermined coefficients between any position of the beam and the left boundary.

The substitution of Eq. (12) into Eq. (10) gives

$$\frac{d[T(x)]}{dx} = [U(x)][T(x)] \tag{13}$$

For Eq. (13), the transfer matrix $[T(x)]$ in any position also can be obtained by using Runge-Kutta method (Kahaner *et al*. 1989, Dormand and Prince 1980), which is derived and given in Appendix A. As can be seen from Eq. (12), the initial value of Runge-Kutta method is an identity matrix.

3. Solutions of natural frequency and mode shape

The sketch of a continuous Timoshenko beam with an arbitrary number of cracks, *S* spring-mass systems and *T* pinned supports which has variable cross-sections is shown in Fig. 1. The continuous beam is divided into $N=T+S-1$ segments by *T* pinned supports and *S* spring-mass systems. For convenience, two kinds of coordinates are unified as one shown in Fig. 1. The positions of the pinned support and spring-mass system are defined by $x_{(r)}$ ($(r)=1{\sim}T$), $x_{[p]}$ ($[p]=1{\sim}S$) , respectively. The symbols of 1, 2, 3,$\cdots$, j-1, j, j+1$\cdots$, N-1, N above the x-axis refer to the numbering of segments, while the symbols of (1), (2),$\cdots$, (r),$\cdots$, (T-1), (T) and those of [1], [2],$\cdots$, [p-1], [p], [p+1],$\cdots$, [S] below the x-axis refer to the numbering of pinned supports and spring-mass systems, respectively. It should be noted that the numbering of pinned supports and spring-mass systems are enclosed in parentheses () and [], respectively. The numbering of segments is without any parentheses. There is an association among the numbering of segments, pinned supports and spring-mass systems. As seen in Fig. 1, the number of the jth segment is $j=[p]+(r-1)$.

Taking the jth segment as an example to illustrate the transfer relationship of undetermined coefficients in the same segment, assuming that the jth segment has r cracks and it is divided into (r+1) sub-segment by the cracks in Fig. 2, the symbols of $b_{s_j}^1$, $b_{s_j}^2$, $\cdots$, $b_{s_j}^{e-1}$, $b_{s_j}^e$, $b_{s_j}^{e+1}$, $\cdots$, $b_{s_j}^{r+1}$ refer to the numbering of (r+1) sub-segments of the jth segment, and their lengths are $l_{s_j}^1$, $l_{s_j}^2$, $\cdots$, $l_{s_j}^{e-1}$, $l_{s_j}^e$, $l_{s_j}^{e+1}$, $\cdots$, $l_{s_j}^{r+1}$.

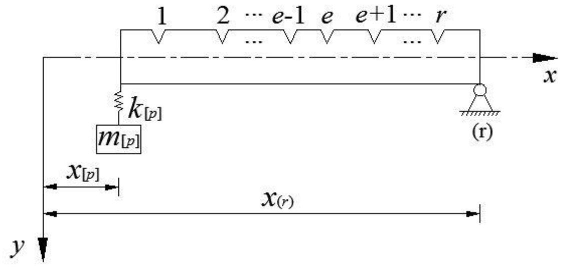


Fig. 2 Equivalent model of the jth segment

$\xi_{s_j}^e = x - \left(x_{[p]} + \sum_{k=1}^{e-1} l_{s_j}^k \right)$, for sub-segment $b_{s_j}^e$, $\xi_{s_j}^e \in [0, l_{s_j}^e]$. Assuming the natural circular frequency of the cracked continuous Timoshenko beam carrying spring-mass systems is $\overline{\omega}$, according to Eq. (12), the undetermined coefficient of the transfer relationship in sub-segment b_{sj}^e can be obtained

$$\left\{Z(\xi_{s_j}^e)\right\} = [T(\xi_{s_j}^e)]\{Z(\xi_{s_j}^e = 0)\} \tag{14}$$

where $[T(\xi_{s_j}^e)]$ is identical to the meaning of $[T(x)]$ in Eq. (12), it is determined by replacing ω of $[U(x)]$ with $\overline{\omega}$ in Eq. (13). $\xi_{s_j}^e = l_{s_j}^e$, Eq. (14) shows the transfer relationship of undetermined coefficient between the left side and right side. It can be written as

$$\{Z\}_{s_j}^{e,R} = [T]_{s_j}^e \{Z\}_{s_j}^{e,L} \tag{15}$$

where $\{Z\}_{s_j}^{e,L}$, $\{Z\}_{s_j}^{e,R}$ represent the undetermined coefficients of the left side ($\xi_{s_j}^e = 0$) and right side ($\xi_{s_j}^e = l_{s_j}^e$) in sub-segment $b_{s_j}^e$, respectively.

$$\{Z\}_{s_j}^{e,L} = \left[\overline{Y}_{s_j}^{e,L} \quad \overline{\varphi}_{s_j}^{e,L} \quad \overline{M}_{s_j}^{e,L} \quad \overline{Q}_{s_j}^{e,L} \right]^T,$$
$$\{Z\}_{s_j}^{e,R} = \left[\overline{Y}_{s_j}^{e,R} \quad \overline{\varphi}_{s_j}^{e,R} \quad \overline{M}_{s_j}^{e,R} \quad \overline{Q}_{s_j}^{e,R} \right]^T.$$

$\overline{Y}_{s_j}^{e,L}$, $\overline{\varphi}_{s_j}^{e,L}$, $\overline{M}_{s_j}^{e,L}$, $\overline{Q}_{s_j}^{e,L}$ refer to the transverse deflection, rotation due to bending moment and shear force of the left side in sub-segment b_{sj}^e, respectively. $\overline{Y}_{s_j}^{e,R}$, $\overline{\varphi}_{s_j}^{e,R}$, $\overline{M}_{s_j}^{e,R}$, $\overline{Q}_{s_j}^{e,R}$ refer to the transverse deflection, rotation due to bending moment and shear force of the right side in sub-segment $b_{s_j}^e$, respectively.

And $\{Z\}_{s_j}^{e,L} = \left\{Z(\xi_{s_j}^e = 0)\right\}$, $\{Z\}_{s_j}^{e,R} = \left\{Z(\xi_{s_j}^e = l_{s_j}^e)\right\}$, $[T]_{s_j}^e = [T(\xi_{s_j}^e = l_{s_j}^e)]$.

A model of massless extensional spring and rotational spring is adopted to describe the local flexibility induced by cracks in this paper. In the eth crack location of the jth segment, there is the discontinuities of the transverse deflection and rotational angle due to bending.

$$\overline{Y}_{s_j}^{e+1,L} - \overline{Y}_{s_j}^{e,R} = SC_{s_j}^e \overline{Q}_{s_j}^{e,R} \tag{16}$$

$$\overline{\varphi}_{s_j}^{e+1,L} - \overline{\varphi}_{s_j}^{e,R} = MC_{s_j}^e \overline{M}_{s_j}^{e,R} \tag{17}$$

where $SC_{s_j}^e$ is the local flexibility constant of the extensional spring in the eth crack location of the jth segment, $MC_{s_j}^e$ is the local flexibility constant of the rotational spring in the eth crack location of the jth segment.

$$SC_{s_j}^e = \frac{h(\xi_{s_j}^e = l_{s_j}^e)}{EA(\xi_{s_j}^e = l_{s_j}^e)} SF_{s_j}^e(\alpha) \tag{18}$$

$$MC_{s_j}^e = \frac{h(\xi_{s_j}^e = l_{s_j}^e)}{EI(\xi_{s_j}^e = l_{s_j}^e)} MF_{s_j}^e(\alpha) \tag{19}$$

where h is the height of the beam, $SF_{s_j}^e$ and $MF_{s_j}^e$ are functions depending on the relative crack deep $\alpha_{s_j}^e = a_{s_j}^e / h(\xi_{s_j}^e = l_{s_j}^e)$ ($a_{s_j}^e$ is the absolute depth of the eth crack in jth segment) and the cross-section geometry. For a rectangular-sectional beam, $SF_{s_j}^e(\alpha)$ can be expressed as (Valiente *et al*. 1990)

$$SF_{s_j}^e(\alpha) = \left(\frac{\alpha_{s_j}^e}{1-\alpha_{s_j}^e} \right)^2 (-0.22 + 3.82\alpha_{s_j}^e + 1.54\left(\alpha_{s_j}^e\right)^2 - 14.64\left(\alpha_{s_j}^e\right)^3 + 9.60\left(\alpha_{s_j}^e\right)^4) \tag{20}$$

And the $MF_{s_j}^e(\alpha)$ can be written as (Tada *et al*. 1985)

$$MF_{s_j}^e(\alpha) = 2\left(\frac{\alpha_{s_j}^e}{1-\alpha_{s_j}^e} \right)^2 (5.93 - 19.69\alpha_{s_j}^e + 37.14\left(\alpha_{s_j}^e\right)^2 - 35.84\left(\alpha_{s_j}^e\right)^3 + 13.12\left(\alpha_{s_j}^e\right)^4) \tag{21}$$

Using the equilibrium conditions of moment and shear in the eth crack location of the jth segment, it has

$$\overline{M}_{s_j}^{e+1,L} = \overline{M}_{s_j}^{e,R} \tag{22}$$

$$\overline{Q}_{s_j}^{e+1,L} = \overline{Q}_{s_j}^{e,R} \tag{23}$$

Eqs. (16), (17), (22), (23) are written as a matrix, it has

$$\{Z\}_{s_j}^{e+1,L} = [G]_{s_j}^{e,R} \{Z\}_{s_j}^{e,R} \tag{24}$$

where $[G]_{s_j}^{e,R}$ is the transfer matrix of the undetermined coefficient between the left side and right side at the eth crack location of the jth segment.

$$[G]_{s_j}^{e,R} = \begin{bmatrix} 1 & 0 & 0 & -SC_{s_j}^e \\ 0 & 1 & -MC_{s_j}^e & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

According to Eqs. (15) and (24), it can obtain the transfer relationship of undetermined coefficients between the right side of the last sub-segment (sub-segment $b_{s_j}^{r+1}$) and the left side of the first sub-segment (sub-segment $b_{s_j}^1$) in the jth segment by using the recursive method.

$$\left\{Z_{s_j}^{r+1,R}\right\} = [TG]_{s_j} \{Z\}_{s_j}^{1,L} \tag{26}$$

where

$$[TG]_{s_j} = [T]_{s_j}^{r+1} [G]_{s_j}^r \cdots [T]_{s_j}^2 [G]_{s_j}^1 [T]_{s_j}^1 \tag{27}$$

For convenience, the undetermined coefficient vector of the last sub-segment right side in the *j*th segment is written as $\{Z\}_{s_j}^{last}$, the undetermined coefficient vector of the first sub-segment left side in the *j*th segment is written as $\{Z\}_{s_j}^{first}$, it has

$$\begin{aligned}\{Z\}_{s_j}^{last} &= \left[\bar{Y}_{s_j}^{last} \quad \bar{\varphi}_{s_j}^{last} \quad \bar{M}_{s_j}^{last} \quad \bar{Q}_{s_j}^{last}\right]^T \\ &= \left[\bar{Y}_{s_j}^{r+1,R} \quad \bar{\varphi}_{s_j}^{r+1,R} \quad \bar{M}_{s_j}^{r+1,R} \quad \bar{Q}_{s_j}^{r+1,R}\right]^T\end{aligned} \tag{28a}$$

$$\begin{aligned}\{Z\}_{s_j}^{first} &= \left[\bar{Y}_{s_j}^{first} \quad \bar{\varphi}_{s_j}^{first} \quad \bar{M}_{s_j}^{first} \quad \bar{Q}_{s_j}^{first}\right]^T \\ &= \left[\bar{Y}_{s_j}^{1,L} \quad \bar{\varphi}_{s_j}^{1,L} \quad \bar{M}_{s_j}^{1,L} \quad \bar{Q}_{s_j}^{1,L}\right]^T\end{aligned} \tag{28b}$$

Eq. (26) also can be expressed as

$$\{Z\}_{s_j}^{last} = [TG]_{s_j}\{Z\}_{s_j}^{first} \tag{29}$$

The equation of motion of the *p*th spring-mass system is

$$m_{[p]}\ddot{v}_{[p]} + k_{[p]}(v_{[p]} - y_{[p]}) = 0 \tag{30}$$

where $v_{[P]}$ is the transverse deflection of the point-mass in the *p*th spring-mass system, $y_{[P]}$ is the transverse deflection of the segment with the *p*th spring-mass system.

The spring-mass system performs a simple harmonic free vibration in the equilibrium position, it has

$$v_{[p]} = V_{[p]}e^{j\bar{\omega}t} \tag{31}$$

where $V_{[P]}$ is the vibration amplitude of the $v_{[P]}$.

The substitution of Eqs. (5a) and (31) into Eq. (30), it has

$$\left(k_{[p]} - \bar{\omega}^2 m_{[p]}\right)V_{[p]} - k_{[p]}\bar{Y}_{[p]} = 0 \tag{32}$$

As seen in Fig. 1, the location of the spring-mass system is the right side of the (*j*-1)th segment, it is also the left side of the *j*th segment. Assuming that there is a crack at the location of the *p*th spring-mass system, the local flexibility constant of the extensional spring at the location of the crack is written as $SC_{[p]}$, the local flexibility constant of rotational spring is written as $MC_{[p]}$. There are the discontinuities of the transverse deflection located at the spring-mass system due to the crack. In this case, the average of transverse deflection of the left and right sides is used instead of the transverse deflection of this position. It has

$$\bar{Y}_{[p]} = \frac{1}{2}\left(\bar{Y}_{s_{j-1}}^{last} + \bar{Y}_{s_j}^{first}\right) \tag{33}$$

The substitution of Eq. (33) into Eq. (32), it has

$$\left(k_{[p]} - \bar{\omega}^2 m_{[p]}\right)V_{[p]} - \frac{k_{[p]}}{2}\left(\bar{Y}_{s_{j-1}}^{last} + \bar{Y}_{s_j}^{first}\right) = 0 \tag{34}$$

According to the discontinuous condition of the transverse deflection and rotation due to the crack at the position of the *p*th spring-mass system, it has

$$\bar{Y}_{s_j}^{first} - \bar{Y}_{s_{j-1}}^{last} = SC_{[p]}\bar{Q}_{s_{j-1}}^{last} \tag{35}$$

$$\bar{\varphi}_{s_j}^{first} - \vec{\varphi}_{s_{j-1}}^{last} = MC_{[p]}\bar{M}_{s_{j-1}}^{last} \tag{36}$$

If no cracks exist at the position of the *p*th spring-mass system, we can think that the crack depth is equal to zero, it has

$SC_{[P]} = 0$, $MC_{[P]} = 0$.

Using the equilibrium conditions of moment and shear at the location of the *p*th spring-mass system, it has

$$\bar{M}_{s_j}^{first} = \bar{M}_{s_{j-1}}^{last} \tag{37}$$

$$\bar{Q}_{s_{j-1}}^{last} - \bar{\omega}^2 V_{[p]}m_{[p]} = \bar{Q}_{s_j}^{first} \tag{38}$$

Eqs. (34)-(38) are written as matrix form, it has

$$[H_{[P]}]\{u_{[P]}\} = 0 \tag{39}$$

where

$$[H_{[p]}] = \begin{bmatrix} -1 & 0 & 0 & -SC_{[p]} & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -MC_{[p]} & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & -\bar{\omega}^2 m_{[p]} & 0 & 0 & 0 & -1 \\ -\frac{k_{[p]}}{2} & 0 & 0 & 0 & k_{[p]} - \bar{\omega}^2 m_{[p]} & -\frac{k_{[p]}}{2} & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

$$\{u_{[p]}\} = \begin{Bmatrix} \{Z\}_{s_{j-1}}^{last} \\ V_{[p]} \\ \{Z\}_{s_j}^{first} \end{Bmatrix} = \left[\bar{Y}_{s_{j-1}}^{last} \quad \bar{\varphi}_{s_{j-1}}^{last} \quad \bar{M}_{s_{j-1}}^{last} \quad \bar{Q}_{s_{j-1}}^{last} \quad V_{[p]} \quad \bar{Y}_{s_j}^{first} \quad \bar{\varphi}_{s_j}^{first} \quad \bar{M}_{s_j}^{first} \quad \bar{Q}_{s_j}^{first}\right]^T$$

Substituting Eq. (29) into Eq. (39), it has

$$[\bar{H}_{[p]}]\{\bar{u}_{[p]}\} = 0 \tag{41}$$

$$[\bar{H}_{[p]}] = [H_{[p]}]\begin{bmatrix} [TG]_{sj-1} & [0] & 0 \\ [0] & diag[I] & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{42}$$

where [0] is zero element matrix of 4×4, *diag*[*I*] is Diagonal matrix of 4×4 with all the diagonal elements are 1.

$[\bar{H}_{[p]}]$ is the matrix of 5×9, it has

$$[\bar{H}_{[p]}] = \begin{array}{c} \begin{matrix} n+1 & n+2 & n+3 & n+4 & n+5 \end{matrix} \\ \begin{bmatrix} \bar{h}_{11}^{[p]} & \bar{h}_{12}^{[p]} & \bar{h}_{13}^{[p]} & \bar{h}_{14}^{[p]} & \bar{h}_{15}^{[p]} \\ \bar{h}_{21}^{[p]} & \bar{h}_{22}^{[p]} & \bar{h}_{23}^{[p]} & \bar{h}_{24}^{[p]} & \bar{h}_{25}^{[p]} \\ \bar{h}_{31}^{[p]} & \bar{h}_{32}^{[p]} & \bar{h}_{33}^{[p]} & \bar{h}_{34}^{[p]} & \bar{h}_{35}^{[p]} \\ \bar{h}_{41}^{[p]} & \bar{h}_{42}^{[p]} & \bar{h}_{43}^{[p]} & \bar{h}_{44}^{[p]} & \bar{h}_{45}^{[p]} \\ \bar{h}_{51}^{[p]} & \bar{h}_{52}^{[p]} & \bar{h}_{53}^{[p]} & \bar{h}_{54}^{[p]} & \bar{h}_{55}^{[p]} \end{bmatrix} \end{array} \tag{43}$$

$$\begin{array}{cccc|c} n+6 & n+7 & n+8 & n+9 & \\ \bar{h}_{16}^{[p]} & \bar{h}_{17}^{[p]} & \bar{h}_{18}^{[p]} & \bar{h}_{19}^{[p]} & m+1 \\ \bar{h}_{26}^{[p]} & \bar{h}_{27}^{[p]} & \bar{h}_{28}^{[p]} & \bar{h}_{29}^{[p]} & m+2 \\ \bar{h}_{36}^{[p]} & \bar{h}_{37}^{[p]} & \bar{h}_{38}^{[p]} & \bar{h}_{39}^{[p]} & m+3 \\ \bar{h}_{46}^{[p]} & \bar{h}_{47}^{[p]} & \bar{h}_{48}^{[p]} & \bar{h}_{49}^{[p]} & m+4 \\ \bar{h}_{56}^{[p]} & \bar{h}_{57}^{[p]} & \bar{h}_{58}^{[p]} & \bar{h}_{59}^{[p]} & m+5 \end{array}$$

where,

$$\left.\begin{array}{ll} m = 5\{[p]-1\}+4\{(r-1)-1\}+2 & (r-1)\ge 2 \\ m = 5\{[p]-1\}+2 & (r-1)=1 \\ n = 5\{[p]-1\}+4\{(r-1)-1\} & \end{array}\right\} \tag{44}$$

where the symbol of [p] refers to the numbering of spring-mass system, the symbol of (r-1) refers to the numbering of pinned point which is closest to the pth spring-mass system in the range from the left side of continuous beam to the pth spring-mass system. As seen in Fig .1, the rth pinned support is located at the right side of the jth segment and it is also located at the left side of the (j+1)th segment. Assuming that there is a crack at that location, the local flexibility constant of rotational spring is written as MC_r. If no cracks exist at the location, we can think that the crack depth is equal to zero, it has MC_r=0.

At the location of the rth pinned support, the transverse deflection is equal to zero, it has

$$\bar{Y}_{s_j}^{last} = 0 \tag{45}$$

$$\bar{Y}_{s_{j+1}}^{first} = 0 \tag{46}$$

The rotational angle is discontinuous caused by a crack, it has

$$\bar{\varphi}_{s_{j+1}}^{first} - \bar{\varphi}_{s_j}^{last} = MC_{(r)}\bar{M}_{s_j}^{last} \tag{47}$$

Using the equilibrium conditions of moment in the rth pinned support location, it has

$$\bar{M}_{s_{j+1}}^{first} = \bar{M}_{s_j}^{last} \tag{48}$$

Eqs. (45)-(48) are written as matrix form, it has

$$[H_{(r)}]\{u_{(r)}\} = 0 \tag{49}$$

where

$$[H_{(r)}] = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & -1 & -MC_{(r)} & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{50}$$

$$\{u_{(r)}\} = \begin{Bmatrix} \{Z\}_{s_j}^{last} \\ \{Z\}_{s_{j+1}}^{first} \end{Bmatrix} = \left[\bar{Y}_{s_j}^{last} \quad \bar{\varphi}_{s_j}^{last} \quad \bar{M}_{s_j}^{last} \quad \bar{Q}_{s_j}^{last} \quad \bar{Y}_{s_{j+1}}^{first} \quad \bar{\varphi}_{s_{j+1}}^{first} \quad \bar{M}_{s_{j+1}}^{first} \quad \bar{Q}_{s_{j+1}}^{first}\right]^T$$

Substituting Eq. (29) into Eq. (49), it has

$$[\bar{H}_{(r)}]\{\bar{u}_{(r)}\} = 0 \tag{51}$$

where

$$[\bar{H}_{(r)}] = [H_{(r)}]\begin{bmatrix} [TG]_{sj} & [0] \\ [0] & diag[I] \end{bmatrix} \tag{52}$$

$$\{\bar{u}_{(r)}\} = \left[\bar{Y}_{s_j}^{first} \quad \bar{\varphi}_{s_j}^{first} \quad \bar{M}_{s_j}^{first} \quad \bar{Q}_{s_j}^{first} \quad \bar{Y}_{s_{j+1}}^{first} \quad \bar{\varphi}_{s_{j+1}}^{first} \quad \bar{M}_{s_{j+1}}^{first} \quad \bar{Q}_{s_{j+1}}^{first}\right]^T$$

$[\bar{H}_{(r)}]$ is the matrix of 4×8, it has

$$[\bar{H}_{(r)}] = \begin{array}{c} \begin{array}{cccc} n'+1 & n'+2 & n'+3 & n'+4 \end{array} \\ \begin{bmatrix} \bar{h}_{11}^{r} & \bar{h}_{12}^{r} & \bar{h}_{13}^{r} & \bar{h}_{14}^{r} \\ \bar{h}_{21}^{r} & \bar{h}_{22}^{r} & \bar{h}_{23}^{r} & \bar{h}_{24}^{r} \\ \bar{h}_{31}^{r} & \bar{h}_{32}^{r} & \bar{h}_{33}^{r} & \bar{h}_{34}^{r} \\ \bar{h}_{41}^{r} & \bar{h}_{42}^{r} & \bar{h}_{43}^{r} & \bar{h}_{44}^{r} \end{bmatrix} \end{array}$$

$$\begin{array}{cccc|c} n'+5 & n'+6 & n'+7 & n'+8 & \\ \bar{h}_{15}^{r} & \bar{h}_{16}^{r} & \bar{h}_{17}^{r} & \bar{h}_{18}^{r} & m'+1 \\ \bar{h}_{25}^{r} & \bar{h}_{26}^{r} & \bar{h}_{27}^{r} & \bar{h}_{28}^{r} & m'+2 \\ \bar{h}_{35}^{r} & \bar{h}_{36}^{r} & \bar{h}_{37}^{r} & \bar{h}_{38}^{r} & m'+3 \\ \bar{h}_{45}^{r} & \bar{h}_{46}^{r} & \bar{h}_{47}^{r} & \bar{h}_{48}^{r} & m'+4 \end{array} \tag{53}$$

where

$$\left.\begin{array}{ll} m' = 5[p]+4\{(r)-2\}+2, & (r)\ge 2 \\ m' = 5[p]+2, & (r)=1 \\ n' = 5[p]+4\{(r)-2\}, & (r)\ge 2 \\ n' = 5[p], & (r)=1 \end{array}\right\} \tag{54}$$

where the symbol of (r) refers to the numbering of pinned support, the symbol of [p] refers to the numbering of spring-mass system which is closest to the rth pinned support in the range from the left side of continuous beam to the rth pinned support as seen in Fig. 1.

According to the left boundary condition of the continuous beam, it has

$$\bar{Y}_{s_1}^{first} = 0 \tag{55}$$

$$\bar{M}_{s_1}^{first} = 0 \tag{56}$$

or

$$[\bar{H}_{(1)}]\{\bar{u}_{(1)}\} = 0 \tag{57}$$

where

$$[\bar{H}_{(1)}] = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{58}$$

$$\{\bar{u}_{(1)}\} = \left[\bar{Y}_{s_1}^{first} \quad \bar{\varphi}_{s_1}^{first} \quad \bar{M}_{s_1}^{first} \quad \bar{Q}_{s_1}^{first}\right]^T$$

According to the right boundary condition of the continuous beam, it has

$$\bar{Y}_{s_N}^{last} = 0 \tag{59}$$

$$\overline{M}_{s_N}^{last}=0 \tag{60}$$

or

$$[H_{(T)}]\{u_{(T)}\}=0 \tag{61}$$

where

$$\left[H_{(T)}\right]=\begin{bmatrix}1 & 0 & 0 & 0\\ 0 & 0 & 1 & 0\end{bmatrix} \tag{62}$$

$$\left\{u_{(T)}\right\}=\left[\overline{Y}_{s_N}^{last} \quad \overline{\varphi}_{s_N}^{last} \quad \overline{M}_{s_N}^{last} \quad \overline{Q}_{s_N}^{last}\right]^T$$

Substituting Eq. (29) into Eq. (61), it has

$$[\overline{H}_{(T)}]\{\overline{u}_{(T)}\}=0 \tag{63}$$

where

$$[\overline{H}_{(T)}]=[H_{(T)}][TG]_{s_N} \tag{64}$$

$$\left\{\overline{u}_{(T)}\right\}=\left[\overline{Y}_{s_N}^{first} \quad \overline{\varphi}_{s_N}^{first} \quad \overline{M}_{s_N}^{first} \quad \overline{Q}_{s_N}^{first}\right]^T$$

$[\overline{H}_{(T)}]$ is the matrix of 2×4, it has

$$\begin{array}{ccccc} & \overline{n}+1 & \overline{n}+2 & \overline{n}+3 & \overline{n}+4 & \\ \left[\overline{H}_{(T)}\right]= & \left[\begin{matrix}\overline{h}_{11}^{(T)}\\ \overline{h}_{21}^{(T)}\end{matrix}\right. & \begin{matrix}\overline{h}_{12}^{(T)}\\ \overline{h}_{22}^{(T)}\end{matrix} & \begin{matrix}\overline{h}_{13}^{(T)}\\ \overline{h}_{23}^{(T)}\end{matrix} & \left.\begin{matrix}\overline{h}_{14}^{(T)}\\ \overline{h}_{24}^{(T)}\end{matrix}\right] & \begin{matrix}\overline{m}+1\\ \overline{m}+2\end{matrix}\end{array} \tag{65}$$

where

$$\left.\begin{aligned}\overline{m}&=5[S]+4\{(T)-2\}+2\\ \overline{n}&=5[S]+4\{(T)-2\}\end{aligned}\right\} \tag{66}$$

For a cracked continuous Timoshenko beam with S spring-mass systems and T pinned supports, it has N (N=T+S-1) segments. There are four undetermined coefficients for each segment, and there are 4N undetermined coefficients for all N segments, which are expressed as $\overline{Y}_{s_1}^{first}$, $\overline{\varphi}_{s_1}^{first}$, $\overline{M}_{s_1}^{first}$, $\overline{Q}_{s_1}^{first}$, $\cdots$, $\overline{Y}_{s_j}^{first}$, $\overline{\varphi}_{s_j}^{first}$, $\overline{M}_{s_j}^{first}$, $\overline{Q}_{s_j}^{first}$, $\cdots$, $\overline{Y}_{s_N}^{first}$, $\overline{\varphi}_{s_N}^{first}$, $\overline{M}_{s_N}^{first}$, $\overline{Q}_{s_N}^{first}$, respectively. There is one undetermined coefficient for each spring-mass system, and there are S undetermined coefficients for all S spring-mass systems, which are expressed as $V_{[1]}$, $\cdots$, $V_{[p]}$, $\cdots$, $V_{[s]}$. Therefore, the total vibration system has 4N+S=5S+4T-4 undetermined coefficients. Each spring-mass system gives five equations and S spring-mass systems give 5S equations. Each intermediate support has four equations and the total number of equations about all the intermediate support is 4[(r)-2]. Besides, the right boundary conditions and the left boundary conditions contain two equations, respectively. Then, total number of equations is 5S+4T-4.

The associated coefficient matrices are given by $[\overline{H}_{[P]}]$, $[\overline{H}_{[r]}]$, $[\overline{H}_{[1]}]$, $[\overline{H}_{[T]}]$ from Eqs. (41), (51), (57) and (63). And each element's identification number of coefficient matrix has been in top side and right side of each matrix. Therefore, using the numerical assembly technique as done by the conventional finite element method one may obtain a matrix equation for all undetermined coefficients of the entire beam.

$$[H]\{u\}=0 \tag{67}$$

Non-trivial solution of Eq. (67) requires that

$$|H|=0 \tag{68}$$

The natural frequencies and mode shapes of a cracked non-uniform continuous Timoshenko beam carrying arbitrary number spring-mass systems are obtained from overall matrix by the combination with half-interval method and Runge-Kutta method. The calculation steps are as follows:

(1) Given an initial value Ω_0 of the circular frequency, it is required that the initial circular frequency is less than the 1st natural circular frequency of the cracked non-uniform continuous Timoshenko beam.

(2) Substituting Ω_0 into Eq. (27), the transfer matrix at the right end of each segment is obtained by the Runge-Kutta method. Form a matrix $[H(\Omega_0)]$, and calculate the determinant value, let $D_0=|H(\Omega_0)|$, $\Omega_1=\Omega_0+\Delta\Omega$, in which $\Delta\Omega$ is the increment of Ω_0. $D_1=|H(\Omega_1)|$ can be obtained by repeating the same calculation. If D_0 and D_1 are the opposite sign. The 1st natural circular frequency of non-uniform Timoshenko beam is in the interval (Ω_0, Ω_1). If D_0 and D_1 are the same sign, then let $\Omega_0=\Omega_1$, $\Omega_1=\Omega_0+\Delta\Omega$. An interval of the 1st natural circular frequency of non-uniform Timoshenko beam can be determined by repeating the calculation process of step 2. The increment should be small enough in order to ensure that there only have the 1st natural circular frequency in an interval (e.g., $\Delta\Omega$=0.5).

(3) Ω_1 in the interval of the 1st natural circular frequency is considered as the initial value of natural circular frequency. According to Step2, an interval of the 2nd natural circular frequency of non-uniform Timoshenko beam also can be obtained. The interval of the lower natural circular frequencies of non-uniform Timoshenko beam can be obtained by using similar calculation procedure, respectively.

(4) According to an interval of a certain natural circular frequency, the accurate value of the natural circular frequency of the non-uniform Timoshenko beam can be obtained by using half-interval method. Substituting the accurate value of the natural circular frequency into Eq. (27), the transfer matrix at the right end of each segment can be obtained by using Runge-Kutta method. The matrix $[H]$ is reformed. The vector of the undetermined coefficient $\{u\}$is obtained by using Eq. (67), and the modal shapes can be obtained according to the vector of the undetermined coefficient.

4. Numerical results and discussions

Unless particularly mentioned, all the numerical results of this paper are obtained based on the non-uniform continuous Timoshenko beam with the following given data: rectangular cross section with uniform width of 0.3m and linearly or parabolically varying height, Young's modulus E=3.25×10^{10}Pa, mass density ρ=2500kg/m^3, Poisson's ratio ν=0.3.

Table 1 The first three order natural frequencies of the wedge beam

Boundary conditions	Crack depth a (mm)	Frequency order	Proposed method (rad/s) ω_1	Torabi's method (rad/s) ω_2	$\frac{\omega_1-\omega_2}{\omega_1}\times 100\%$
Clamped-clamped	5	1st	71200.7051	71131.9396	0.0966
		2nd	142914.2303	142841.9323	0.0506
		3rd	228537.4451	228406.3484	0.0574
	10	1st	68381.3086	67820.7010	0.8198
		2nd	131261.7188	130797.0663	0.3540
		3rd	202773.4375	202086.0856	0.3390
cantilever	5	1st	23347.2600	23342.0330	0.0224
		2nd	80380.6534	80324.2396	0.0702
		3rd	159633.4953	159523.7890	0.0687
	10	1st	20210.3252	20219.2900	-0.0444
		2nd	71105.7510	70748.6654	0.5022
		3rd	138399.5781	137639.4550	0.5492

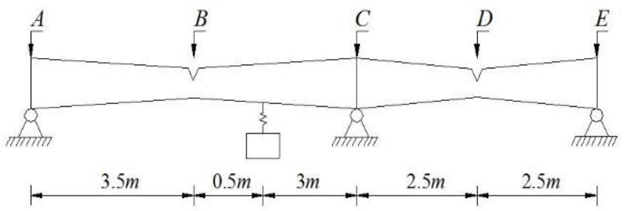


Fig. 3 A two-span pinned-pinned Timoshenko beam with two cracks and one spring-mass system

4.1 Reliability of the proposed method

4.1.1 Cracked wedge Timoshenko beam without spring-mass system

Torabi *et al*. (2014) obtained the natural frequencies of cracked wedge Timoshenko beam without spring mass systems using the differential quadrature element method (DQEM). Material properties of the wedge beam are: Young's modulus E=210 GPa, material mass density ρ=7860 kg/m^3 and Poisson ratio v=0.3. Geometric parameters of the wedge beam are left side height, h_0=50mm, right side height, h_1=25 mm, thickness b=10 mm and length, L=100 mm.

Torabi considered two cases of the crack depth (a=5 mm, 10 mm), three equal depth cracks at different positions (x_1=25 mm, x_2=50 mm, x_3=75 mm), and two kind of boundary conditions (clamped-clamped and cantilever) in the process of calculating the natural frequencies.

For the convenience of comparative analysis, the first three order natural frequencies of the wedge beam are calculated by the proposed method under the condition of the same crack location, crack depth and boundary conditions (Table 1). As can been seen from Table 1, the results obtained by proposed method in this paper are in good agreement with the results obtained by the Torabi's method, which reveals the validity and reliability of the method presented in this paper

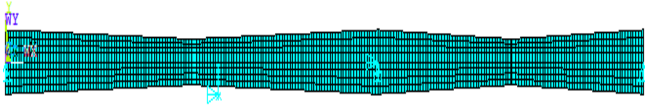
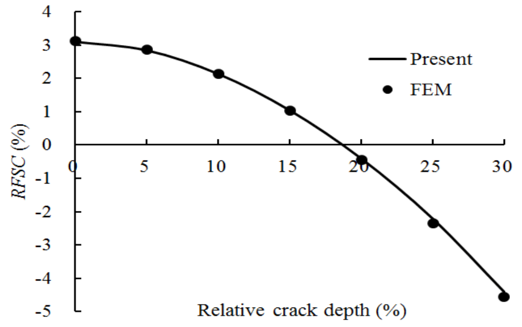
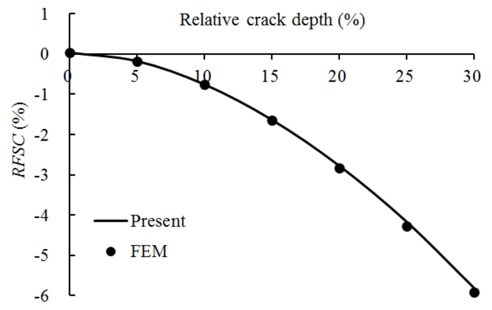


Fig. 4 FEM model of a two-span pinned-pinned Timoshenko beam with two cracks and one spring-mass system



(a) *RFSC* for the first natural frequency



(b) *RFSC* for the second natural frequency

Fig. 5 Change ratio for natural frequency with respect to relative crack depth

4.1.2 Two span Timoshenko beam with two cracks and one spring-mass system

The second example (shown in Fig. 3) is a two-span continuous Timoshenko beam with a variable cross section. The height of the beam varies linearly and the height h_A=h_C=h_E=0.7m, h_B=h_D=0.5m. The beam with one spring-mass system and two cracks which have the same depth. For the parameters of spring-mass system, point mass is 1500kg and spring coefficient is 2×10^{6}N/m. The lowest two natural frequencies of the two-span continuous beam with respect to the different relative crack depths are calculated by the proposed method and FEM, respectively.

Finite element analysis is carried out by ANSYS software. A 2D-FEM model of the non-uniform Timoshenko beam is built with 8-node PLANE183 elements. Each element of PLANE183 has the length of 0.1m and the width of 0.1m. Therefore, there are 860 PLANE183 elements. Spring and mass are built by COMBINE14 and MASS21, respectively. The change ratio for natural frequency is defined by Eq. (69) and the results are presented in Fig. 5.

$$RFSC = \frac{\hat{\omega}-\omega_0}{\omega_0}\times 100\% \tag{69}$$

where $\hat{\omega}$ is the natural frequency of cracked beam with spring-mass systems, ω_0 is the natural frequency of intact Timoshenko beam without spring-mass systems.

As can be seen from Fig. 5, when relative crack depth is 0% (i.e., this is an intact beam), the lowest two natural frequencies of continuous Timoshenko beam increase by the influence of spring-mass system, in which the change of the first natural frequency is more significant than that of the second one. This is because the spring-mass system is located at different points with respect to the lowest two mode shapes. Then the lowest two natural frequencies of continuous Timoshenko beam decrease with relative crack depth increasing. It's also seen that the good agreement between the results of presented method and FEM is achieved and the validity and reliability of the proposed method is verified.

4.2 A three-span cracked continuous Timoshenko beam with variable cross section carrying one spring-mass system

A three-span cracked continuous Timoshenko beam carrying one spring-mass system is shown in Fig. 6. The height of the beam in each span varies parabolically and the height $h_A=h_C=h_E=h_G=0.7$ m, $h_B=h_D=h_F=0.5$ m. For the parameters of spring-mass system, point mass is 1500 kg and spring coefficient is 2×10^6 N/m. The relative depths of three cracks are 30%. Natural frequencies and associated mode shapes for four cases are calculated by the proposed method, i.e.,

Case I: Intact beam without spring-mass system;

Case II: Intact beam with one spring-mass system;

Case III: Cracked beam without spring-mass system;

Case IV: Cracked beam with one spring-mass system.

The first three natural frequencies for these cases are shown in Table 2 and the associated mode shapes are plotted in Fig. 7.

It can be seen from Table 2 that: (1) For Cases I and II, the first two natural frequencies have more variations, and the changes of 3rd natural frequencies are not obvious. This is because the spring-mass system is located at different

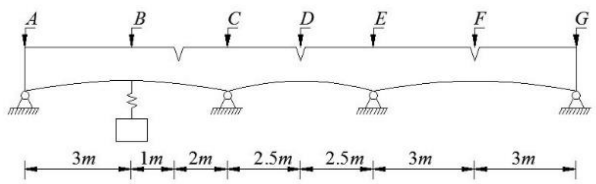
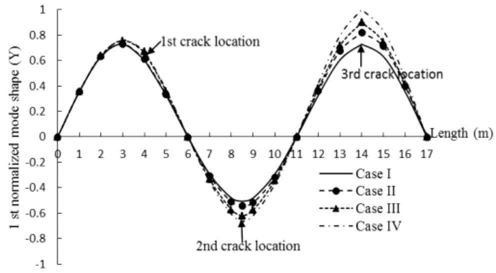


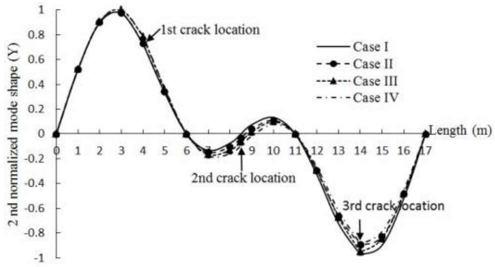
Fig. 6 A three-span cracked continuous Timoshenko beam with variable cross section carrying one spring-mass system

Table 2 The lowest three natural frequencies for four cases, respectively

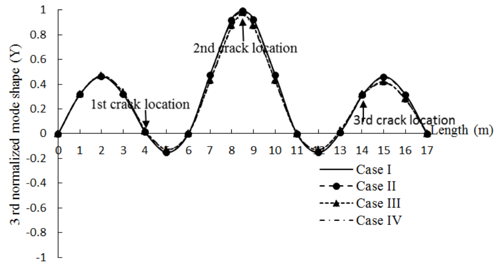
Cases	1st (rad/s)	2nd (rad/s)	3rd (rad/s)
Case I	161.5705	206.7160	382.7652
Case II	163.7750	209.0126	382.9788
Case III	150.4670	200.5637	367.3904
Case IV	152.4713	203.0942	367.6712



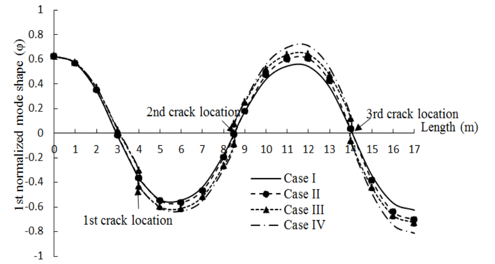
(a) 1st normalized mode shape (Y)



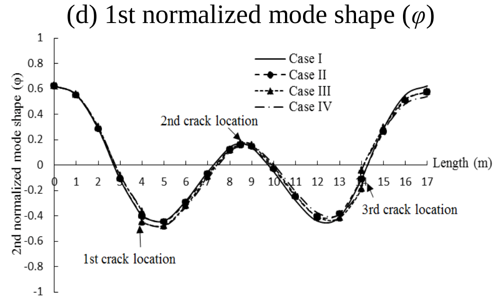
(b) 2nd normalized mode shape (Y)



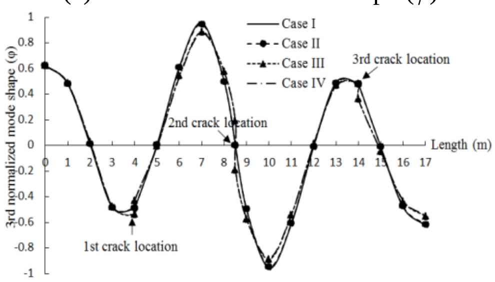
(c) 3rd normalized mode shape (Y)



(d) 1st normalized mode shape (φ)



(e) 2nd normalized mode shape (φ)



(f) 3rd normalized mode shape (φ)

Fig. 7 The lowest three mode shapes for four cases points with respect to the lowest three mode shapes; (2) For comparison of Cases I and III, there are bigger differences for the lowest three natural frequencies. It lies that one crack occurs in each span and then cracks are located at peaks or troughs of the lowest three mode shapes all the time. (3) The variation between Cases I and IV is less than that between Cases I and III, because there are the effects of spring-mass systems and cracks on natural frequencies of continuous Timoshenko beam, in which spring-mass systems lead to the increase of natural frequencies of continuous Timoshenko beam, on the contrary, cracks make natural frequencies decrease.

In Fig. 7, it is seen that the lowest three mode shapes of continuous beam have changed due to spring-mass systems and cracks, in which the lowest three mode shapes of φ have more obvious changes than the lowest three mode shapes of Y. The slopes of mode shapes are not consistent at left and right sides of crack due to the jump of slope caused by crack.

4.3 Influences of parameters of spring-mass system and crack on natural frequencies of continuous Timoshenko beam with variable cross section

In order to demonstrate the effects of parameters of spring-mass systems and cracks on natural frequencies of continuous Timoshenko beam with variable cross section, a two-span continuous Timoshenko beam with variable cross-section studied is shown in Fig. 8. The height of the beam in each span varies parabolically and the height h_A=h_C=h_E=0.7 m, h_B=h_D=0.5 m. The parameters mainly refer to positions and natural frequencies (i.e., various combinations of point masses and spring constants) for spring-mass system, and positions and relative depth for crack, respectively.

For intact continuous Timoshenko beam without spring-mass system, the first natural frequency is 146.9718rad/s and the second one is 253.1410rad/s, the associated mode shapes are shown in Fig. 9.

4.3.1 Position of spring-mass system

Change ratio for natural frequency caused by spring-mass system is defined by Eq. (70).

$$RFS = \frac{\omega_s - \omega_0}{\omega_0} \times 100\% \tag{70}$$

where ω_s is natural frequency of intact Timoshenko beam with spring-mass system.

The position of spring-mass system changes from 0.5 m to 11.5 m distance from the left end of continuous Timoshenko beam with variable cross beam by a step of 0.5 m as shown in Fig. 10. RFS with respect to different positions of spring-mass system is calculated by the proposed method and the results are presented in Fig. 11.

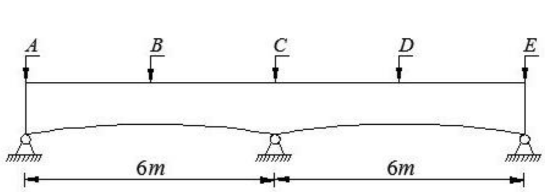
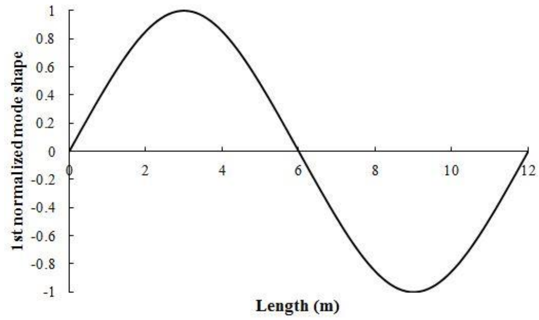
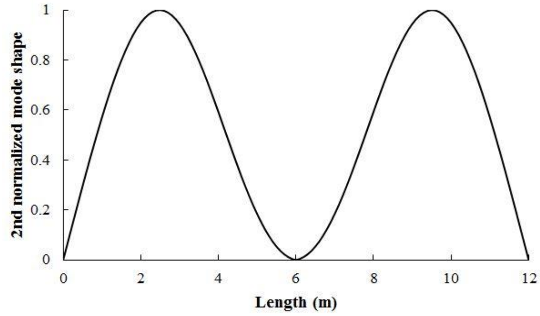


Fig. 8 A two-span continuous Timoshenko beam with variable cross section



(a) 1st normalized mode shape



(b) 2nd normalized mode shape

Fig. 9 The lowest two mode shapes for intact continuous Timoshenko beam without spring-mass system

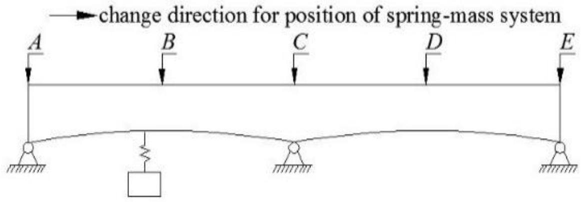


Fig. 10 Schematic diagram of changing position for spring-mass system

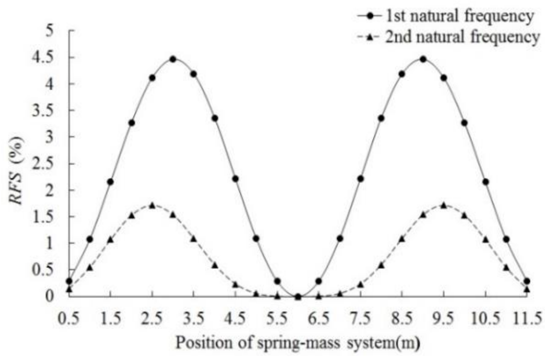


Fig. 11 Change ratios for natural frequency with respect to different positions of spring-mass system

As can be seen from Fig. 11, RFS reaches the largest value when spring-mass system is located at peak or trough of the associated mode shape, while the value of RFS is zero for spring-mass system located at zero point for this mode and it is evident that spring-mass system has little influence on the associated natural frequency.

Table 3 The effect of different parameters of spring-mass system on natural frequency of continuous Timoshenko beam

Systems	Spring-mass system			Continuous beam			
	Parameter		Natural frequency (rad/s)	1st modal		2nd modal	
	Point mass (kg)	Spring coefficient (N/m)		Natural frequency (rad/s)	*RFS* (%)	Natural frequency (rad/s)	*RFS* (%)
System I	2000	4×10^6	44.7214	152.9683	4.0800	256.9069	1.4877
System II	1500	4×10^6	51.6398	153.1449	4.2002	256.9496	1.5045
System III	1500	4×10^6	63.2456	153.5397	4.4688	257.0320	1.5371
System IV	150	4×10^6	163.2993	133.4314	-9.2129	259.2995	2.4328
System V	150	6×10^6	200.0000	138.1446	-6.0061	266.4559	5.2599
System VI	150	1×10^7	258.1989	140.3532	-4.5033	229.6154	-9.2935
System VII	150	1.2×10^7	282.8427	140.7624	-4.2249	234.6748	-7.2948

4.3.2 Natural frequency of spring-mass system

In order to demonstrate the effect of natural frequency of spring-mass system on the continuous Timoshenko beam with variable cross section, a two-span continuous Timoshenko beam with one spring-mass system located at 3m distance from the left end is studied and calculated by the proposed method, in which seven kinds of spring-mass system with different natural frequencies (various combinations of point masses and spring constants) are adopted, respectively. The parameters of these spring-mass systems and the calculated *RFS* of the first two natural frequencies are listed in Table 3.

From Table 3, it can be seen that *RFS* for systems I, II and III are positive which indicates that natural frequency of continuous Timoshenko beam induced by these systems increase. For systems IV and V, *RFS* of 1st natural frequencies are negative and *RFS* of 2nd natural frequencies are positive. It means that the first natural frequency decrease and the second one increase due to spring-mass system. Besides, *RFS* of the lowest two natural frequencies are negative due to spring-mass systems, which mean that spring-mass system leads to the decrease of natural frequencies. Based on the above analysis, it can be concluded the qualitative judgment rule of how spring-mass system influences natural frequency of continuous Timoshenko beam, and the procedure is as follows:

For comparison of ω_s and ω_0^i (i.e., natural frequency of spring-mass system and continuous Timoshenko beam), $\omega_s > \omega_0^i$, *RFS* of the *i*th natural frequency is negative; $\omega_s < \omega_0^i$, *RFS* is positive.

For systems I, II and III, *RFS* of 2nd natural frequencies are smaller than of 1st ones. This is because compared with natural frequency of spring-mass system, the difference of the 2nd natural frequency of continuous Timoshenko beam is larger than that of the 1st one, i.e., the natural frequency of spring-mass system is more close to the first one of continuous Timoshenko beam. In addition, *RFS* of the lowest two natural frequencies increases with the increasing of natural frequency of spring-mass system. This is because the natural frequency of spring-mass system is close to that of continuous Timoshenko beam. Thus, we can see that when the difference between natural frequencies of spring-mass system and continuous Timoshenko beam is smaller, the influence of spring-mass on beam is more obvious. It can be also obtained from comparison between systems IV and V or VI and VII. Thus, when natural frequency of spring-mass system is close to one of natural frequencies of continuous Timoshenko beam, the influence of spring-mass system on corresponding natural frequency is more obvious.

4.3.3 Position and relative depth of crack

Change ratio for natural frequency caused by crack is defined by Eq. (71).

$$RFC = \frac{\omega_c - \omega_0}{\omega_0} \times 100\% \tag{71}$$

where ω_c is natural frequency of cracked beam.

The position of crack changes from 0.5 m to 11.5 m distance from the left end of beam by a step of 0.5 m as shown in Fig. 12. *RFC* with respect to different positions of crack is calculated by the proposed method and the results are presented in Fig. 13.

As can be seen in Fig. 13, *RFC* of the first natural frequency has the same change rule as *RFS*, i.e., *RFC* reaches the largest value when crack is located at peak or trough of the associated mode shape, while the value of *RFC* is the smallest for crack located at zero point for this mode. For *RFC* of the second natural frequency, the value is close to zero when crack is located at 4m or 8m distance from the left; while crack is located at intermediate support, value of *RFC* reaches the lowest point, and at this time, crack has the most influence on the second natural frequency of continuous Timoshenko beam with variable cross section. This is because for the second mode shape,

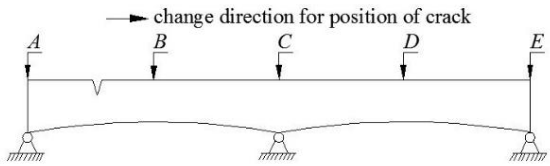


Fig. 12 Schematic diagram of changing position for crack

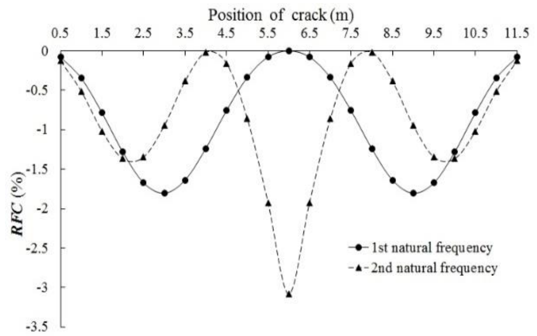
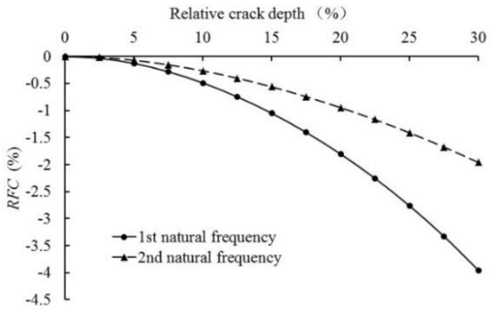
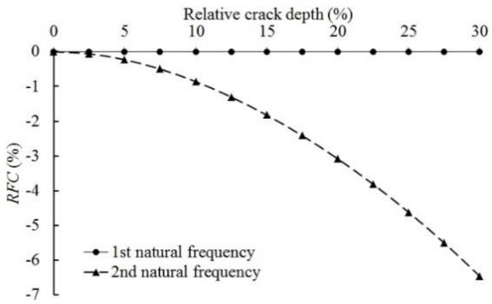


Fig. 13 Change ratios for natural frequency with respect to different positions of crack



(a) *RFC* for crack located at 3 m distance from the left end of beam



(b) *RFC* for crack located at 6m distance from the left end of beam

Fig. 14 Change ratios for natural frequency with respect to relative crack depth

the bending moment of intermediate support is the largest and those located at 4m and 8m are close to zero.

The lowest two natural frequencies for two cases which crack is located at 3m and 6m distance from the left end respectively are calculated by the proposed method, and the associated change ratios with respect to relative crack depth are presented in Fig. 14.

As can be seen from Fig. 14, *RFC* of the first two natural frequencies decreases nonlinearly with the increasing of relative crack depth for crack located at 3m distance from the left end. *RFC* of 1st natural frequency is zero while the crack is located at 6m distance from the left end (i.e., crack is at the intermediate support), and that of 2nd decreases nonlinearly with the increasing of relative crack depth. This is because the beam at the intermediate support can rotate freely in the first mode i.e., the bending moment is equal to zero; however, as for the second mode, the beam at the intermediate support couldn’t rotate freely, which indicates that there is a bending moment at this point. Therefore, based on the above analysis, the present of crack will lead to the change of modal properties if the bending moment of crack position is not equal to zero; while the bending moment of crack position is zero, crack has little effect on the associated modal properties.

5. Conclusions

This paper presents an approach for the free vibration analysis of cracked non-uniform continuous Timoshenko beam with an arbitrary number of spring-mass systems. Firstly, the beam is considered to be divided into several segments, and the transfer relationship of the undetermined coefficients in a segment is built. Then, the equations of motion of spring-mass systems and the compatibility conditions at intermediate supports are used to establish the characteristic equation of the continuous Timoshenko beam with any number of cracks and spring-mass systems. Finally, a previous method (DQEM) and FEM is used to validate the proposed method. The dynamic characteristics of a three-span cracked non-uniform continuous Timoshenko beam carrying one spring-mass system are calculated and the effects of parameters for cracks and spring-mass systems on natural frequencies of non-uniform continuous Timoshenko beams are studied.

Based on the numerical results, the following conclusions are drawn:

(1) In this paper, an analytical approach for the free vibration analysis of a cracked non-uniform continuous Timoshenko beam with any number spring-mass systems is presented which is applicable for any variant cross-section beam. And it is found that the agreement between the calculated results and other methods (DQEM and FEM) results is good, which verifies the validity and reliability of the proposed method.

(2) The first three mode shapes of φ have more obvious changes than the first three mode shapes of Y. The spring-mass systems located at intermediate supports have no influence on natural frequencies of non-uniform continuous Timoshenko beams. However, the influences of cracks located at intermediate supports on natural frequencies of beams depend on whether there are bending moments at the points of corresponding mode shape.

(3) When natural frequency of spring-mass system is less than that of non-uniform continuous Timoshenko beam, the corresponding order natural frequency of the non-uniform continuous Timoshenko beam will increase and vice versa. Moreover, the smaller the difference between natural frequencies of spring-mass system and non-uniform continuous Timoshenko beam is, the more obvious the influence of spring-mass system is.

Acknowledgments

The authors express their appreciation for financial supports of National Natural Science Foundation of China (Grant Number 51478203) and Training Program for Outstanding Young Teachers of Jilin University.

References

Aydin, K. (2007), “Vibratory characteristics of axially-loaded timoshenko beams with arbitrary number of cracks”, *J. Vib. Acoust.*, **129**(3), 341-354.

Dormand, J.R. and Prince, P.J. (1980), “A family of embedded Runge-Kutta formulae”, *J. Comput. Appl. Math.*, **6**(1), 19-26.

El-Sayed, T. and Farghaly, S.H. (2016), “Exact vibration of Timoshenko beam combined with multiple mass spring sub-systems”, *Struct. Eng. Mech.*, **57**(6), 989-1014.

Kahaner, D., Moler, C. and Nash, S. (1989), *Numerical Methods* *and Software*, Prentice Hall, Englewood Cliffs, New Jersey, USA.

Khiem, N.T. and Lien, T.V. (2001), "A simplified method for natural frequency of a multiple cracked beam", *J. Sound Vib.*, **245**(4), 737-751.

Kisa, M., Brandon, J. and Topcu, M. (1998), "Free vibration analysis of cracked beams by a combination of finite elements and component mode synthesis methods", *Comput. Struct.*, **67**(4), 215-223.

Krawczuk, M., Palacz, M. and Ostachowicz, W. (2003), "The dynamic analysis of a cracked Timoshenko beam by the spectral element method", *J. Sound Vib.*, **264**(5), 1139-1153.

Li, Q.S. (2003), "Vibratory characteristics of timoshenko beams with arbitrary number of cracks", *J. Eng. Mech.*, **129**(11), 1355-1359.

Lin, H.P. (2004), "Direct and inverse methods on free vibration analysis of simply supported beams with a crack", *Eng. Struct.*, **26**(4), 427-436.

Lin, H.Y. (2009), "On the natural frequencies and mode shapes of a multispan Timoshenko beam carrying a number of various concentrated elements", *J. Sound Vib.*, **319**(1-2), 593-605.

Loya, J.A., Rubio, L. and Fernández-Sáez, J. (2006), "Natural frequencies for bending vibrations of Timoshenko cracked beams", *J. Sound Vib.*, **290**(3-5), 640-653.

Mazanoglu, K., Yesilyurt, I. and Sabuncu, M. (2009), "Vibration analysis of multiple-cracked non-uniform beams", *J. Sound Vib.*, **320**(4), 977-989.

Rossi, R.E., Laura, P.A.A., Avalos, D.R. and Larrondo, H. (1993), "Free Vibrations of Timoshenko Beams Carrying Elastically Mounted, Concentrated Masses", *J. Sound Vib.*, **165**(2), 209-223.

Swamidas, A.S.J., Seshadri, R. and Yang, X. (2004), "Identification of cracking in beam structures using Timoshenko and Euler formulations", *J. Eng. Mech.*, **130**(11), 1297-1308.

Tada, H., Paris, P. and Irwin, G. (1985), *The Stress Analysis of Cracks Handbook*, 2nd Edition, Paris Productions, St. Louis, USA.

Torabi, K., Afshari, H. and Haji Aboutalebi, F. (2014), "A DQEM for transverse vibration analysis of multiple cracked non-uniform Timoshenko beams with general boundary conditions", *Comput. Math. Appl.*, **67**(3), 527-541.

Valiente, A., Elices, M. and Ustariz, F. (1990), "Determinacion de esfuerzos y movimientos en estructuras lineales con secciones fisuradas", *Anales de Mecanica de la Fractura*, **7**, 272-277. (in Spanish)

Viola, E., Federici, L. and Nobile, L. (2001), "Detection of crack location using cracked beam element method for structural analysis", *Theor. Appl. Fract. Mech.*, **36**(1), 23-35.

Wang, J.R., Liu, T.L. and Chen, D.W. (2007), "Free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems with the effects of shear deformation and rotary inertia", *Struct. Eng. Mech.*, **26**(1), 1-14.

Wu, J.S. and Chang, B.H. (2013) "Free vibration of axial-loaded multi-step Timoshenko beam carrying arbitrary concentrated elements using continuous-mass transfer matrix method" *Eur. J. Mech.-A/Solid.*, **38**(3), 20-37.

Wu, J.S. and Chen, D.W. (2001), "Free vibration analysis of a Timoshenko beam carrying multiple spring-mass systems by using the numerical assembly technique", *Int. J. Numer. Meth. Eng.*, **50**(5), 1039-1058.

Yesilce, Y. and Demirdag, O. (2008), "Effect of axial force on free vibration of Timoshenko multi-span beam carrying multiple spring-mass systems", *Int. J. Mech. Sci.*, **50**(6), 995-1003.

Yesilce, Y., Demirdag, O. and Catal, S. (2008), "Free vibrations of a multi-span Timoshenko beam carrying multiple spring-mass systems", *Sādhanā*, **33**(4), 385-401.

Zheng, D.Y. and Fan, S.C. (2001), "Natural frequencies of a non-uniform beam with multiple cracks via modified Fourier series", *J. Sound Vib.*, **242**(4), 701-717.

Zheng, D.Y. and Fan, S.C. (2001), "Natural frequency changes of a cracked Timoshenko beam by modified Fourier series", *J. Sound Vib.*, **246**(2), 297-317.

Zheng, T. and Ji, T. (2012), "An approximate method for determining the static deflection and natural frequency of a cracked beam", *J. Sound Vib.*, **331**(11), 2654-2670.

CC

Appendix A: The solution of transfer matrix using Runge-Kutta method

As shown in Fig. A1, non-uniform Timoshenko beam is divided into n equally spaced segments, and the length of each segment is h, that is, the step length of Runge-Kutta method. The symbols of x_1, x_2, $\cdots$, x_{i-1}, x_i, $\cdots$, x_n. refer to the right position of each segment.

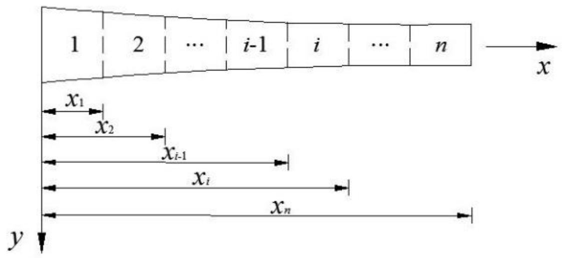


Fig. A1 non-uniform Timoshenko beam

Let

$$\begin{cases} u_1(x) = \dfrac{1}{KGA(x)} \\ u_2(x) = \dfrac{-1}{EI(x)} \\ u_3(x) = \omega^2 \rho I(x) \\ u_4(x) = -\omega^2 \rho A(x) \end{cases} \tag{A1}$$

Using $T_{\alpha,\lambda}(x)$to represent any elements in the transfer matrix $[T(x)]$. α, λ represent rows and columns of the elements in the matrix, respectively, and α=1,2,3,4; λ=1,2,3,4.

Calculate the right side of the Eq. (13) in this paper, it has

$$\left[f(x,\{TT_{\alpha,\lambda}\}\right] = [U(x)][T(x)] = \begin{bmatrix} f_{1,1}(x,\{TT_{\alpha,\lambda}\}) & f_{1,2}(x,\{TT_{\alpha,\lambda}\}) & f_{1,3}(x,\{TT_{\alpha,\lambda}\}) & f_{1,4}(x,\{TT_{\alpha,\lambda}\}) \\ f_{2,1}(x,\{TT_{\alpha,\lambda}\}) & f_{2,2}(x,\{TT_{\alpha,\lambda}\}) & f_{2,3}(x,\{TT_{\alpha,\lambda}\}) & f_{2,4}(x,\{TT_{\alpha,\lambda}\}) \\ f_{3,1}(x,\{TT_{\alpha,\lambda}\}) & f_{3,2}(x,\{TT_{\alpha,\lambda}\}) & f_{3,3}(x,\{TT_{\alpha,\lambda}\}) & f_{3,4}(x,\{TT_{\alpha,\lambda}\}) \\ f_{4,1}(x,\{TT_{\alpha,\lambda}\}) & f_{4,2}(x,\{TT_{\alpha,\lambda}\}) & f_{4,3}(x,\{TT_{\alpha,\lambda}\}) & f_{4,4}(x,\{TT_{\alpha,\lambda}\}) \end{bmatrix} \tag{A2}$$

where, $\{TT_{\alpha,\lambda}\}$ is a row vector consisting of elements $T_{\alpha,\lambda}(x)$ in $[T(x)]$.

$$\{TT_{\alpha,\lambda}\} = [T_{1,1} \quad T_{1,2} \quad \cdots \quad T_{\alpha,\lambda} \quad \cdots \quad T_{4,3} \quad T_{4,4}] \tag{A3}$$

$$\begin{cases} f_{1,1}(x,\{TT_{\alpha,\lambda}\}) = T_{2,1}(x) + T_{4,1}(x)u_1(x) \\ f_{1,2}(x,\{TT_{\alpha,\lambda}\}) = T_{2,2}(x) + T_{4,2}(x)u_1(x) \\ f_{1,3}(x,\{TT_{\alpha,\lambda}\}) = T_{2,3}(x) + T_{4,3}(x)u_1(x) \\ f_{1,4}(x,\{TT_{\alpha,\lambda}\}) = T_{2,4}(x) + T_{4,4}(x)u_1(x) \end{cases} \tag{A4a}$$

$$\begin{cases} f_{2,1}(x,\{TT_{\alpha,\lambda}\}) = T_{3,1}(x)u_2(x) \\ f_{2,2}(x,\{TT_{\alpha,\lambda}\}) = T_{3,2}(x)u_2(x) \\ f_{2,3}(x,\{TT_{\alpha,\lambda}\}) = T_{3,3}(x)u_2(x) \\ f_{2,4}(x,\{TT_{\alpha,\lambda}\}) = T_{3,4}(x)u_2(x) \end{cases} \tag{A4b}$$

$$\begin{cases} f_{3,1}(x,\{TT_{\alpha,\lambda}\}) = T_{2,1}(x)u_3(x) + T_{4,1}(x) \\ f_{3,2}(x,\{TT_{\alpha,\lambda}\}) = T_{2,2}(x)u_3(x) + T_{4,2}(x) \\ f_{3,3}(x,\{TT_{\alpha,\lambda}\}) = T_{2,3}(x)u_3(x) + T_{4,3}(x) \\ f_{3,4}(x,\{TT_{\alpha,\lambda}\}) = T_{2,4}(x)u_3(x) + T_{4,4}(x) \end{cases} \tag{A4c}$$

$$\begin{cases} f_{4,1}(x,\{TT_{\alpha,\lambda}\}) = T_{1,1}(x)u_4(x) \\ f_{4,2}(x,\{TT_{\alpha,\lambda}\}) = T_{1,2}(x)u_4(x) \\ f_{4,3}(x,\{TT_{\alpha,\lambda}\}) = T_{1,3}(x)u_4(x) \\ f_{4,4}(x,\{TT_{\alpha,\lambda}\}) = T_{1,4}(x)u_4(x) \end{cases} \tag{A4d}$$

The Eq. (13) can also be expressed as

$$\frac{d\{TT_{\alpha,\lambda}\}}{dx} = f_{\alpha,\lambda}(x,\{TT_{\alpha,\lambda}\}) \tag{A5}$$

According to the basic principle of the fourth-order Runge-Kutta method, the recursive relation of the elements $T_{\alpha,\lambda}(x)$ of the transfer matrix can be obtained from Eq. (A5).

$$T_{\alpha,\lambda}(x_{i+1}) = T_{\alpha,\lambda}(x_i) + \frac{h}{6}(K^1_{\alpha,\lambda} + 2K^2_{\alpha,\lambda} + 2K^3_{\alpha,\lambda} + K^4_{\alpha,\lambda}) \tag{A6}$$

where $K^1_{\alpha,\lambda}$ $K^2_{\alpha,\lambda}$ $K^4_{\alpha,\lambda}$ are the estimated value of slope obtaining by Euler method at x_i, $x_i+h/2$, x_{i+1}; $K^3_{\alpha,\lambda}$ is the estimated value of slope obtaining by improved Euler method at $x_i+h/2$.

Through the analysis of Eq. (A4), we can obtain that

$$\begin{cases} f_{1,\lambda}(x,\{TT_{\alpha,\lambda}\}) = f_{1,\lambda}(x,T_{2,\lambda},T_{4,\lambda}) \\ f_{2,\lambda}(x,\{TT_{\alpha,\lambda}\}) = f_{2,\lambda}(x,T_{3,\lambda}) \\ f_{3,\lambda}(x,\{TT_{\alpha,\lambda}\}) = f_{3,\lambda}(x,T_{2,\lambda},T_{4,\lambda}) \\ f_{4,\lambda}(x,\{TT_{\alpha,\lambda}\}) = f_{4,\lambda}(x,T_{1,\lambda}) \end{cases} \tag{A7}$$

According to Eq. (A7), the Eq. (A5) can also be expressed as

$$\begin{cases} \dfrac{dT_{1,\lambda}(x)}{dx} = f_{1,\lambda}(x,T_{2,\lambda},T_{4,\lambda}) \\ \dfrac{dT_{2,\lambda}(x)}{dx} = f_{2,\lambda}(x,T_{3,\lambda}) \\ \dfrac{dT_{3,\lambda}(x)}{dx} = f_{3,\lambda}(x,T_{2,\lambda},T_{4,\lambda}) \\ \dfrac{dT_{4,\lambda}(x)}{dx} = f_{4,\lambda}(x,T_{1,\lambda}) \end{cases} \tag{A8}$$

According to Eq. (A8), $K^1_{1,\lambda}$ $K^2_{1,\lambda}$ $K^3_{1,\lambda}$ $K^4_{1,\lambda}$; $K^1_{2,\lambda}$ $K^2_{2,\lambda}$ $K^3_{2,\lambda}$; $K^1_{3,\lambda}$ $K^2_{3,\lambda}$ $K^3_{3,\lambda}$ $K^4_{3,\lambda}$; $K^1_{4,\lambda}$ $K^2_{4,\lambda}$ $K^3_{4,\lambda}$ $K^4_{4,\lambda}$ corresponding to the elements $T_{1,\lambda}(x)$, $T_{2,\lambda}(x)$, $T_{3,\lambda}(x)$, $T_{4,\lambda}(x)$ respectively in the 1st, 2nd, 3rd and 4th row of the transfer matrix can be given. The relative computational formulas are as follows

$$\begin{cases} K^1_{1,\lambda} = f_{1,\lambda}(x_i,T_{2,\lambda}(x_i),T_{4,\lambda}(x_i)) \\ K^2_{1,\lambda} = f_{1,\lambda}(x_i + \dfrac{h}{2},T_{2,\lambda}(x_i) + \dfrac{h}{2}K^1_{2,\lambda},T_{4,\lambda}(x_i) + \dfrac{h}{2}K^1_{4,\lambda}) \\ K^3_{1,\lambda} = f_{1,\lambda}(x_i + \dfrac{h}{2},T_{2,\lambda}(x_i) + \dfrac{h}{2}K^2_{2,\lambda},T_{4,\lambda}(x_i) + \dfrac{h}{2}K^2_{4,\lambda}) \\ K^4_{1,\lambda} = f_{1,\lambda}(x_i + h,T_{2,\lambda}(x_i) + hK^3_{2,\lambda},T_{4,\lambda}(x_i) + hK^3_{4,\lambda}) \end{cases} \tag{A9}$$

$$\begin{cases} K_{2,\lambda}^{1} = f_{2,\lambda}(x_i, T_{3,\lambda}(x_i)) \\ K_{2,\lambda}^{2} = f_{2,\lambda}(x_i + \frac{h}{2}, T_{3,\lambda}(x_i) + \frac{h}{2}K_{3,\lambda}^{1}) \\ K_{2,\lambda}^{3} = f_{2,\lambda}(x_i + \frac{h}{2}, T_{3,\lambda}(x_i) + \frac{h}{2}K_{3,\lambda}^{2}) \\ K_{2,\lambda}^{4} = f_{2,\lambda}(x_i + h, T_{3,\lambda}(x_i) + hK_{3,\lambda}^{3}) \end{cases} \tag{A10}$$

$$\begin{cases} K_{3,\lambda}^{1} = f_{3,\lambda}(x_i, T_{2,\lambda}(x_i), T_{4,\lambda}(x_i)) \\ K_{3,\lambda}^{2} = f_{3,\lambda}(x_i + \frac{h}{2}, T_{2,\lambda}(x_i) + \frac{h}{2}K_{2,\lambda}^{1}, T_{4,\lambda}(x_i) + \frac{h}{2}K_{4,\lambda}^{1}) \\ K_{3,\lambda}^{3} = f_{3,\lambda}(x_i + \frac{h}{2}, T_{2,\lambda}(x_i) + \frac{h}{2}K_{2,\lambda}^{2}, T_{4,\lambda}(x_i) + \frac{h}{2}K_{4,\lambda}^{2}) \\ K_{3,\lambda}^{4} = f_{3,\lambda}(x_i + h, T_{2,\lambda}(x_i) + hK_{2,\lambda}^{3}, T_{4,\lambda}(x_i) + hK_{4,\lambda}^{3}) \end{cases} \tag{A11}$$

$$\begin{cases} K_{4,\lambda}^{1} = f_{4,\lambda}(x_i, T_{1,\lambda}(x_i)) \\ K_{4,\lambda}^{2} = f_{4,\lambda}(x_i + \frac{h}{2}, T_{1,\lambda}(x_i) + \frac{h}{2}K_{1,\lambda}^{1}) \\ K_{4,\lambda}^{3} = f_{4,\lambda}(x_i + \frac{h}{2}, T_{1,\lambda}(x_i) + \frac{h}{2}K_{1,\lambda}^{2}) \\ K_{4,\lambda}^{4} = f_{4,\lambda}(x_i + h, T_{1,\lambda}(x_i) + hK_{1,\lambda}^{3}) \end{cases} \tag{A12}$$

The Eqs. (A6), (A9), (A10), (A11) and (A12) express the transfer relationship between any elements $T_{\alpha,\lambda}(x_{i+1})$ at x_{i+1} and $T_{\alpha,\lambda}(x_i)$ at x_i of the transfer matrix. From Eq. (12), the transfer matrix value at x=0 is determined as the identity matrix. The value of the transfer matrix $[T(x)]$ at x=0 is taken as the initial condition, and the value of transfer matrix at x_1, x_2, $\cdots$, x_n can be obtained using the transfer relationship between $T_{\alpha,\lambda}(x_{i+1})$ and $T_{\alpha,\lambda}(x_i)$.