Structural Engineering and Mechanics, *Vol. 62, No. 3 (2017) 365-372*
DOI: https://doi.org/10.12989/sem.2017.62.3.365

Reliability index for non-normal distributions of limit state functions

Seyed Hooman Ghasemi*[1,2] and Andrzej S. Nowak[3a]

[1]*Department of Civil Engineering, Islamic Azad University, Qazvin Branch. Iran*
[2]*Auburn University, Auburn, AL 36849, USA*
[3]*Department of Civil Engineering, Auburn University, Auburn, USA*

(Received August 9, 2015, Revised December 8, 2016, Accepted February 24, 2017)

Abstract. Reliability analysis is a probabilistic approach to determine a safety level of a system. Reliability is defined as a probability of a system (or a structure, in structural engineering) to functionally perform under given conditions. In the 1960s, Basler defined the reliability index as a measure to elucidate the safety level of the system, which until today is a commonly used parameter. However, the reliability index has been formulated based on the pivotal assumption which assumed that the considered limit state function is normally distributed. Nevertheless, it is not guaranteed that the limit state function of systems follow as normal distributions; therefore, there is a need to define a new reliability index for no-normal distributions. The main contribution of this paper is to define a sophisticated reliability index for limit state functions which their distributions are non-normal. To do so, the new definition of reliability index is introduced for non-normal limit state functions according to the probability functions which are calculated based on the convolution theory. Eventually, as the state of the art, this paper introduces a simplified method to calculate the reliability index for non-normal distributions. The simplified method is developed to generate non-normal limit state in terms of normal distributions using series of Gaussian functions.

Keywords: reliability; convolution theorem; non-normal distribution; probability of failure; limit state function

1. Introduction

To determine the reliability level of the structures, instead of using the deterministic values of capacity of the structure and applied loads, it is required to utilize statistical parameters of the load and/or resistance. The objective of the reliability analysis is to calculate the safety level of the structures or its components. The definition of reliability index was first given by Basler (1961), Cornell (1969) as

$$\beta = -\Phi^{-1}(P_f) \tag{1}$$

where,

$\beta =$ reliability index

$\Phi^{-1} =$ inverse of the cumulative distribution of the standard normal function

Eq. (1) is widely used to determine the reliability index of systems/structures. Okasha *et al*. (2012) utilized Eq. (1) to assess the lifetime reliability index for bridges. Der Kiureghian and Song, (2008) expanded the development of the reliability index analysis for the different aspects of the safety analysis such as Multi-scale reliability analysis. Shi *et al*. (2014) probabilistic response modeling, cost analysis. Yanaka *et al*. (2016) used Eq. (1) to determine the safety level for chloride penetration into pressteressed girder bridges. However, according to the proposed definition of the reliability index in Eq. (1), the reliability indices for all cases are determined with the inherently assumption of the normally distributed limit state function for the given limit state function. Ghasemi and Ashatri (2014), Ashtari and Ghasemi (2013), Ghasemi *et al*. (2013) found that there are several phenomenon which are not follow normal distributions. Leira (2013) established a framework to consider the structures with time-varying load and resistance properties. In his framework he utilized the conventional methods to estimate the reliability index for non-normal distribution. Several researches have been conducted to determine the reliability index for different conditions. For example, Li and *et al* (2010) introduced a new high-order response surface method to calculate the structural reliability index. Following, Chowdhury and Rao (2011) proposed a method to predict the structural limit state function using Multiple High Dimensional Model Representation (Multicut-HDMR). Accordingly, Fang *et al* (2013) presented analytical method to determine structural dynamic reliability based on the probability density evolution. However, all of them considered Eq. (1) as a fundamental equation to compute the reliability index. Also, in an engineering application the Eq. (1) is wildly used to evaluate the safety level of systems or structures. For instance, Zhu and Frangopol (2014) investigated the material postfailure behavior on system reliability with consideration of the first given definition of reliability index for normal distribution. Jalayer and Zhou (2016) presented a new 24 approach to evaluating the safety risk of roadside features for rural two-lane roads based on reliability analysis. Ayyub *et al.* (2015) considered the reliability analysis to propose the optimal design case for steel box structures using the conventional reliability method. Ghasemi *et al*. (2016a, 2016b), Ghasemi and Nowak

*Corresponding author, Assistant Professor
E-mail: Hooman.Ghasemi@auburn.edu
[a]Professor

http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

(2016a) used the conventional reliability index to evaluate the reliability level of highway systems. Nevertheless, it cannot be assured that the limit states of all case studies behave as a normal distributions. Lewis (1996) exhibited that there are several structural failure scenarios which their distribution of them follow Weibull destruction. For instance, Brake (2011) found that the represented historical data of fatigue failure of bearings mentioned by Moubray (2002) follows as the exponential distribution. Ghasemi and Nowak (2016b) developed a new approach to determine mean maximum values of non-normal distribution. However, there is a need to derive the reliability index for the limit states which their limit state functions of them are not normally distributed. The main objective of this paper is to propose a methodology to determine the reliability index for a non-normal distribution of the limit state function.

2. Reliability analysis

In structural engineering, probability failure is defined as a probability of structural failure during its design period (life-cycle). In other words, the reliability index can be defined as the probability of structural performance during its life-cycle. To assess the level of structural reliability, the reliability index, β, is a commonly used quantify. The reliability index has been defined as the inverse of the coefficient of variation of the Cumulative Distribution Function (CDF) of the limit state function. The graphical definition of the reliability index is the shortest distance of the limit state function from the origin of the reduced variables space state. For instance, if R represents the reduced variables of resistance, Q indicates the reduced variables of load and the limit state is defined by g (see Fig. 1).

$$g = R - Q \tag{2}$$

Nowak and Collins (2013) summarized the procedure to determine the reliability index as follows

Structural loading and load effect

Consider the applied load on the structure

Determine the statistical parameters of load

Structural resistance

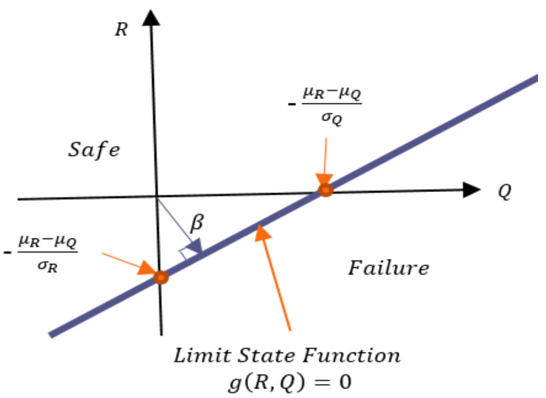


Fig. 1 Graphical definition of the reliability index, by Nowak and Collins (2013)

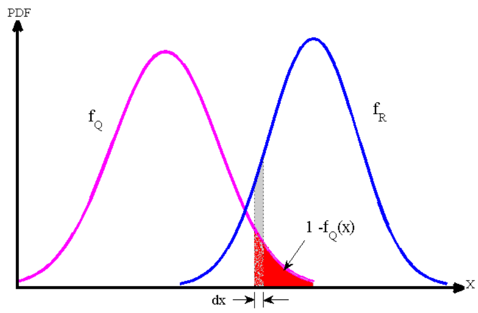


Fig. 2 Graphical integration approach to evaluate P_f, by Nowak and Collins (2013)

Consider the structural resistance

Determine the statistical parameters of resistance

Balance between load effect and structural resistance

Establish the limit state function $g = R - Q$

Determine the reliability index

Depending on the limit state function, several approaches were introduced to compute the reliability index. For instance, if the variables follow normal distributions, the Hasofer-Lind (1974) method is one of the appropriate approaches; however, if one of the variables is treated as a non-normal distribution, the Rackwitz-Fiessler (1978) method is an alternative approach. If the limit state consists of several random variables with different distributions, the Monte Carlo method has been recommended. The state of the failure (P_F) is the condition when the limit state function is less than zero $g < 0$.

$$P_F = P(R - Q < 0) = P(g < 0) \tag{3}$$

Assuming statistical independence between R and Q leads to the joint PDF by way of the multiplication of the PDFs of R and Q ($f_{RQ}(r,q) = f_Q(q)f_R(r)$). Then, taking the integration of $f_Q(q)$ with respect to q from r to infinity associated with computing the integration with respect to r over the entire possible domain, the probability of failure (P_f) can be determined. Fig. 2 illustrates the graphical approach to compute P_f.

$$\mathrm{P_f} = 1 - \int_{-\infty}^{+\infty} \mathrm{f_R(r)F_Q(r)dr} \tag{4}$$

where,

$f_R(r)$ = PDF of resistance,

$F_Q(r)$ = CDF of load.

where f_R and f_Q are the probability density functions, and F_Q and F_R is the cumulative distribution functions (CDF). r and q are the random variables of the load and resistance, respectively..Alternately, first, by taking the integration over $f_R(r)$ from negative infinity to infinity with respect to q, P_F can be calculated as follows

$$\mathrm{P_f} = \int_{-\infty}^{+\infty} \mathrm{f_Q(q)F_R(q)dq} \tag{5}$$

where,

$f_R(r)$ = PDF of load,

$F_Q(r)$ = CDF of resistance.

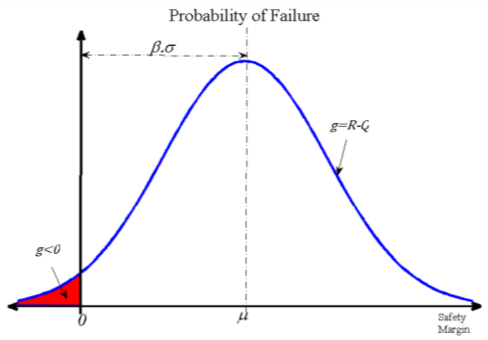


Fig. 3 Graphical relationship between the reliability index and P_f, Nowak and Collins (2013)

Based on the reliability function definition, the reliability is the complementary function of the probability function; therefore, the reliability function can be demonstrated as follows (Nowak and Collins 2013)

$$R = \int_{-\infty}^{+\infty} f_R(r) F_Q(r) dr = 1 - \int_{-\infty}^{+\infty} f_Q(q) F_R(q) dq \quad (6)$$

Mathematically, if both functions are normally distributed, it can be proven that the limit state function also follows a normal distribution and the reliability index can be computed using Hasofer-Lind approach, which has been also shown by Ditlevsen and Madsen (1996), Nowak and Collins (2013).

$$\beta = \frac{\mu_R - \mu_Q}{\sqrt{{\sigma_R}^2 + {\sigma_Q}^2}} \quad (7)$$

where,

$\mu_R =$ mean value of resistance,

$\mu_Q =$ mean value of load,

$\sigma_R =$ standard deviation of resistance,

$Q =$ standard deviation of load.

Another technique to determine the reliability index for a normal distribution is a graphical method. Basically, as far as the distribution of the limit state function is given and follows a normal distribution, the graphical method would be beneficial (Fig. 3). Based on the graphical method, the reliability index is defined as the distance between the mean value and the safety margin ($g = 0$) in terms of the standard deviation. Eq. (8) shows the formula to compute the reliability index using the graphical method.

$$\beta = \frac{\mu_g}{\sigma_g} \quad (8)$$

where,

$\mu_g =$ mean of the limit state function,

$\sigma_g =$ standard deviation of the limit state function.

However, if any of the aforementioned distributions (load and resistance) do not behave as a normal distribution, it is necessary to apply a different method to find the reliability index. Rackwitz-Fiessler (1978) method is a commonly used to determine the reliability. As an example, if the resistance has a log-normal distribution and the load has a normal distribution, the reliability index can be written as follows (Nowak and Collins 2013).

$$\beta = \frac{\mu_R \left(1 - k\frac{\sigma_R}{\mu_R}\right)\left[1 - \ln\left(1 - k\frac{\sigma_R}{\mu_R}\right)\right] - \mu_Q}{\sqrt{\left(\mu_R \left(1 - k\frac{\sigma_R}{\mu_R}\right)\left(\frac{\sigma_R}{\mu_R}\right)\right)^2 + {\sigma_Q}^2}} \quad (9)$$

where k is the multiplication factor of the standard deviation, indicating the distance between the design point, x^*, and the mean value (Eq. (10)).

where

$$k = \left(\frac{\mu_r - x^*}{r}\right) \quad (10)$$

It is worth mentioning that in this approach, the design point is a point on the failure distribution's boundary; the CDF and the PDF of the investigated distribution are approaching the CDF and the PDF of a normal distribution, respectively (Eqs. (11) and (12)).

$$F(x^*) = \Phi\left(\frac{x^* - {\mu^e}_x}{{\sigma^e}_x}\right) \quad (11)$$

$$f(x^*) = \frac{1}{{\sigma^e}_x}\varphi\left(\frac{x^* - {\mu^e}_R}{{\sigma^e}_x}\right) \quad (12)$$

where,

$F_Q(r)$ = CDF of Probability of failure,

$f_R(r)$ = PDF of probability of failure,

${\mu^e}_x =$ equivalent normal mean value,

${\sigma^e}_x =$ equivalent normal standard deviation.

Rackwitz-Fiessler method's result is sensitive to the k value because the distribution depends on the probability failure function, and there is no closed-form approach to recommend the k value.

The other popular method to compute the reliability index is Monte Carlo simulation. There are two traditions to determine the reliability index from Monte Carlo simulation:

Approach 1: The probability of failure is the ratio of the number of failures, where $g < 0$, to the total number of samples, therefore, the reliability index is the inverse function of the CDF with respect to obtained probability.

Approach 2: The reliability index is the intersection of the limit state and the vertical coordinate.

According to the definition of the reliability index in Eq. (1), which has been defined the reliability index as the result of the negative inverse function of normally distributed of CDF of the considered limit state function. However, the given limit state function does not necessarily behaves as a normal distribution function. Therefore, there is a need to define the reliability index for non-normal distributions. Hence, herein, authors of this paper introduce a new notion to compute the reliability index for a non-normal limit state function as a negative inverse function of the CDF of limit state function as

$$\beta = -{F_X}^{-1}(P_f) \quad (13)$$

where, ${F_X}^{-1}$ is the inverse function of the CDF of a non-normal distribution for the intended limit state function. In the next section based on the convolution theorem the probability of failure is first defined, and then a formula is proposed to determine the reliability index for non-normal distributions.

3. Convolution theorem for determination of the probability of failure of the limit states function

If the Probability of failure, does not behave as a normal distribution, the current available methods to determine the reliability indices do not conclude to the accurate result. Based on the examples shown by Nowak and Collins (2013), Hasofer-Lind methods would not be an accurate method to determine the reliability index for non-normal distributions, also Rackwitz-Fiessler gives a rough approximation and needs tedious iterations, and Monte-Carlo requires numerous sampling to approach to the acceptable results. The aforementioned methods compute the reliability index based on the inverse of the CDF of normal distributions, which means the actual distributions of the probability of failure of the given limit state functions are ignored. As long as the PDF of limit state behaves as a normal distribution, achieved result would be precise; however, if the PDF of the limit state does not behave as a normal distribution, the obtained reliability index is not accurate. Therefore, in order to compute the precise reliability index for non-normally distributed limit state functions, it is required to propose a new approach with a closed-form formula. In doing so, the probability of failure first should be defined (Ghasemi 2014). In this research the convolution operation is applied to define the probability of failure.

In mathematics world, convolution $(f * g)$ is defined as the integration of the product of the two functions (f and g) after reversing and shifting one of them (with respect to τ variable) (Dimovski 1990 and Weisstein 2014a).

$$(f * g)(t) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} f(\tau)g(t-\tau)d\tau = \int_{-\infty}^{+\infty} f(t-\tau)g(\tau)d\tau \tag{14}$$

Based on the definition of convolution operator, in probability analysis, the convolution can be interpreted as the joint distribution function. In this research, by changing the variables the probability of failure is defined with regard to the convolution theorem. Therefore, for a given limit state function in Eq. (2), the probability of failure is expressed as follows

$$P_f = \big(R(x) * Q(-x)\big) \stackrel{\text{def}}{=} \int_{-\infty}^{+\infty} R(\tau)Q(x+\tau)d\tau = \int_{-\infty}^{+\infty} R(x-\tau)Q(-\tau)d\tau \tag{15}$$

Accordingly, in order to determine the reliability index for those limit state functions which their probability of failures are not normally distributed, the following equation (Eq. (16)) is proposed.

$$\beta = -\,inv\left(\left(\int_{-\infty}^{+\infty} \big(R(x) * Q(-x)\big)d\tau\right)\right)\Bigg|_{(R>Q)} \tag{16}$$

where inv denotes the inverse function. Eq. (16) is a proposed closed-form solution to compute the reliability index for non-normal distributions. The validity and accuracy of the proposed closed-form formulas is accomplished using Monte Carlo simulation for different type of limit states. As an illustration, in this paper two examples are examined.

The first example (E.g., No. 1) assumes that a beam subjected to the dead and live loads which their distributions follow normal distributions. The mean value and standard deviation of the dead load stresses are 1.5 and 1 MPa. And, the mean value and standard deviation of the dead live stresses are 0.5 and 1 MPa.

$$Q_1 = Normal(x, 1.5,1),\ Q_2 = Normal(x, 0.5,1)$$

Accordingly the load distribution is

$$Q = Normal(x, 2,1)$$

The resistance also is normally distributed with mean and standard deviation equal to 5.0 and 1.0 MPa, respectively.

$$R = Normal(x, 5,1)$$

where,

$$Normal(x, \mu, \sigma) = \frac{1}{\sigma\sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2\right)$$

Accordingly the limit state function can be written as follows.

$$g = R - Q = Normal(x, 5,1) - Normal(x, 2,1)$$

Fig. 4(a) illustrates the load and resistance distributions for E.g., No. 1. The probability of failure is determined based on the Eq. (15), which graphically displays in Fig. 4(b).

$$\mathrm{P_f} = 1.7 \times 10^{-2}$$

Accordingly, the reliability index using the proposed method is equaled to $\beta = 2.12$. However, the second example (E.g., No. 2) is the same as the first example, but assumed that the distribution of the loads behave as a Gamma distribution, also, the resistance distribution still behaves as a normal one.

$$Q = Gamma(x, a = 2, b = 1) = \frac{1}{b^a\Gamma(a)} x^{a-1} \exp\left(\frac{-x}{b}\right),$$
$$R = Normal(x, 5,1)$$

where,

$$\text{Gamma function} = \Gamma(a) = \int_0^\infty \exp(-t)t^{a-1}dt$$
$$= \Gamma(a) = (a-1)!$$
$$! = \text{ factorial operator}$$

Accordingly, the limit state function for E.g., No. 2 can be expressed as follows

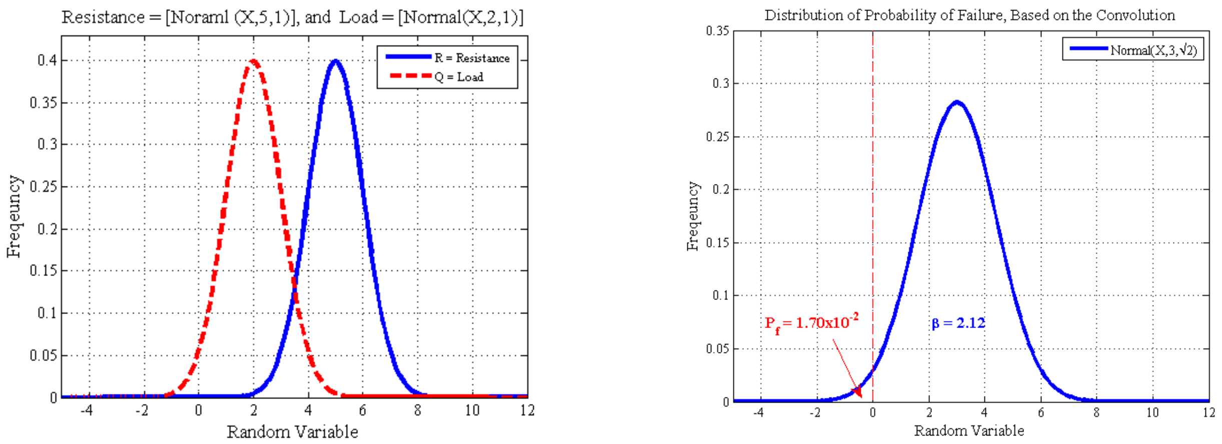
$$g = R - Q = Normal(x, 5,1) - Gamma(x, a = 2, b = 1)$$

Fig. 5(a) shows the probability distributions of load and resistance for E.g., No. 2. Using Eqs. (15)-(16), the probability of failure and reliability index is calculated. The limit state function of E.g., No. 2 displays in Fig. 5(b).

$$P_f = \int_{-\infty}^{0} \big(R(x) * Q(-x)\big) = 5.55 \times 10^{-2}$$

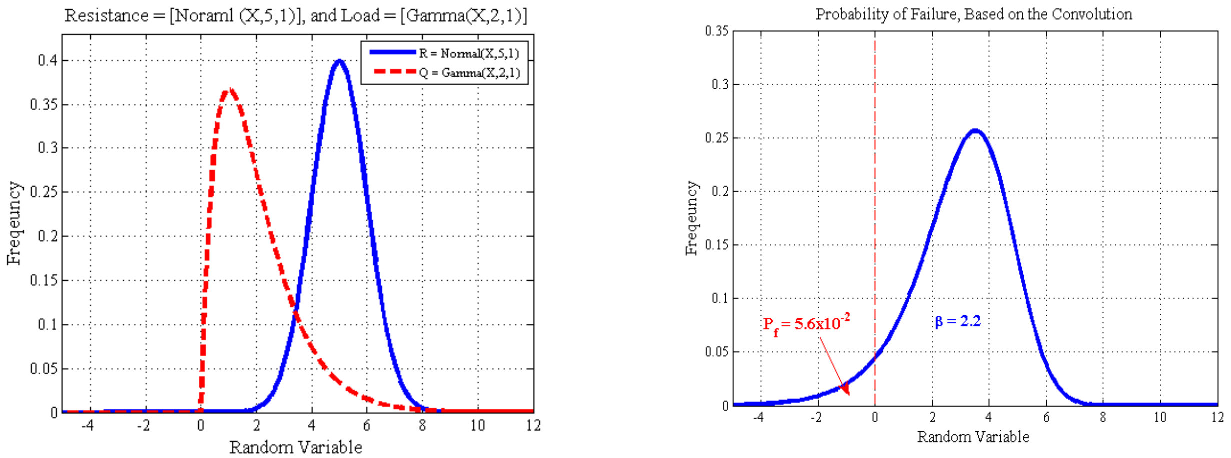
$$\beta = -inv\left(\left(\int_{-\infty}^{+\infty} \big(R(x) * Q(-x)\big)d\tau\right)\right) = 2.20$$

Comparison between E.g., No. 1 and E.g., No. 2 indications that although the resistance distributions are similar in both examples, the load distribution for E.g., No. 1 behaves as a normal distribution, which does not have any



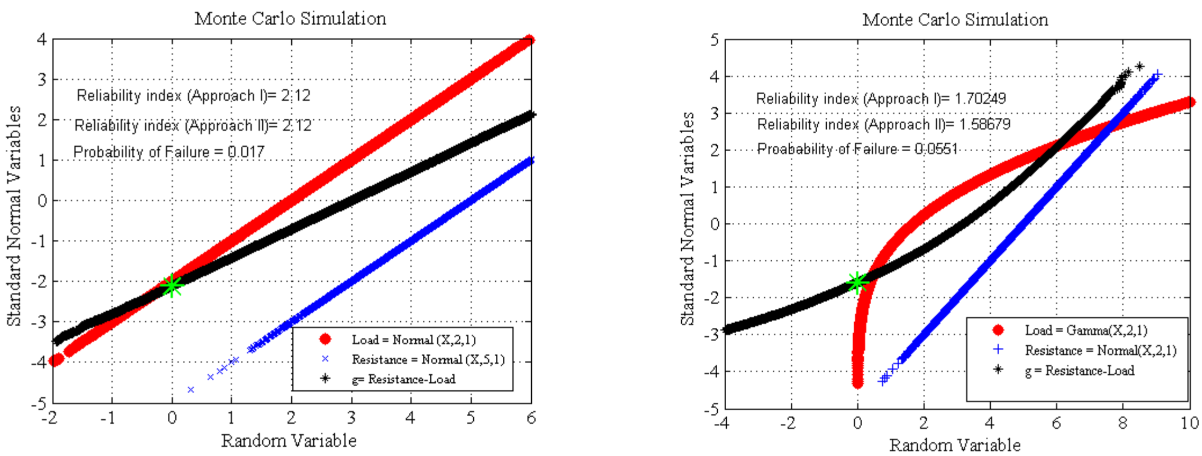
(a) Load and resistance distributions (b) Graphical illustration of P_f

Fig. 4 $\boldsymbol{P_f}$ and reliability index while $\mathbf{R} = \boldsymbol{Normal}(\boldsymbol{x}, \mathbf{5}, \mathbf{1})$ and $\boldsymbol{Q} = \boldsymbol{Normal}(\boldsymbol{x}, \mathbf{2}, \mathbf{1})$



(a) Load and resistance distributions (b) Graphical illustration of P_f

Fig. 5 $\boldsymbol{P_f}$ and reliability index **while** $\boldsymbol{R} = \boldsymbol{Normal}(\boldsymbol{x}, \mathbf{5}, \mathbf{1})$ **and** $\boldsymbol{Q} = \boldsymbol{Gamma}(\boldsymbol{x}, \boldsymbol{a} = \mathbf{2}, \boldsymbol{b} = \mathbf{1})$



(a) Reliability index based on the Monte Carlo simulation, R=$Normal(x, 5, 1)$ and $Q = Normal(x, 2, 1)$

(b) Reliability index based on the Monte Carlo simulation, R=$Normal(x, 5, 1)$ and $Q = Gamma(x, 2, 1)$

Fig. 6 Reliability index based on the Monte

skewness, and the load distribution for E.g., No. 2 behaves as Gamma distribution. Since most of the frequency content of Gamma distribution is focused on the left side, it is expected that Gamma distribution for load terms results to the higher safety value, In other words, it is expected that the limit state function of E.g., No. 2 results to the greater the reliability index. In order to validate the results, these two examples are also examined using Monte Carlo simulation (Janssen 2013).

As can be seen in Fig. 6, the results from Monte Carlo simulation of E.g., No. 1 (load and resistance are normally distributed, for 100,000 sampling (Fig. 6(a)) is similar to the results from the convolution method,
which can be claimed as a statistical validity for the proposed derived equations (Eq. (15) and Eq. (16)).

However, in E.g., No. 2 load distribution behaves as Gamma distribution, the obtained probability of failure from Monte Carlo simulations was equaled to the probability of failure resulting from the convolution method (see Fig. 6(b)). Although the probability of failure of Monte Carlo simulation is accurate, the reliability index from this method is not necessary accurate. Since in order to determine the reliability index using Monte Carlo simulations Eq. (1) was utilized, this inherently assumed a normal distribution for the limit state functions. However, as discerned, the obtained limit state function was not normally distributed. Therefore, using the proposed equation (Eq. 16) the non-normality distribution of limit state functions can be taken into the account, accordingly, the reliability index can be precisely determined. Nevertheless, it should be mentioned that although using Eq. (16) leads to the closed-form solution, it requires the advanced mathematical knowledge which may not user friendly for engineering applications. Consequently, there is a need to introduce a simplified proposed method.

4. Simplified procedure to calculate reliability index based on the proposed method

This section proposes a simplified procedure to calculate the reliability index for non-normal distributions. As a creative idea, it is possible to formulate any function in terms of the infinite sum of known functions. Taylor (1715) developed an approach to formulate the function, which today is called as Taylor's series. Based on Taylor's series, under certain conditions (differentiable function), it is possible to formulate any function in terms of a polynomial function. Also, Fourier (1822) expressed that: under particular conditions (periodic function), any function can be formulated with regard to the summation of the series of the periodic functions (such as: *sine* and *cosine*). In this research, it is stated that under the certain conditions (continuity condition/continuous distribution) any distribution can be formulated in terms of the series of the Gaussian functions, Eq. (17), (see Weisstein 2014b) to see the properties of the Gaussian function). This assumption also was introduced by Hall (2013) in forms of bootstrap and Edgeworth expansion.

$$f_X(x) = \sum_{i=1}^{n} a_i Normal(x, \mu_i, \sigma_i) \tag{17}$$

where,

$a_i =$ constant coefficient, which can be positive or negative and can be determined using the regression methods

Therefore, the limit state function can be written in terms of the series of Gaussian functions. Since Gaussian functions can be converted to the fraction of normal

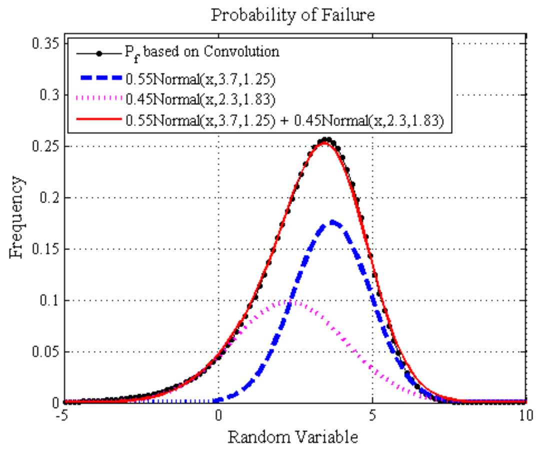


Fig. 7 Curve Fitting over the limit state function of while R=$Normal(x, 5,1)$ and $Q = Gamma(x, 2,1)$

distributions. Accordingly, the reliability index is defined based on the summation of the reliability indices of normal distributions.

$$\beta = a_1\beta_1 + a_2\beta_2 + \cdots + a_i\beta_i + \cdots + a_n\beta_n \tag{18}$$

where,

a_i= constant coefficients

β_i= reliability indices

Currently there is an accessibility to the advanced mathematical software, such as MATLAB, herein, the procedure to determine the reliability index is summarized as follows

1- Using the convolution operator to compute the probability of failure based on the proposed Eq. (15).

2- Fitting a distribution by utilizing the Gaussian function, this was defined in MATLAB curve fitting documentary. In this step the constant coefficients will be determined based on the regression methods.

3- Calculating the reliability index based on the Eq. 18.

To illustrate the proposed simplified method, herein, the presented examples (E.g., No. 1 and No. 2) from the previous section are considered. Considering E.g., No. 1, where both load and resistance are normally distributed, where it was observed that the limit state function is normally distributed for the given limit state function. Therefore, there is no need to fit Gaussian series over its limit state function (which was normally distributed). However, for the second example (E.g., No. 2), where the resistance is normally distributed and load follows Gamma distribution, the obtained probability of distribution for the intended limit state function does not behave as a normal distribution, therefore, in order to simply determine the reliability index, it is decent to utilize the proposed simplified approach (Eq. (18)). Following is the procedure to determine the reliability index for E.g., No. 2 using the simplified approach. As it was stated, the first step is to determine the probability of failure by taking the advantage of the convolution operation.

$$P_f = Gamma(x, 2,1) * Normal(x, 5,1)$$
$$= \int_{-\infty}^{+\infty} \tau\, exp(-\tau) \frac{1}{\sqrt{2\pi}} exp\left(-\frac{1}{2}(x+\tau-5)^2\right) d\tau$$

The equation above is implicitly solved by MATLAB. The second step is to fit a curve over the obtained distribution of limit state function using Gaussian series. This step is also accomplished by MATLAB. As can be seen in Fig. 7, using only the combination of two Gaussian functions can sufficiently provide an appropriate fitting curve.

Eventually, the probability of failure and the reliability index is computed. The reliability index is calculated based on summation of the reliability index of two normal distributions (Eq. (18)).

$$f_X(x) \approx 0.55Normal(x, 3.7, 1.25)$$
$$+0.45Normal(x, 2.3, 1.83)$$
$$P_f = 0.056$$
$$\beta = a_1\beta_1 + a_2\beta_2 = 0.55\left(\frac{3.7}{1.25}\right) + 0.45\left(\frac{2.3}{1.83}\right) = 2.19$$

5. Conclusions

The conventional definition of the reliability index was given by Basler based on normal distribution behavior of the limit state function, which is widely used in structural reliability analysis. However, if the distribution for the considered limit state function does not necessary behaves as a normal distribution, therefore, there is a need to define a new methodology to determine the reliability index for non-normal distributions. In this research, primarily, by taking the advantage of the convolution theorem, a new formula was defined to determine the probability of failure for the non-normal distribution of the limit states function (Eq. (15)). Accordingly, a closed-form solution was introduced to compute the reliability index. The proposed approach was verified using Monte Carlo simulation. It was observed that although the proposed formula can be concluded to the accurate result, because of the need for the advance mathematical knowledge, this approach is not perfectly convenient for engineering applications. Therefore, this paper introduced a simplified method to calculate the reliability index. The simplified method was established based on the fitting curve over the distribution of the limit state function. The fitting curve was generated with regard to the series of Gaussian functions. Eventually, as the state of the art, using a series of normal distributions is recommended to determine the reliability index for the non-normal limit state functions (Eq. (16) and Eq. (18)). The advantage of the proposed formulas in this paper is to determine the probability of failure and reliability index for the given limit state function which its distribution does not behave as normal distribution. In future, the influence of the proposed reliability index for non-normal distributions on design code calibration should be investigated for different limit state functions. In order to achieve this fact, it is required to perform extensive studies to first determine the reliability index for the intended limit state function and validity of the results based on past structural failure observations, which can be targeted for future studies.

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