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Accurate semi-analytical solution for nonlinear vibration of conservative mechanical problems

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Abstract. In this paper, it has been tried to propose a new semi analytical approach for solving nonlinear vibration of conservative systems. Hamiltonian approach is presented and applied to high nonlinear vibration systems. Hamiltonian approach leads us to high accurate solution using only one iteration. The method doesn't need any small perturbation and sufficiently accurate to both linear and nonlinear problems in engineering. The results are compared with numerical solution using Runge-Kutta-algorithm. The procedure of numerical solution are presented in detail. Hamiltonian approach could be simply apply to other powerfully non-natural oscillations and it could be found widely feasible in engineering and science.

Keywords: Hamiltonian approach; nonlinear vibration; numerical solution

1. Introduction

One of the most important an interesting area in nonlinear phenomena is to prepare an easy to apply approach for solving them especially in engineering science. Many practical engineering problems are modeled as oscillatory systems. Generally, it's very difficult o have an exact solution for nonlinear problems, therefore in recent years many scientists have been working on semi-analytical methods to solve nonlinear problems. The traditional methods has some limitations. To overcome the limitations of the traditional methods, some new approaches have been presented in recent years such as: Hamiltonian approach (He 2010, Xu 2010), Adomian decomposition method (Luo 2005, Ramana 2014), Differential transformation method (Arikoglu and Ozkol,2005), Energy balance method (Jamshdi and Ganji 2010), Max-Min approach (Shen and Mo 2009); improved Amplitude-frequency Formulation (He 2008) and other analytical methods (Bayat *et al*. 2013a, b, 2014, 2016a,.b, Bayat and Pakar 2011, 2012, 2013, 2015, Bayat and Abdollahzade 2011, Kutanaei *et al*. 2011, Cveticanin 2012, 2015, Edalati *et al*. 2016)were used to handle strongly nonlinear systems. Among these methods, Hamiltonian approach (HA) is considered to solve the nonlinear vibration of a solid circular sector object in this paper.

The paper has been collocated as follows: section 1is the introduction on the recent advances in nonlinear vibrations. In section 2 we consider the mathematical formulation of the problem, we describe basic concept of Hamiltonian approach in Section 3. Then for section 4, applications of Hamiltonian approach have been studied, to demonstrate the applicability and preciseness of the method. In section 5, some comparisons between analytical and numerical solutions are presented. Eventually we show that Hamiltonian approach can converge to a precise cyclic solution for nonlinear systems.

2. Solid circular sector object formulation

A homogeneous solid circular sector object with angle (α) and radius (R) as shown in Fig. 1 that rolls in an oscillatory motion back and for thon a flat stationary support, with no sliding effect. Obviously when α becomes radian, no oscillatory swinging motion will occur. It may be easily verified that the governing equation of the oscillation is as follow

$$\left(\frac{3}{2}R^2-\frac{4\sin(\alpha)}{3\alpha}R\cos(\theta)\right)\ddot{\theta}+ R\left(\frac{2R\sin(\alpha)}{3\alpha}\sin(\theta)\right)\dot{\theta}^2+\left(\frac{2\sin(\alpha)}{3\alpha}g\right)\sin(\theta)=0 \quad (1)$$
$$\theta(0)=A,\quad \dot{\theta}(0)=0,$$

Where the geometrical parameters are shown in Fig. 1. The height of Mass center obtained as below

$$\bar{y}=\frac{2R\sin(\alpha)}{3\alpha} \quad (2)$$

Introducing the dimensionless time variable

$$\bar{t}=\sqrt{\frac{1}{\bar{y}}t}=\left(\sqrt{\frac{2R\sin(\alpha)}{3\alpha}}\right)^{-1}t. \quad (3)$$

Eq. (1) becomes

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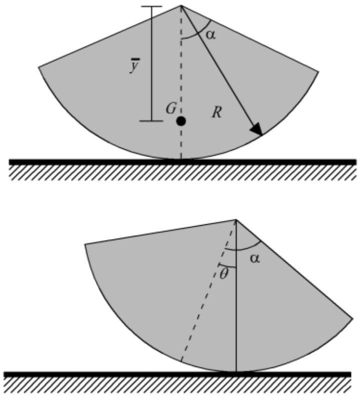


Fig. 1 Geometric parameters of the homogeneous solid circular sector object

$$\left(\frac{3}{2}R^2-\cos(\theta)\right)\ddot{\theta}+R\,\bar{y}\sin(\theta)\dot{\theta}^2+g\,\bar{y}\sin(\theta)=0 \qquad (4)$$
$$\theta(0)=A,\qquad \dot{\theta}(0)=0.$$

And by introducing the dimensionless geometrical parameter

$$\lambda=\frac{\bar{y}}{R}=\frac{2\sin(\alpha)}{3\alpha} \qquad (5)$$

Eq. (4) becomes

$$\left(\frac{3}{2\lambda}-2\cos(\theta)\right)\ddot{\theta}+\sin(\theta)\dot{\theta}^2+\frac{g\,\lambda}{R}\sin(\theta)=0 \qquad (6)$$
$$\theta(0)=A,\quad \dot{\theta}(0)=0.$$

3. Basic idea of Hamiltonian approach

Recently, He (2010) has proposed the Hamiltonian approach to overcome the shortcomings of the energy balance method. This approach is a kind of energy method with a vast application in conservative oscillatory systems. In order to clarify this approach, consider the following general oscillator

$$\ddot{\theta}+f\,(\theta,\dot{\theta},\ddot{\theta})=0 \qquad (7)$$

With initial conditions

$$\theta(0)=A,\qquad \dot{\theta}(0)=0. \qquad (8)$$

Oscillatory systems contain two important physical parameters, i.e., the frequency ω and the amplitude of oscillation A. It is easy to establish a variational principle for Eq. (7), which reads

$$J(\theta)=\int_0^{T/4}\left\{-\frac{1}{2}\dot{\theta}^2+F(\theta)\right\}dt \qquad (9)$$

Where T is period of the nonlinear oscillator, $\partial F/\theta\partial=f$;. In the Eq. (9), $\frac{1}{2}\dot{\theta}^2$ is kinetic energy and $F(\theta)$ potential energy, so the Eq. (9) is the least Lagrangian action, from which we can immediately obtain its Hamiltonian, which reads

$$H(\theta)=\frac{1}{2}\dot{\theta}^2+F(\theta)=\text{constant} \qquad (10)$$

From Eq. (10), we have

$$\frac{\partial H}{\partial A}=0 \qquad (11)$$

Introducing a new function, $\bar{H}(\theta)$, defined as

$$\bar{H}(u)=\int_0^{T/4}\left\{\frac{1}{2}\dot{\theta}^2+F(\theta)\right\}dt=\frac{1}{4}TH \qquad (12)$$

Eq. (11) is, then, equivalent to the following one

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial T}\right)=0 \qquad (13)$$

Or

$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right)=0 \qquad (14)$$

From Eq. (14) we can obtain approximate frequency-amplitude relationship of a nonlinear oscillator.

4. Application of Hamiltonian approach

In this section, the Hamiltonian approach is applied to the governing equation of solid circular sector object. By using the Taylor's series expansion for cos($\theta(t)$), sin($\theta(t)$) and applying them in Eq. (6) we can re-write Eq. (6) in the following form

$$\left(\frac{3}{2}R^2-\left(1-\tfrac{1}{2}\theta^2+\tfrac{1}{24}\theta^4\right)\right)\ddot{\theta}+ \qquad (15)$$
$$R\,\bar{y}\left(\theta-\tfrac{1}{6}\theta^3\right)\dot{\theta}^2+g\,\bar{y}\left(\theta-\tfrac{1}{6}\theta^3\right)=0$$
$$\theta(0)=A,\qquad \dot{\theta}(0)=0.$$

The Hamiltonian of Eq. (15) is constructed as

$$H=\frac{3}{4\lambda}\ddot{\theta}-\dot{\theta}^2+\frac{1}{2}\dot{\theta}^2\theta^2-\frac{1}{24}\dot{\theta}^2\theta^4 + \frac{1}{2}\frac{\lambda}{r}g\,\theta^2-\frac{1}{24}\frac{\lambda g\theta^4}{r} \qquad (16)$$

Integrating Eq. (16) with respect to t from 0 to $T/4$, we have

$$\bar{H}=\int_0^{T/4}\left(\begin{array}{l}\frac{3}{4\lambda}\ddot{\theta}-\dot{\theta}^2+\frac{1}{2}\dot{\theta}^2\theta^2-\frac{1}{24}\dot{\theta}^2\theta^4\\ +\frac{1}{2}\frac{\lambda}{r}g\,\theta^2-\frac{1}{24}\frac{\lambda g\theta^4}{r}\end{array}\right)dt \qquad (17)$$

Assume that the solution can be expressed as

$$\theta(t)=A\cos(\omega t) \qquad (18)$$

Table 1 Comparison of frequency corresponding to various parameters of system

A	α	g	R	HA	RKM	Error %
$\pi/12$	$\pi/4$	10	4	2.011	2.052	1.993
$\pi/12$	$\pi/3$	10	2	1.903	1.891	0.610
$\pi/6$	$\pi/2$	10	1	1.567	1.576	0.575
$\pi/6$	$\pi/4$	10	1.5	2.469	2.502	1.328
$\pi/4$	$\pi/3$	10	0.5	3.163	3.245	2.522
$\pi/4$	$\pi/2$	10	3	0.844	0.842	0.252
$\pi/3$	$\pi/12$	10	1	2.689	2.742	1.951
$\pi/3$	$\pi/6$	10	3.5	1.346	1.360	1.080
$\pi/2$	$\pi/4$	10	2.5	1.037	1.059	2.011
$\pi/2$	$\pi/12$	10	1.5	1.518	1.545	1.753

Substituting Eq. (18) into Eq. (17), we obtain

$$\bar{H}=\int_0^{T/4}\begin{pmatrix}\frac{1}{2}\frac{\lambda g}{r}A^2\cos^2(\omega t)-\frac{1}{24}\frac{\lambda g}{r}A^4\cos^4(\omega t)+\\ \frac{3}{4\lambda}A^2\omega^2\sin^2(\omega t)-A^2\omega^2\sin^2(\omega t)+\\ \frac{1}{2}A^4\omega^2\sin^2(\omega t)\cos^2(\omega t)-\\ \frac{1}{24}A^6\omega^2\sin^2(\omega t)\cos^4(\omega t)\end{pmatrix}dt$$

$$=\int_0^{\pi/2}\begin{pmatrix}\frac{1}{2}\frac{\lambda g}{r}A^2\cos^2 t-\frac{1}{24}\frac{\lambda g}{r}A^4\cos^4 t+\\ \frac{3}{4\lambda}A^2\omega^2\sin^2 t-A^2\omega^2\sin^2 t+\\ \frac{1}{2}A^4\omega^2\sin^2 t\cos^2 t-\frac{1}{24}A^6\omega^2\sin^2 t\cos^4 t\end{pmatrix}dt \tag{19}$$

$$=\frac{1}{8}\frac{\lambda g\pi}{\omega r}A^2-\frac{1}{128}\frac{\lambda g\pi}{\omega r}A^4+\frac{3}{16}\frac{\omega\pi}{\lambda}A^2-\frac{1}{4}\omega\pi A^2+\frac{1}{32}\omega\pi A^4-\frac{1}{768}\omega\pi A^6$$

Setting

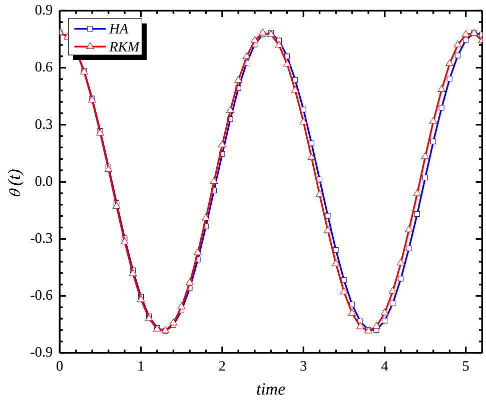
$$\frac{\partial}{\partial A}\left(\frac{\partial\bar{H}}{\partial(1/\omega)}\right)=\frac{1}{4}\frac{\lambda g\pi}{r}A-\frac{1}{32}\frac{\lambda g\pi}{r}A^3+\frac{3}{8}\frac{\pi}{\lambda}\omega^2A-\frac{1}{2}\pi\omega^2A+\frac{1}{8}\pi\omega^2A^3-\frac{1}{128}\pi\omega^2A^5 \tag{20}$$

Solving the above equation, an approximate frequency as a function of amplitude equal to

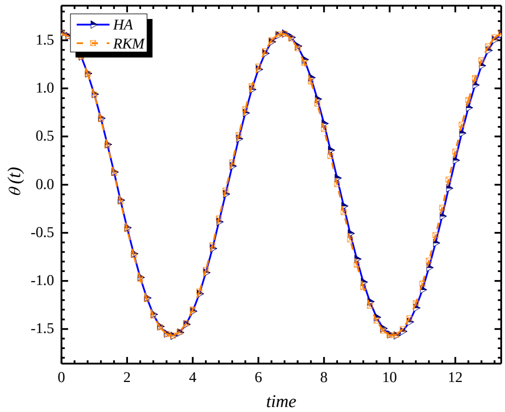
$$\omega_{HA}=\sqrt{\frac{4\lambda^2 g\left(A^2-8\right)}{r\left(A^4\lambda-16A^2\lambda+64\lambda-48\right)}} \tag{21}$$

Hence, the approximate solution can be readily obtained

$$\theta(t)=A\cos\left(\sqrt{\frac{4\lambda^2 g\left(A^2-8\right)}{r\left(A^4\lambda-16A^2\lambda+64\lambda-48\right)}}\;t\right) \tag{22}$$



(I)



(II)

Fig. 2 Comparison of analytical solution of $\theta(t)$ based on time with the numerical solution for (I): $\alpha=\pi/6$, $\alpha=\pi/4$, g=10, R=1.5 (II): $\alpha=\pi/3$, $\alpha=\pi/2$, g=10, R=2.5

5. Results and discussions

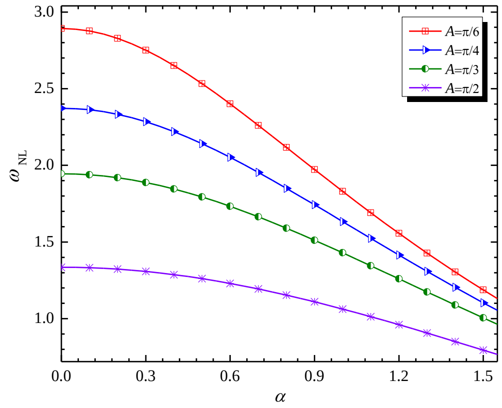
In this section, to show the accuracy of the presented approach for different parameters value, it has been considered different cases. The results of the Hamiltonian approach and numerical solution are compared in Table 1 for different parameters point values of A, α, R_1, R_2, g. The maximum relative error between the Energy balance method results and numerical results is 2.54%. Fig. 2 is comparison of analytical solution of $\theta(t)$ based on time with the numerical solution for two different cases:

(I): $\alpha=\pi/6,\ A=\pi/4,\ g=10,\ R=1.5$

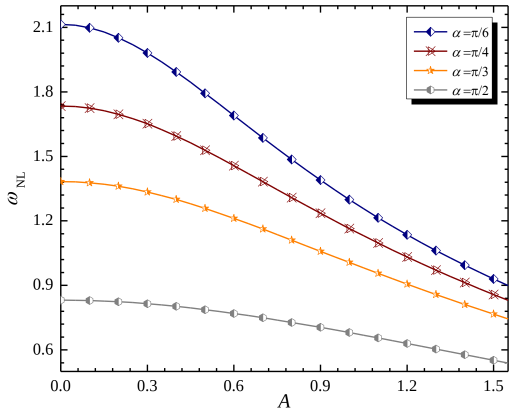
(II): $\alpha=\pi/3,\ A=\pi/2,\ g=10,\ R=2.5$.

Fig. 3 is shown the effect of amplitude and α on nonlinear frequency for (a): g=10, R=1.5 (b): g=10, R=4. Fig. 4 is the effect of α and R on nonlinear frequency for g=10, $A=\pi/2$.

To show the simultaneous effect of the parameters, a sensitive analysis has been carried out for three different case in 3D plots. It is obvious from the figures the Hamiltonian approach has an excellent agreement with the numerical solution and quickly convergent and valid for a wide range of vibration amplitudes and initial conditions. The accuracy of the results shows that the Hamiltonian approach can be potentiality used for the analysis of strongly nonlinear oscillation problems accurately.



(a)



(b)

Fig. 3 Effect of amplitude and α on nonlinear frequency for (a): g=10, R=1.5 (b): g=10, R=4

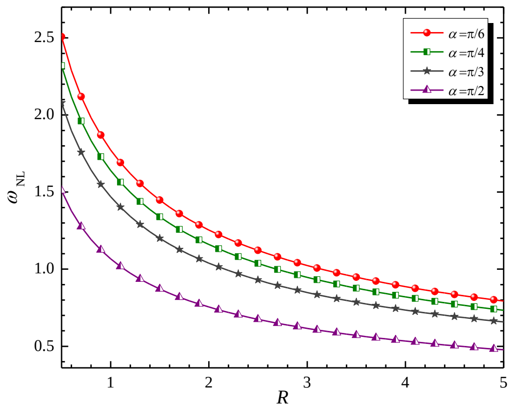


Fig. 4 Effect of α and R on nonlinear frequency for g=10, A=π/2

6. Conclusions

In this study, a new quite uncomplicated approach has been presented. Hamiltonian approach is utilized for analyzing the nonlinear vibration of a solid circular sector object. Only one iteration leads us to high accurate analytical solution with the maximum error less than 3%. It has been shown that the Hamiltonian approach is very efficient, comfortable and sufficiently exact in engineering problems. It has been demonstrated that the Hamiltonian approach can be simply extended to any nonlinear

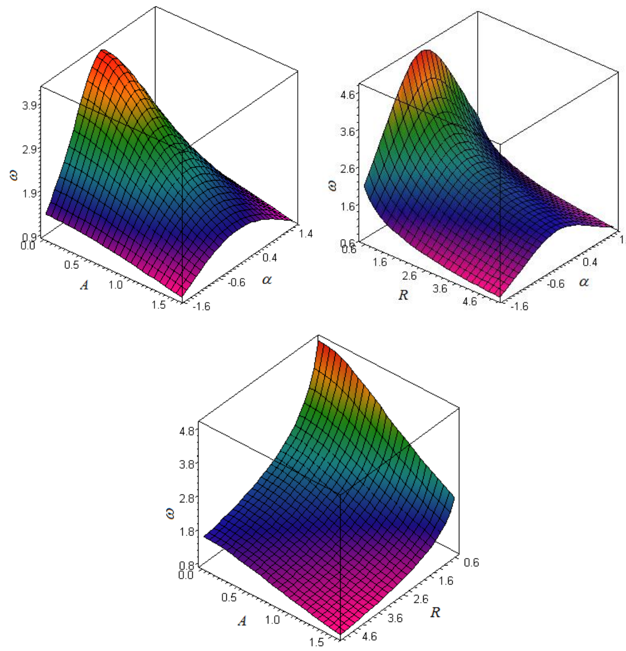


Fig. 5 Sensitivity analysis of nonlinear frequency for different parameters of the system

conservative equation for the analysis of nonlinear systems. The obtained results from the approximate analytical solutions are in excellent agreement with the corresponding numerical solutions.

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Appendix: Basic idea of Runge-Kutta's Method (RKM)

The Runge-Kutta method is an important iterative method for the approximation solutions of ordinary differential equations. These methods were developed by the German mathematician Runge and Kutta around 1900. For simplicity, we explain one of the important methods of Runge-Kutta methods, called forth-order Runge-Kutta method.

Consider an initial value problem be specified as follows

$$\dot{\theta} = f\ (t, \theta), \quad \theta(t_0) = \theta_0 \tag{A.1}$$

θ is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} \theta_{n+1} &= \theta_n + \frac{1}{6} h (k_1 + 2k_2 + 2k_3 + k_4), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{A.2}$$

for n=0, 1, 2, 3, . . . , using

$$\begin{aligned} k_1 &= f\ \left(t_n, \theta_n\right), \\ k_2 &= f\ \left(t_n + \frac{1}{2}h,\ \theta_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\ \left(t_n + \frac{1}{2}h,\ \theta_n + \frac{1}{2}hk_3\right), \\ k_4 &= f\ \left(t_n + h,\ \theta_n + hk_3\right). \end{aligned} \tag{A.3}$$

Where θ_{n+1} is the RK4 approximation of $\theta(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right hand side per step h.