Structural Engineering and Mechanics, *Vol. 61, No. 2 (2017) 209-220*
DOI: https://doi.org/10.12989/sem.2017.61.2.209

Large amplitude free vibration analysis of functionally graded nano/micro beams on nonlinear elastic foundation

AliReza Setoodeh* and Mohammad Rezaei[a]

Department of Mechanical and Aerospace Engineering, Shiraz University of Technology, Shiraz 71555, Iran

(Received June 25, 2016, Revised August 15, 2016, Accepted August 15, 2016)

Abstract. The purpose of this paper is to study the geometrically nonlinear free vibration of functionally graded nano/micro beams (FGNBs) based on the modified couple stress theory. For practical applications, some analytical expressions of nonlinear frequencies for FGNBs on a nonlinear Pasternak foundation are developed. Hamilton's principle is employed to obtain nonlinear governing differential equations in the context of both Euler-Bernoulli and Timoshenko beam theories for a comprehensive investigation. The modified continuum theory contains one material length scale parameter to capture the size effect. The variation of two-constituent material along the thickness is modeled using Reddy's power-law. Also, the Mori-Tanaka method as an accurate homogenization technique is implemented to estimate the effective material properties of the FGNBs. The results are presented for both hinged-hinged and clamped-clamped boundary conditions. The nonlinear partial differential equations are reduced to ordinary differential equations using Galerkin method and then the powerful method of homotopy analysis is utilized to obtain the semi-analytical solutions. Eventually, the presented analytical expressions are used to examine the influences of the length scale parameter, material gradient index, and elastic foundation on the nonlinear free vibration of FGNBs.

Keywords: modified couple stress theory; nonlinear vibration; functionally graded nano/micro-beam; homotopy analysis method

1. Introduction

Recently a new class of composite materials known as functionally graded materials (FGMs) has attracted considerable attention in many various industrial fields. These inhomogeneous composites usually are made from a mixture of metals and ceramics. In these materials, the mechanical properties change from one surface to another. The capability of functionally graded (FG) materials can be used in nano/microstructures by employing modern spattering machines. Meanwhile, nano/micro-beams have been widely used in biosensors, atomic force microscope and many other micro/nano-electro-mechanical systems (Schmid *et al*. 2009, Kahrobaiyan *et al*. 2010, Younis *et al*. 2003). However, the properties of nano/micro-beams are closely related to their microstructures. To understand the mechanical behavior of such beams, it is significant to consider the size effect that resulting from their microstructures. Since the classical continuum theory could not captures the size effects, thus the non-classical theories such as classical couple stress theory (Mindlin and Tiersten 1962), the nonlocal elasticity theory (Eringen 1972), and the strain gradient theory (Lam *et al.* 2003) have been proposed.

Linear/nonlinear vibration is very common for nano/micro-beams subjected to external forces in some basic components of new nanoscale devices such as oscillators, and actuators. In this regard, some studies have been performed by employing various modified continuum theories together with different numerical or analytical solutions (Janghorban and Zare 2011, Bayat *et al.* 2013, Thai and Choi 2015, Sedighi *et al.* 2014, Bagdatli 2015, Setoodeh *et al.* 2015, Malekzadeh and Shojaee 2015, Setoodeh *et al.* 2016, Ehyaei *et al.* 2016, Ebrahimi and Shafiei 2016). Specifically, Malekzadeh and shojaee (2013) studied surface and nonlocal effects on the nonlinear flexural free vibration of elastically supported non-uniform nano-beams using differential quadrature method (DQM) based on Euler-Bernoulli and Timoshenko beam theories. Shenas and Malekzadeh (2016) presented the influences of thermal environment together with the geometrical parameters on the free vibration characteristics of the FG quadrilateral micro-plates based on the modified strain gradient theory using the Chebyshev-Ritz method. Ansari *et al*. (2016) investigated the coupled longitudinal-transverse-rotational free vibration of post-buckled FG first-order shear deformable micro/nano-beams employing generalized differential quadrature (GDQ) method. They used Mindlin's strain gradient theory to capture the size dependent features of the nanostructures. Jia *et al*. (2015) examined the size effect on the free vibration of geometrically nonlinear FG micro-beams under electrical actuation and temperature change in the context of Euler-Bernoulli beam theory using DQM. Taeprasartsit (2013) developed the large amplitude free vibration of thin Euler-Bernoulli FG beams based on finite element method.

However, only few researchers have paid attention to

*Corresponding author, Associate Professor
E-mail: setoodeh@sutech.ac.ir
[a]Graduate Student

ISSN: 1225-4568 (Print), 1598-6217 (Online)

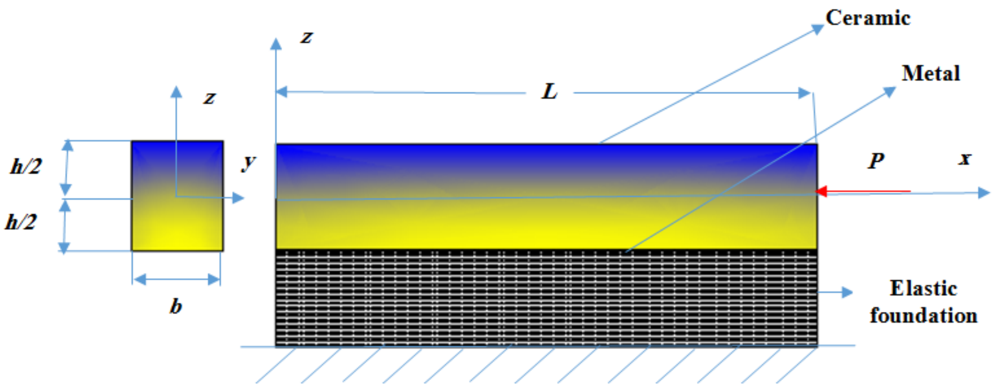

Fig. 1 Schematic configuration of FG nano/micro-beam

the nonlinear vibration of FG nano/micro-beams as a new potential application of nanostructures. Asghari *et al.* (2011) presented a size-dependent formulation for Timoshenko micro-beams made of FG material using modified couple stress theory (MCST). They obtained closed-form analytic expressions for the bending and axial deformations of beams and also investigated the free vibration of simply supported FG beams utilizing the Fourier series expansions as a case study. Ke *et al.* (2012) determined the nonlinear vibration frequencies of FG Timoshenko micro-beams with different boundary conditions employing DQM together with an iterative algorithm. They investigated size effect based on MCST and showed that the size effect on the nonlinear vibration is significant only when the thickness of beam has a similar value compared to the length scale parameter. Nateghi and Salamat-talab (2013) presented thermal effect on the size-dependent behavior of FG micro-beams using classical and first order shear deformation theories in context of MCST using GDQ method. Setoodeh and Afrahim (2014) studied nonlinear vibrational behavior of FG Euler-Bernoulli micro-pipes conveying fluid based on strain gradient theory. They used homotopy analysis method (HAM) to obtain the results. According to the available literature, no analytical expressions for the nonlinear frequencies of FG nano/micro-beams have been derived so far.

The main target of this paper is to develop size-dependent analytical expressions for the nonlinear vibration of FG nano/micro-beams using homotopy analysis method. A microstructure-dependent nonlinear Euler-Bernoulli (EBT) and Timoshenko beam (TBT) theories which account for through-thickness power-law variation of a two-constituent material are developed in the context of modified couple stress theory. The effects of nonlinear elastic foundation and boundary conditions are taken into account.

2. Nonlinear size-dependent equations of motion

Fig. 1 shows a FG nano/micro-beam with length L, width b, and thickness h made from a mixture of ceramic and metal. In this investigation the top surface of micro-beam ($z=h/2$) is ceramic-rich and the bottom surface ($z=-h/2$) is metal-rich. The beam is resting on a nonlinear Pasternak foundation with linear coefficient k_l, nonlinear coefficient k_{nl} and shear coefficient k_G. The effective material properties of FGNBs are estimated through the Mori-Tanaka homogenization technique as follow (Ke *et al.* 2012)

$$\frac{K_e-K_m}{K_c-K_m}=\frac{V_c}{1+V_m\left(K_c-K_m\right)/\left(K_m+4\mu_m/3\right)} \tag{1}$$

$$\frac{\mu_e-\mu_m}{\mu_c-\mu_m}=\frac{V_c}{1+V_m\left(\mu_c-\mu_m\right)/\left[\mu_m+\mu_m\left(9K_m+8\mu_m\right)/6\left(K_m+2\mu_m\right)\right]} \tag{2}$$

where K, μ and V denote the bulk modulus, the shear modulus and the volume fraction of the materials, respectively. The subscripts m and c stand for metal and ceramic phases, respectively and e denotes the corresponding effective property. The volume fractions of ceramic and metal phases are related by

$$V_m+V_c=1 \tag{3}$$

$$V_c(z)=\left(0.5+z/h\right)^n \tag{4}$$

In above formula n is the material gradient index. The effective values of the Young's modulus E and Poisson's ratio ν can be expressed in terms of K_e and μ_e as follows

$$E(z)=\frac{9K_e\mu_e}{3K_e+\mu_e} \tag{5}$$

$$\nu(z)=\frac{3K_e-2\mu_e}{6K_e+2\mu_e} \tag{6}$$

The mass density of the beam is also given by the rule of mixture as

$$\rho(z)=\rho_c V_c+\rho_m V_m \tag{7}$$

The equations of motion of the FGNBs are derived using Hamilton's principle. The principle can be stated as

$$\int_{t_1}^{t_2}\delta\left(T-U\right)dt=0 \tag{8}$$

where T is the kinetic energy, and U is the potential energy including the strain and elastic foundation energies. According to the modified couple stress theory, the strain energy can be written as

$$U_e = \frac{1}{2}\int_V \left(\sigma_{ij}\varepsilon_{ij} + m_{ij}\chi_{ij}\right) dV \qquad (i, j = x, y, z) \quad (9)$$

where σ_{ij} and ε_{ij} denote the components of the stress and strain tensors; m_{ij} *and* χ_{ij} represent respectively the deviatoric part of the couple stress tensor and symmetric curvature tensor defined as follows

$$\sigma_{ij} = \lambda\varepsilon_{kk}\delta_{ij} + 2\mu\varepsilon_{ij}, \ \lambda = E\nu/(1+\nu)(1-2\nu), \quad \mu = E/2(1+\nu) \quad (10)$$

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}) \quad (11)$$

$$\chi_{ij} = \frac{1}{2}(\theta_{i,j} + \theta_{j,i}) \quad (12)$$

$$m_{ij} = 2l^2\mu\chi_{ij} \quad (13)$$

where in all relations (i, j, k=x, y, z), λ and μ are the Lame's constants, l denotes the material length scale parameter. Also, u_i are the components of the displacement vector and θ_i are the components of the rotation vector which defined as

$$\theta_x = \frac{1}{2}\left(\frac{\partial u_z}{\partial y} - \frac{\partial u_y}{\partial z}\right), \theta_y = \frac{1}{2}\left(\frac{\partial u_x}{\partial z} - \frac{\partial u_z}{\partial x}\right), \quad \theta_z = \frac{1}{2}\left(\frac{\partial u_y}{\partial x} - \frac{\partial u_x}{\partial y}\right) \quad (14)$$

2.1 Euler-Bernoulli FGNB theory

The displacement field (u_x, u_z) along the coordinate directions (x, z) for an Euler-Bernoulli beam can be given in terms of (u, w) which are the displacements along the (x, z) coordinate directions of a point on the mid-plane of the beam

$$u_x(x,z,t) = u(x,t) - z\,\frac{\partial w(x,t)}{\partial x}, \quad u_y(x,z,t) = 0, \ u_z(x,z,t) = w(x,t) \quad (15)$$

According to Eqs. (11) and (15), the only nonzero nonlinear strain based on the von Kármán assumptions is

$$\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\,\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2 \quad (16)$$

In view of Eqs. (12)-(15), the nonzero components of rotation vector, curvature tensor and couple stress tensor can be obtained as

$$\theta_y = -\frac{\partial w}{\partial x}, \quad \chi_{xy} = -\frac{1}{2}\frac{\partial^2 w}{\partial x^2}, \quad m_{xy} = -\mu l^2\frac{\partial^2 w}{\partial x^2} \quad (17)$$

Thus, the potential energy can be expressed as

$$U = \frac{1}{2}\int_0^L\int_A \left(\sigma_{xx}\varepsilon_{xx} + 2m_{xy}\chi_{xy}\right) dA\,dx + \frac{1}{2}\int_0^L\left(k_l w^2 + k_G\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}k_{nl}w^4\right)dx \quad (18)$$

After substituting appropriate components in Eq. (18), the potential energy can be written as

$$U = \frac{1}{2}\int_0^L\left[A_1\left\{\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{4}\left(\frac{\partial w}{\partial x}\right)^4 + \frac{\partial u}{\partial x}\left(\frac{\partial w}{\partial x}\right)^2\right\} - 2B_1\left\{\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2} + \frac{1}{2}\frac{\partial^2 w}{\partial x^2}\left(\frac{\partial w}{\partial x}\right)^2\right\} + C_1\left(\frac{\partial^2 w}{\partial x^2}\right)^2 + D_1 l^2\left(\frac{\partial^2 w}{\partial x^2}\right)^2\right] dx + \frac{1}{2}\int_0^L\left(k_l w^2 + k_G\left(\frac{\partial w}{\partial x}\right)^2 + \frac{1}{2}k_{nl}w^4\right)dx \quad (19)$$

where

$$\{A_1, B_1, C_1\} = b\int_{-\frac{h}{2}}^{+\frac{h}{2}}\left(\lambda(z) + 2\mu(z)\right)\{1, z, z^2\}dz, \quad D_1 = b\int_{-\frac{h}{2}}^{\frac{h}{2}}\mu(z)\,dz \quad (20)$$

The kinetic energy can be obtained as

$$T = \frac{1}{2}\int_0^L m_0\left[\left(\frac{\partial u}{\partial t}\right)^2 + \left(\frac{\partial w}{\partial t}\right)^2\right]dx, \quad m_0 = b\int_{-\frac{h}{2}}^{\frac{h}{2}}\rho\,dx \quad (21)$$

Finally after employing Hamilton's principle and using calculus of variations and then collecting the coefficients of (δu, δw), two nonlinear differential equations are obtained as follows

$$m_0\frac{\partial^2 u}{\partial t^2} = A_1\frac{\partial^2 u}{\partial x^2} + \frac{A_1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^2\right] - B_1\frac{\partial^3 w}{\partial x^3} \quad (22)$$

$$m_0\frac{\partial^2 w}{\partial t^2} = \frac{A_1}{2}\frac{\partial}{\partial x}\left[\left(\frac{\partial w}{\partial x}\right)^3\right] + A_1\frac{\partial}{\partial x}\left(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x}\right) + B_1\frac{\partial^3 u}{\partial x^3} + \frac{B_1}{2}\frac{\partial^2}{\partial x^2}\left[\left(\frac{\partial w}{\partial x}\right)^2\right] - B_1\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial w}{\partial x}\right) - C_1\frac{\partial^4 w}{\partial x^4} - D_1 l^2\frac{\partial^2}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}\right) - k_l w + k_G\frac{\partial^2 w}{\partial x^2} - k_{nl}w^3 \quad (23)$$

The stress resultants for the nano/micro-beam are defined as

$$Y_{xy} = \int_A m_{xy}\,dA = -l^2 D_1\frac{\partial^2 w}{\partial x^2} \quad (24)$$

$$N_x = \int_A \sigma_{xx}\,dA = A_1\left(\frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial w}{\partial x}\right)^2\right) - B_1\frac{\partial^2 w}{\partial x^2} \quad (25)$$

$$M_x = \int_A \sigma_{xx} z\, dA = B_1\left(\frac{\partial u}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2\right) - C_1 \frac{\partial^2 w}{\partial x^2} \qquad (26)$$

In view of Eqs. (25)-(26), Eqs. (22)-(23) can be rewritten as below

$$m_0 \frac{\partial^2 u}{\partial t^2} = \frac{\partial N_x}{\partial x} \qquad (27)$$

$$\begin{aligned} m_0 \frac{\partial^2 w}{\partial t^2} &= \frac{\partial^2 M_x}{\partial x^2} + \frac{\partial}{\partial x}\left(N_x \frac{\partial w}{\partial x}\right) \\ &- k_l w + k_G \frac{\partial^2 w}{\partial x^2} - k_{nl} w^3 - D_1 l^2 \frac{\partial^4 w}{\partial x^4} \end{aligned} \qquad (28)$$

As the value of longitudinal inertia is very small then Eq. (27) can be simplified as $N_x = N_{x0} = cte$. Thus, Eq. (25) can be written as (Ke *et al.* 2012)

$$\frac{\partial u}{\partial x} = \frac{1}{A_1}\left(N_{x0} + B_1 \frac{\partial^2 w}{\partial x^2} - \frac{1}{2} A_1 (\frac{\partial w}{\partial x})^2\right) \qquad (29)$$

By integrating Eq. (29) over the length of the micro-beam, one obtains

$$u(L) - u(0) = \frac{1}{A_1}\int_0^L \left[N_{x0} - \frac{A_1}{2}(\frac{\partial w}{\partial x})^2 + B_1 \frac{\partial^2 w}{\partial x^2}\right] dx \qquad (30)$$

If two ends of the beam are immovable, i.e., $u(L)=u(0)=0$, the following relations are respectively resulted from Eqs. (30) and (26)

$$N_{x0} = \frac{1}{L}\int_0^L \left(\frac{A_1}{2}(\frac{\partial w}{\partial x})^2 - B_1 \frac{\partial^2 w}{\partial x^2}\right) dx \qquad (31)$$

$$M_x = \frac{B_1}{A_1} N_{x0} + \left(\frac{B_1^2}{A_1} - C_1\right)\frac{\partial^2 w}{\partial x^2} \qquad (32)$$

By inserting Eq. (31) into (32) and then substituting the result into Eq. (28), two nonlinear governing equations are reduced to only one nonlinear equation as follow

$$\begin{aligned} m_0 \frac{\partial^2 w}{\partial t^2} &= \left\{\frac{1}{L}\int_0^L \left(\frac{A_1}{2}(\frac{\partial w}{\partial x})^2 - B_1 \frac{\partial^2 w}{\partial x^2}\right) dx + k_G\right\}\frac{\partial^2 w}{\partial x^2} \\ &+ \left(\frac{B_1^2}{A_1} - C_1 - D_1 l^2\right)\frac{\partial^4 w}{\partial x^4} - k_l w - k_{nl} w^3 \end{aligned} \qquad (33)$$

2.2 Timoshenko FGNB theory

The displacement field of the Timoshenko beam theory can be written as

$$\begin{aligned} &u_x(x,z,t) = u(x,t) - z\psi(x,t) \quad , \\ &u_y(x,z,t) = 0 \quad , \quad u_z(x,z,t) = w(x,t) \end{aligned} \qquad (34)$$

where u, ψ and w, are respectively the axial displacement, rotation and deflection of the FGNB. According to Eqs. (10)-(11) and the above displacement field, the nonzero components of von Kármán nonlinear strain and stress tensors are given by

$$\begin{aligned} &\varepsilon_{xx} = \frac{\partial u}{\partial x} - z\frac{\partial \psi}{\partial x} + \frac{1}{2}(\frac{\partial w}{\partial x})^2, \quad \varepsilon_{xz} = \frac{1}{2}(\frac{\partial w}{\partial x} - \psi) \\ &\sigma_{xx} = \left[\lambda(z) + 2\mu(z)\right]\varepsilon_{xx} \quad , \quad \sigma_{xz} = 2k_s \mu \varepsilon_{xz} \end{aligned} \qquad (35)$$

where k_s=5/6 denotes the shear correction factor. In a similar manner, the nonzero components of the rotation vector, curvature tensor, and couple stress tensor are

$$\begin{aligned} &\theta_2 = \frac{1}{2}(-\psi - \frac{\partial w}{\partial x}), \chi_{xy} = -\frac{1}{4}(\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}), \\ &m_{xy} = -\frac{1}{2} l^2 \mu (\frac{\partial \psi}{\partial x} + \frac{\partial^2 w}{\partial x^2}) \end{aligned} \qquad (36)$$

In view point of Eqs. (35)-(36), the potential energy is expressed as

$$\begin{aligned} U &= \frac{1}{2}\int_0^L \int_A \left(\sigma_{xx}\varepsilon_{xx} + 2\sigma_{xz}\varepsilon_{xz} + 2m_{xy}\chi_{xy}\right) dA\, dx \\ &+ \frac{1}{2}\int_0^L \left(k_l w^2 + k_G (\frac{\partial w}{\partial x})^2 + \frac{1}{2} k_{nl} w^4\right) dx \end{aligned} \qquad (37)$$

Using Eqs. (35)-(36), the potential energy is extended as

$$\begin{aligned} U = \frac{1}{2}\int_0^L \Bigg[& A_1\left\{(\frac{\partial u}{\partial x})^2 + \frac{1}{4}(\frac{\partial w}{\partial x})^4 + \frac{\partial u}{\partial x}(\frac{\partial w}{\partial x})^2\right\} \\ & + B_1\left\{-2\frac{\partial u}{\partial x}\frac{\partial \psi}{\partial x} - \frac{\partial \psi}{\partial x}(\frac{\partial w}{\partial x})^2\right\} \\ & + C_1 (\frac{\partial \psi}{\partial x})^2 + k_s D_1 \left\{(\frac{\partial w}{\partial x})^2 + \psi^2 - 2\psi\frac{\partial w}{\partial x}\right\} \\ & + \frac{D_1 l^2}{4}\left\{(\frac{\partial \psi}{\partial x})^2 + (\frac{\partial^2 w}{\partial x^2})^2 + 2\frac{\partial \psi}{\partial x}\frac{\partial^2 w}{\partial x^2}\right\} \\ & + k_l w^2 + k_G (\frac{\partial w}{\partial x})^2 + \frac{1}{2} k_{nl} w^4 \Bigg] dx \end{aligned} \qquad (38)$$

The kinetic energy in terms of displacement and rotation components is given by

$$\begin{aligned} T = \frac{1}{2}\int_0^L \Bigg\{ & I_1\left[(\frac{\partial u}{\partial t})^2 + (\frac{\partial w}{\partial t})^2\right] \\ & - 2I_2 \frac{\partial u}{\partial t}\frac{\partial \psi}{\partial t} + I_3 (\frac{\partial \psi}{\partial t})^2 \Bigg\} dx \end{aligned} \qquad (39)$$

where

$$\left\{I_1, I_2, I_3\right\} = b\int_{-\frac{h}{2}}^{\frac{h}{2}} \rho\left\{1, z, z^2\right\} dz \qquad (40)$$

Substituting expressions for δU, and δT into Eq. (8) and integrating-by-parts with respect to t as well as x to relieve the virtual variations (δu, $\delta \psi$, δw) of any differentiations, three coupled nonlinear governing equations are obtained as

$$I_2 \frac{\partial^2 \psi}{\partial t^2} - I_1 \frac{\partial^2 u}{\partial t^2} = B_1 \frac{\partial^2 \psi}{\partial x^2} - A_1 (\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}) \qquad (41)$$

$$I_2 \frac{\partial^2 u}{\partial t^2} - I_3 \frac{\partial^2 \psi}{\partial t^2} = B_1\left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}\right) \qquad (42)$$

$$-C_1\frac{\partial^2\psi}{\partial x^2}+k_sD_1\psi-k_sD_1\frac{\partial w}{\partial x}-\frac{D_1l^2}{4}\left(\frac{\partial^2\psi}{\partial x^2}+\frac{\partial^3 w}{\partial x^3}\right)$$

$$I_1\frac{\partial^2 w}{\partial t^2}=\frac{3}{2}A_1(\frac{\partial w}{\partial x})^2\frac{\partial^2 w}{\partial x^2}+$$
$$A_1\frac{\partial}{\partial x}(\frac{\partial u}{\partial x}\frac{\partial w}{\partial x})-B_1\frac{\partial}{\partial x}(\frac{\partial \psi}{\partial x}\frac{\partial w}{\partial x})$$
$$+k_sD_1(\frac{\partial^2 w}{\partial x^2}-\frac{\partial\psi}{\partial x})-\frac{D_1l^2}{4}(\frac{\partial^4 w}{\partial x^4}+\frac{\partial^3\psi}{\partial x^3})$$
$$-k_lw+k_G\frac{\partial^2 w}{\partial x^2}-k_{nl}w^3 \tag{43}$$

Eqs. (41)-(43) can be expressed in terms of the stress resultants as

$$I_1\frac{\partial^2 u}{\partial t^2}-I_2\frac{\partial^2\psi}{\partial t^2}=\frac{\partial N_x}{\partial x} \tag{44}$$

$$I_2\frac{\partial^2 u}{\partial t^2}-I_3\frac{\partial^2\psi}{\partial t^2}=\frac{\partial M_x}{\partial x}-Q_x+\frac{1}{2}\frac{\partial Y_{xy}}{\partial x} \tag{45}$$

$$I_1\frac{\partial^2 w}{\partial t^2}=\frac{\partial Q_x}{\partial x}+\frac{\partial}{\partial x}(N_x\frac{\partial w}{\partial x})+$$
$$\frac{1}{2}\frac{\partial^2 Y_{xy}}{\partial x^2}-k_lw+k_G\frac{\partial^2 w}{\partial x^2}-k_{nl}w^3 \tag{46}$$

where

$$N_x=\int_A\sigma_{xx}\,dA=A_1\left(\frac{\partial u}{\partial x}+\frac{1}{2}(\frac{\partial w}{\partial x})^2\right)-B_1\frac{\partial\psi}{\partial x}$$
$$M_x=\int_A\sigma_{xx}z\,dA=B_1\left(\frac{\partial u}{\partial x}+\frac{1}{2}(\frac{\partial w}{\partial x})^2\right)-C_1\frac{\partial\psi}{\partial x}$$
$$Y_{xy}=\int_A m_{xy}\,dA=-\frac{l^2}{2}D_1(\frac{\partial\psi}{\partial x}+\frac{\partial^2 w}{\partial x^2})$$
$$Q_x=\int_A\sigma_{xz}\,dA=k_sD_1(\frac{\partial w}{\partial x}-\psi) \tag{47}$$

In order to solve the above coupled nonlinear governing equation, they are rewritten in the following forms after some mathematical manipulations

$$\frac{\partial^2 u}{\partial t^2}=\bar{u}_1\frac{\partial^2 u}{\partial x^2}+\bar{u}_2\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}$$
$$+\bar{u}_3\frac{\partial^2\psi}{\partial x^2}+\bar{u}_4\psi+\bar{u}_5\frac{\partial w}{\partial x}+\bar{u}_6\frac{\partial^3 w}{\partial x^3} \tag{48}$$

$$\frac{\partial^2\psi}{\partial t^2}=\bar{\psi}_1\frac{\partial^2\psi}{\partial x^2}+\bar{\psi}_2\frac{\partial^2 u}{\partial x^2}+\bar{\psi}_3\frac{\partial w}{\partial x}\frac{\partial^2 w}{\partial x^2}$$
$$+\bar{\psi}_4\psi+\bar{\psi}_5\frac{\partial w}{\partial x}+\bar{\psi}_6\frac{\partial^3 w}{\partial x^3} \tag{49}$$

$$\frac{\partial^2 w}{\partial t^2}=\bar{w}_1\frac{\partial^2 w}{\partial x^2}(\frac{\partial w}{\partial x})^2+\bar{w}_2\frac{\partial w}{\partial x}\frac{\partial^2 u}{\partial x^2}+$$
$$\bar{w}_3\frac{\partial u}{\partial x}\frac{\partial^2 w}{\partial x^2}+\bar{w}_4\frac{\partial w}{\partial x}\frac{\partial^2\psi}{\partial x^2}+\bar{w}_5\frac{\partial\psi}{\partial x}\frac{\partial^2 w}{\partial x^2}$$
$$+\bar{w}_6\frac{\partial^2 w}{\partial x^2}+\bar{w}_7\frac{\partial\psi}{\partial x}+\bar{w}_8\frac{\partial^4 w}{\partial x^4}$$
$$+\bar{w}_9\frac{\partial^3\psi}{\partial x^3}+\bar{w}_{10}w+\bar{w}_{11}w^3 \tag{50}$$

where coefficients $\bar{u}_i,\bar{\psi}_i,\bar{w}_i$ are defined in Appendix A.

2.3 Boundary conditions

After using Hamilton's principle, the boundary conditions in terms of stress resultants for Euler-Bernoulli FGNB can be obtained as

$$\begin{aligned}&x=0,L:\\&u=0\quad or\quad N_x=0\\&w=0\quad or\quad N_x\frac{\partial w}{\partial x}+\frac{\partial M_x}{\partial x}+\frac{\partial Y_{xy}}{\partial x}+k_G\frac{\partial w}{\partial x}=0\\&\frac{\partial w}{\partial x}=0\quad or\quad M_x+Y_{xy}=0\end{aligned} \tag{51}$$

which are specified respectively for the hinged and clamped end supports according to Eqs. (52) and (53)

$$x=0,L:\quad u=0,\ w=0,\ M_x+Y_{xy}=0\ \text{(hinged-hinged)} \tag{52}$$

$$x=0,L:\quad u=0,\ w=0,\ \frac{\partial w}{\partial x}=0\ \text{(clamped-clamped)} \tag{53}$$

It can be noted that the third boundary condition in Eq. (52) is simply reduced to $\frac{\partial^2 w}{\partial x^2}=0$ by ignoring the nonlinear term.

In a similar way, the boundary conditions for Timoshenko FGNB can be written as

$$\begin{aligned}&x=0,L:\\&u=0\quad or\quad N_x=0\\&w=0\quad or\quad N_x\frac{\partial w}{\partial x}+Q_x+\frac{1}{2}\frac{\partial Y_{xy}}{\partial x}+k_G\frac{\partial w}{\partial x}=0\\&\psi=0\quad or\quad M_x+\frac{Y_{xy}}{2}=0\\&\frac{\partial w}{\partial x}=0\quad or\quad Y_{xy}=0\end{aligned} \tag{54}$$

In view of Eq. (54), the associated boundary conditions of hinged and clamped nano/micro-beams are identified as

$$x=0,L:\quad u=0,\ w=0,\ M_x+\frac{Y_{xy}}{2}=0,\ Y_{xy}=0 \quad \text{(hinged-hinged)} \tag{55}$$

$$x=0,L:\quad u=0,\ w=0,\ \psi=0,\ \frac{\partial w}{\partial x}=0 \quad \text{(clamped-clamped)} \tag{56}$$

Similarly, it should be noted that the third and fourth boundary conditions in Eq. (55) are suppressed to $\frac{\partial^2 w}{\partial x^2}=0$, $\frac{\partial \psi}{\partial x}=0$ when the nonlinear terms are neglected.

It is found from Eqs. (51) and (54) that the number of boundary conditions is increased to eight for the case of the non-classical Timoshenko beam model based on the modified couple stress theory in comparison to the six boundary conditions of classical beam theory (Ma *et al.* 2008).

3. Galerkin method

Separation of variable analysis and Galerkin procedure are used to obtain uncoupled nonlinear ordinary differential equations. The transverse displacement of FGNBs can be written as

$$w(x,t)=Q(x)W(t) \tag{57}$$

where $W(t)$ is a time dependent unknown function. For hinged-hinged beams, $Q(x)$ is defined as (Rao 2007)

$$Q(x)=\sin(\frac{\pi x}{L}) \tag{58}$$

And for the case of clamped-clamped beams, it can be written as follows

$$Q(x)=c\left[\cosh(\frac{qx}{L})-\cos(\frac{qx}{L})-\frac{\cosh(q)-\cos(q)}{\sinh(q)-\sin(q)}\left(\sinh(\frac{qx}{L})-\sin(\frac{qx}{L})\right)\right] \tag{59}$$

here, c=0.6297 and q=4.730041.

After substituting Eq. (57) into (33), multiplying the result by $Q(x)$ and then integrating over the length of the nano/micro-beam, one obtains the following nonlinear time-dependent ordinary differential equation for the Euler-Bernoulli beam theory

$$\ddot{W}+\alpha W+\beta W^2+\gamma W^3=0 \tag{60}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t. Also, α, β, γ are defined in Appendix A.

Similar to the Euler-Bernoulli theory, separation of variables can be used for Timoshenko nano/micro-beam by considering

$$u(x,t)=P(x)U(t),\quad \psi(x,t)=S(x)\Psi(t),\quad w(x,t)=Q(x)W(t) \tag{61}$$

The functions $P(x)$, $S(x)$, $Q(x)$ can be respectively selected for hinged-hinged and clamped-clamped boundary conditions according to Eqs. (62) and (63)

$$Q(x)=\sin(\frac{\pi x}{L}) \quad,\quad S(x)=\cos(\frac{\pi x}{L}) \quad P(x)=\sin(\frac{\pi x}{L}) \tag{62}$$

$$Q(x)=c\left[\cosh(\frac{qx}{L})-\cos(\frac{qx}{L})-\frac{\cosh(q)-\cos(q)}{\sinh(q)-\sin(q)}\left(\sinh(\frac{qx}{L})-\sin(\frac{qx}{L})\right)\right]$$
$$S(x)=\sin(\frac{2\pi x}{L})\;,\quad P(x)=\sin(\frac{\pi x}{L}) \tag{63}$$

By substituting Eqs. (61) into Eqs. (48)-(50) and multiplying the resulted equations respectively by $P(x)$, $S(x)$, $Q(x)$ and integrating over the length of the beam, three nonlinear time-dependent ordinary differential equations are obtained as

$$\begin{aligned}&\alpha_1 U+\alpha_2 W^2+\alpha_3\psi+\alpha_4 W=0\\&\ddot{\Psi}+\beta_1\Psi+\beta_2 U+\beta_3 W^2+\beta_4 W=0\\&\ddot{W}+\gamma_1 W+\gamma_2 W^3+\gamma_3\psi+\gamma_4 UW+\gamma_5\Psi W=0\end{aligned} \tag{64}$$

where α_i, β_i, γ_i are defined in Appendix A.

However, it is more convenient to convert the set of aforementioned equations into two coupled ordinary differential equations as follows

$$\begin{aligned}&\ddot{\Psi}+\bar{z}_1\Psi+\bar{z}_2 W^2+\bar{z}_3 W=0\\&\ddot{W}+\bar{z}_4 W+\bar{z}_5 W^3+\bar{z}_6\Psi W+\bar{z}_7\Psi+\bar{z}_8 W^2=0\end{aligned} \tag{65}$$

The coefficients $\bar{z}_i$ are introduced in Appendix A.

4. Analytical solution

The developed nonlinear governing differential equations are analytically solved for two different boundary conditions using homotopy analysis method. The HAM initially introduced by Liao (2004). The HAM is a powerful and computationally cost-effective method which is capable of solving strongly nonlinear differential equations. At first, the solution procedure is briefly explained. For further details one can refer to Setoodeh *et al.* (2016). Consider a series of time dependent nonlinear differential equations as follows

$$N_i\left[z_i(t)\right]=0 \quad i=1,2,...,n \tag{66}$$

In Eq. (66), N_i are nonlinear operators, t denotes an independent variable and $z_i(t)$ are unknown functions. Liao (2004) constructed the so-called zero-order deformation equations as

$$(1-q)L\left[\phi_i(t;q)-z_{i,0}(t)\right]=q\hbar_i h_i(t)N_i\left[\phi_i(t;q)\right] \tag{67}$$

where q is an embedding parameter which changes in the range of [0,1], $\hbar_i$ are nonzero auxiliary parameters and $h_i(t)$ denote nonzero auxiliary functions. The function $z_{i,0}(t)$ are the initial guesses of $z_i(t)$, $\phi_i(t;q)$ are unknown functions and the selected auxiliary linear operator is designated by L. There are some freedoms to select auxiliary linear operator and $h_i(t)$. The parameters $\hbar_i$ and $h_i(t)$ are important and adjust the convergence region of the solution. Here, when q increases from 0 to 1, the solutions $\phi_i(t;q)$ alters from the

$z_{i,0}(t)$ to the $z_i(t)$ solutions. In other words, ϕ_i take the following forms for q=0 and q=1, respectively

$$\phi_i(t;0) = z_{i,0}(t), \quad \phi_i(t;1) = z_i(t) \tag{68}$$

By differentiating Eq. (68) with respect to q, the first-order deformation equation can be obtained as

$$L\left[z_{i,1}(t)\right] = \hbar_i h_i(t) N_i\left[\phi_i(t;q)\right]_{q=0} \tag{69}$$

By expanding $\phi_i(t;q)$ in the form of Taylor series, one has

$$\begin{gathered}\phi_i(t;q) = z_{i,0}(t) + \sum_{m=1}^{+\infty} z_{i,m}(x,t)q^m, \\ z_{i,m} = \frac{1}{m!}\frac{\partial^m \phi_i(t;q)}{\partial q^m}\bigg|_{q=0}\end{gathered} \tag{70}$$

4.1 Nonlinear frequencies of EBT

It is convenient to transform from the frequency domain to the time domain by setting a new variable $\tau=\omega t$ in Eq. (33), wherein ω denotes the nonlinear frequency. Accordingly, it yields

$$\omega^2 \frac{d^2V(\tau)}{d\tau^2} + \alpha V(\tau) + \beta V^2(\tau) + \gamma V^3(\tau) = 0 \tag{71}$$

subject to the following initial condition at the center of the FG beam

$$V_0(0) = W_{\max}, \qquad \frac{dV_0(0)}{d\tau} = 0 \tag{72}$$

W_{max} denotes the maximum amplitude of the vibration. The initial guess $V_0(\tau)$ should be selected such that the initial condition is satisfied (Jafari-Talookolaei *et al.* 2011)

$$V_0(\tau) = W_{max}\cos(\tau) \tag{73}$$

The linear and nonlinear operators can be written as

$$L[\phi(\tau;q)] = \omega_0^2\left(\frac{d^2\phi(\tau;q)}{d\tau^2} + \phi(\tau;q)\right) \tag{74}$$

$$\begin{gathered}N[\phi(\tau;q),\omega(q)] = \omega^2(q)\frac{d^2\phi(\tau;q)}{d\tau^2} + \\ \alpha\phi(\tau;q) + \beta\phi^2(\tau;q) + \gamma\phi^3(\tau;q)\end{gathered} \tag{75}$$

Subsequently, the first-order deformation equation is found out as below

$$\begin{gathered}\omega_0^2\left(\frac{d^2V_1(\tau)}{d\tau^2} + V_1(\tau)\right) = \\ \left(\omega_0^2\frac{d^2V_0(\tau)}{d\tau^2} + \alpha V_0(\tau) + \beta V_0^2(\tau) + \gamma V_0^3(\tau)\right)\end{gathered} \tag{76}$$

with initial conditions of

$$V_1(0) = \frac{dV_1(0)}{d\tau} = 0 \tag{77}$$

After inserting the initial guess from Eq. (73) into Eq. (76), solving the resulted equation and then equating the coefficient of the secular term, $\tau\sin(\tau)$ to zero, leads to the first approximation of the vibration response and the nonlinear frequency as

$$\begin{gathered}V_1(\tau) = \beta'\left(2 - \cos(\tau) - \cos^2(\tau)\right) + \gamma'\left(\cos(\tau) - \cos^3(\tau)\right) \\ \beta' = \frac{\beta W_{\max}^2}{3\omega_0^2}, \quad \gamma' = \frac{\gamma W_{\max}^3}{8\omega_0^2}\end{gathered} \tag{78}$$

$$\omega_0 = \sqrt{\alpha + \frac{3}{4}\gamma W_{max}^2} \tag{79}$$

The second stage of formulations can be written by setting m=2 as

$$\begin{gathered}L[V_2(\tau) - V_1(\tau)] = \hbar h(t)\frac{\partial N[\phi(t;q)]}{\partial q}\bigg|_{q=0}, \\ V_2(0) = \frac{dV_2(0)}{d\tau} = 0\end{gathered} \tag{80}$$

Similarly the second analytical approximation for FGNBs is obtained

$$\begin{gathered}V_2(\tau) = \left(\frac{8}{15}A_2 + \frac{2}{3}A_4 + A_6\right) + \\ \left(\frac{7}{48}A_1 - \frac{1}{5}A_2 + \frac{1}{8}A_3 - \frac{1}{3}A_4 - A_6\right)\cos(\tau) \\ -\left(\frac{64A_2 + 80A_4}{240}\right)\cos^2(\tau) - \left(\frac{25A_1 + 30A_3}{240}\right) \\ \cos^3(\tau) - \left(\frac{A_2}{15}\right)\cos^4(\tau) - \left(\frac{A_1}{24}\right)\cos^5(\tau)\end{gathered} \tag{81}$$

$$\begin{gathered}\omega_1 = -(384)^{-1}\left(160\beta^2 W_{\max}^2 - 96\gamma\beta W_{\max}^3 + \frac{9}{2}\gamma^2 W_{\max}^4\right) \\ \left(\alpha + \frac{3}{4}\gamma W_{max}^2\right)^{-1.5}\end{gathered} \tag{82}$$

where A_i are presented in Appendix A.

Finally, the analytical expressions for linear (ω_L) and nonlinear (ω_{NL}) natural frequencies as well as vibration response of FGNBs are developed by collecting the related terms as

$$\begin{gathered}\omega_L = \sqrt{\alpha}, \quad \omega_{NL} = \omega_0 + \omega_1 \\ V(\tau) = V_0(\tau) + V_1(\tau) + V_2(\tau)\end{gathered} \tag{83}$$

4.2 Nonlinear frequencies of TBT

Initial conditions of deflection and rotation of the beam can be expressed as

$$W(0) = a, \; \dot{W}(0) = 0 \quad , \quad \Psi(0) = a', \; \dot{\Psi}(0) = 0 \tag{84}$$

In Eq. (84) parameter a’ is the maximum initial rotation that must be determined. After applying initial conditions, the first approximations for the deflection and rotation functions can be expressed as

$$W_0(t) = a\cos(\Omega_{NL}t) \quad , \quad \Psi_0(t) = a'\cos(\Omega_{NL}t) \tag{85}$$

Table 1 Comparison of frequency ratio (Λ) for FGNBs (n=2, L/h=12, ζ_{max}=0.8)

B.C.s	h/l	Λ		
		EBT	TBT	Ke, Wang *et al*. (2012)
H-H	Classic	1.6453	1.7369	1.7136
	10	1.6201	1.7144	1.6934
	6	1.5798	1.6778	1.6598
	3	1.4456	1.5480	1.5370
	2	1.3226	1.4177	1.4119
	1.5	1.2331	1.3150	1.3103
	1	1.1302	1.1871	1.1827
C-C	Classic	1.1790	1.2127	1.2202
	10	1.1711	1.2058	1.2128
	6	1.1588	1.1947	1.2009
	3	1.1187	1.1556	1.1589
	2	1.0836	1.1172	1.1183
	1.5	1.0591	1.0876	1.0877
	1	1.0332	1.0515	1.0514

where Ω_{NL} is the nonlinear frequency of the Timoshenko beam. Also, The linear frequency is designated by Ω_L.The linear and nonlinear operators can be respectively expressed as

$$L\left[V_1(t;q)\right] = \frac{\partial^2 V_1(t;q)}{\partial t^2} + \Omega_{NL}{}^2 V_1(t;q),$$
$$L\left[V_2(t;q)\right] = \frac{\partial^2 V_2(t;q)}{\partial t^2} + \Omega_{NL}{}^2 V_2(t;q) \tag{86}$$

$$N_1[V_1(t;q),V_2(t;q)] =$$
$$\ddot{V}_1 + \bar{z}_4 V_1 + \bar{z}_5 V_1{}^3 + \bar{z}_6 V_1 V_2 + \bar{z}_7 V_2 + \bar{z}_8 V_1{}^2$$
$$N_2[V_1(t;q),V_2(t;q)] =$$
$$\ddot{V}_2 + \bar{z}_1 V_2 + \bar{z}_2 V_1{}^2 + \bar{z}_3 V_1 \tag{87}$$

where

$$V_1(t) = W(t)\ ,\ V_2(t) = \Psi(t) \tag{88}$$

The first-order deformation equation is constructed as below

$$L_1[V_1(t;q)] = \hbar_1 h_1(t) N_1[V_1(t;q),V_2(t;q)]_{q=0}\Big|$$
$$L_2[V_2(t;q)] = \hbar_2 h_2(t) N_2[V_1(t;q),V_2(t;q)]_{q=0}\Big| \tag{89}$$

In Eq. (89) after considering $h_1(t)=h_2(t)=1$ and $\hbar_1 = \hbar_2 = 1$, one can get

$$\ddot{W}_1 + \Omega_{NL}{}^2 W_1 = \left(\ddot{W}_0 + \bar{z}_4 W_0 + \bar{z}_5 W_0{}^3\right.$$
$$\left. + \bar{z}_6 \Psi_0 W_0 + \bar{z}_7 \Psi_0 + \bar{z}_8 W_0{}^2\right),\ W_1(0) = \dot{W}_1(0) = 0$$
$$\ddot{\Psi}_1 + \Omega_{NL}{}^2 \Psi_1 = \left(\ddot{\Psi}_0 + \bar{z}_1 \Psi_0 + \bar{z}_2 W_0{}^2 + \bar{z}_3 W_0\right),$$
$$\Psi_1(0) = \dot{\Psi}_1(0) = 0 \tag{90}$$

By substituting $W_0(t)$ and $\Psi_0(t)$, and their derivatives from Eq. (85) into (90) and solving the resulted equations and then equating the coefficients of the secular terms $t\sin(\Omega_{NL} t)$ to zero, two equations are obtained as

$$\Omega_{NL}\left[\bar{z}_5 a^3 + \left(-\frac{4}{3}\Omega_{NL}{}^2 + \frac{4}{3}\bar{z}_4\right)a + \frac{4}{3}\bar{z}_7 a'\right] = 0 \tag{91}$$

$$-a'\Omega_{NL}{}^2 + a'\bar{z}_1 + a\bar{z}_3 = 0 \tag{91}$$

After combining the related equations and eliminating parameter a' from two relations in Eq. (91), the nonlinear natural frequencies are developed

$$\Omega_{NL1} = \frac{1}{4}\sqrt{\left[2\sqrt{\begin{pmatrix} 9\bar{z}_5{}^2 a^4 + 24\bar{z}_4\bar{z}_5 a^2 - 24\bar{z}_1\bar{z}_5 a^2 + 16\bar{z}_4{}^2 - 32\bar{z}_1\bar{z}_4 \\ +16\bar{z}_1{}^2 + 64\bar{z}_3\bar{z}_7 \end{pmatrix}} + 6\bar{z}_5 a^2 + 8\bar{z}_4 + 8\bar{z}_1\right]}$$
$$\Omega_{NL2} = \frac{1}{4}\sqrt{\left[-2\sqrt{\begin{pmatrix} 9\bar{z}_5{}^2 a^4 + 24\bar{z}_4\bar{z}_5 a^2 - 24\bar{z}_1\bar{z}_5 a^2 + 16\bar{z}_4{}^2 - 32\bar{z}_1\bar{z}_4 \\ +16\bar{z}_1{}^2 + 64\bar{z}_3\bar{z}_7 \end{pmatrix}} + 6\bar{z}_5 a^2 + 8\bar{z}_4 + 8\bar{z}_1\right]} \tag{92}$$

It is worth noting that among the obtained solutions for the frequency, only two positive values are acceptable.

5. Numerical results

The FGNB considered here composed of ceramic (SiC) and aluminum (Al). The mechanical properties of the beam are listed as; Metal (Al): E_m=70 GPa, ρ_m=2702 kg/m^3, ν_m=0.3; Ceramic (SiC): E_c=427 GPa, ρ_c=3100 kg/m^3, ν_c=0.17. The following dimensionless parameters are considered through the results for the convenience.

$$\zeta_{\max} = \frac{W_{\max}}{h},$$
$$K_L = \frac{k_l L^2}{A_{1m}}\ ,\ K_G = \frac{k_G}{A_{1m}}\ ,\ K_{NL} = \frac{k_{nl} h^2 L^2}{A_{1m}}$$
$$\Lambda = \frac{\omega_{NL}}{\omega_L}\ \text{(for EBT)},\ \Lambda = \frac{\Omega_{NL}}{\Omega_L}\ \text{(for TBT)}$$

where A_{1m} is the corresponding value of the A_1 for a homogenous metal nano-beam.

At first, the present analytical model is validated by

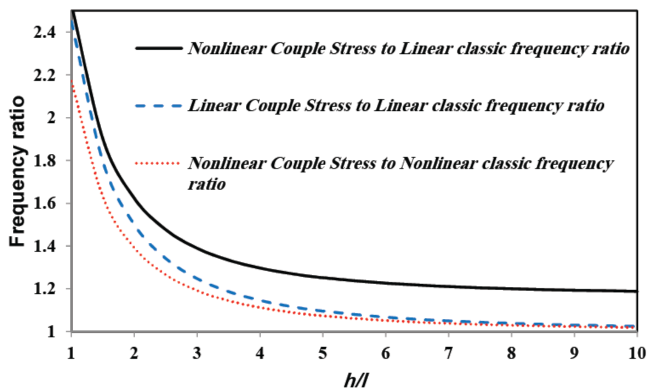

Fig. 2 The influence of size effect parameter and nonlinearity on the frequency ratios of FGNBs (n=2, L/h=12)

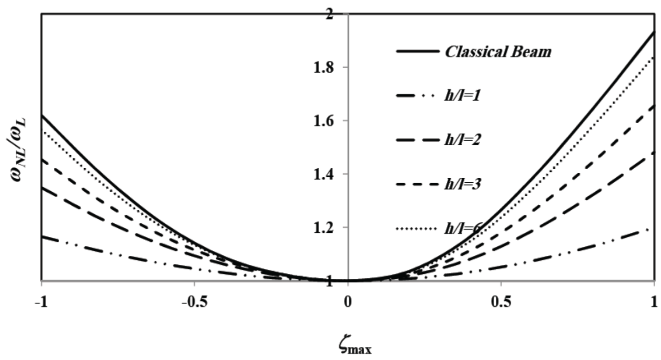

Fig. 3 Effects of dimensionless length scale parameter on the frequency ratio for hinged-hinged FGNBs (n=2, L/h=12)

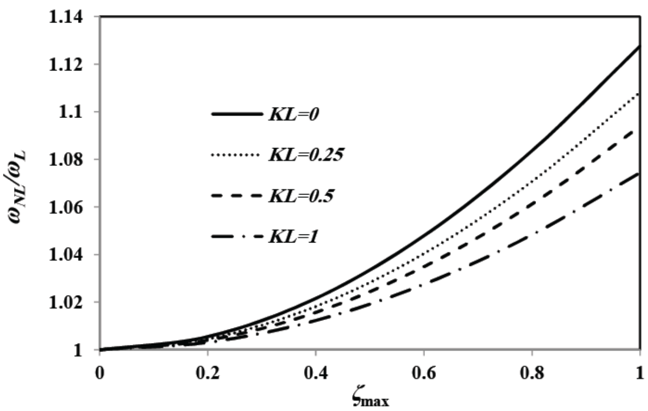

Fig. 4 Effects of linear coefficient on the frequency ratio of clamped FGNBs (n=2, h/l=2 and L/h=12)

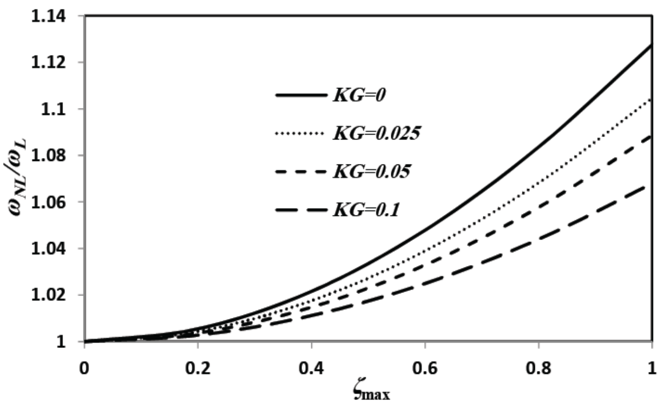

Fig. 5 Effects of shear coefficient on the frequency ratio of clamped FGNBs (n=2, h/l=2 and L/h=12)

comparing the results with those of Ke *et al.* (2012) in Table 1. Ke *et al.* (2012) studied the nonlinear vibration of FG micro-beams using numerical method of differential quadrature. The frequency ratios (Λ) are provided for both of hinged-hinged (H-H) and clamped-clamped (C-C) boundary conditions as well as two different theories of beams. The present results exhibit good agreement in comparison with the numerical results in the aforementioned reference. It is observed that the frequency ratio increases by increasing the dimensionless length scale parameter. It can be noted that although TBT exhibits more accurate results, however the solutions correspond to EBT are also within an acceptable accuracy.

Fig. 2 demonstrates the importance of capturing the size effect and also the influence of geometric nonlinearity on the nonlinear frequency ratios of hinged-hinged FGNBs.

The small scale parameter is set to zero in formulations to obtain the corresponding classic frequencies. It is seen that the impact of size effect is maximum for h/l=1 while its influence is negligible for h/l ratios higher than 8.

In Fig. 3, the effects of dimensionless length scale parameter on the frequency ratio are demonstrated for beam with hinged ends. It should be noted that the frequency ratios are different and the resulted curves are unsymmetrical for identical values of ζ_{max} but with opposite signs. The reason is due to bending-stretching coupling effect.

The effects of linear, shear and nonlinear coefficients of the Pasternak foundation on the frequency ratio versus dimensionless maximum amplitude for clamped FGNBs are studied in Figs. 4-6. It is seen that the frequency ratio increases monotonically with increasing the nonlinear coefficient of the foundation and this effect is amplified for higher values of maximum amplitude. It is interesting that this effect is vice-versa for the linear and shear coefficients of the foundation.

The effects of material gradient index n on the frequency ratio of hinged-hinged as well as clamped micro-beams are shown in Figs. 7-8. Similar to the trend observed in Fig. 3, the value of frequency ratio is changed in the case of hinged beams for ζ_{max} with opposite signs, however the related curve is symmetric for the clamped beams. In other words, the frequency ratio of clamped beams is independent of the sign of the vibration amplitude.

6. Conclusions

In this study, the nonlinear free vibration of FG nano/micro-beams is investigated based on modified couple stress theory in presence of nonlinear Pasternak foundation. The Mori-Tanaka homogenization technique is employed

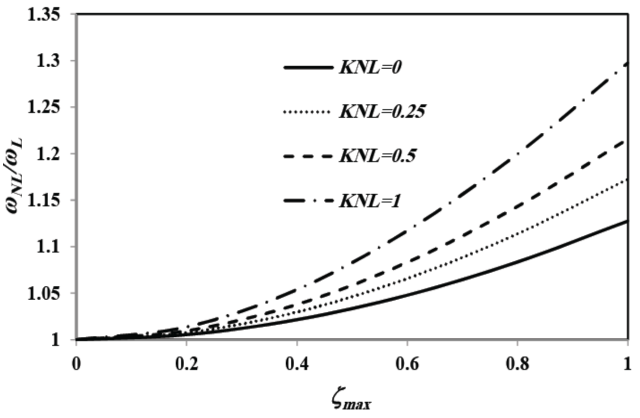

Fig. 6 Effects of nonlinear coefficient on the frequency ratio of clamped FGNBs (n=2, h/l=2 and L/h=12)

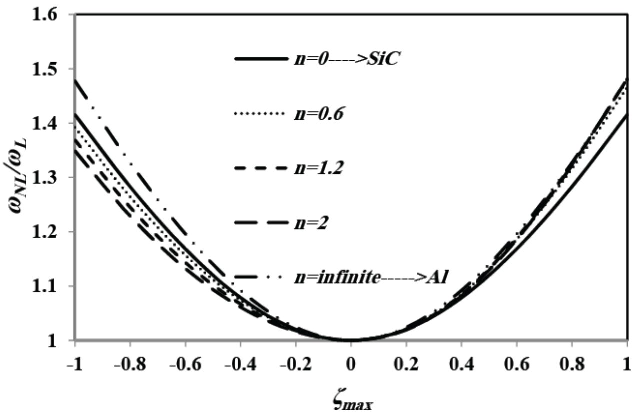

Fig. 7 Effects of material gradient index n on the frequency ratio of hinged-hinged FGNBs (h/l=2 and L/h=12)

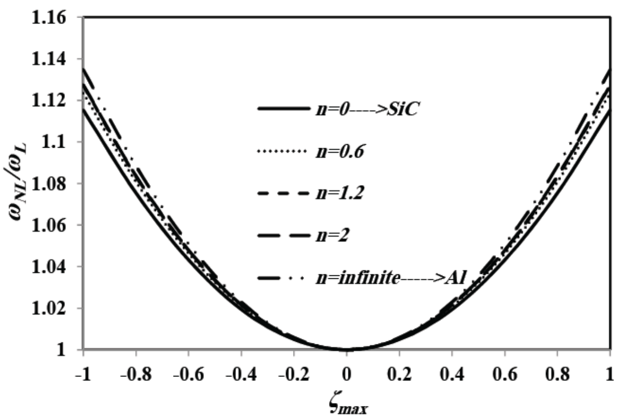

Fig. 8 Effects of material gradient index n on the frequency ratio of clamped FGNBs (h/l=2 and L/h=12)

for material properties of the beams. The nonlinear governing differential equations are derived by Hamilton's principle and the HAM is successfully utilized to obtain the nonlinear frequencies. The comparisons between the present analytical solutions and the available numerical results exhibit the accuracy and efficacy of the method. The influences of the length scale parameter, material gradient index and elastic foundation on the nonlinear free vibration of FGNBs are discussed. The main obtained results are:

- The frequency ratio increases by increasing the dimensionless length scale parameter.
- The results demonstrate the necessity of performing a nonlinear analysis even for small values of the vibration amplitude.
- The maximum difference between the couple stress and classical theories is observed when $h=l$.
- The frequency ratio of hinged-hinged beams is dependent on both the magnitude and sign of the vibration amplitude.

References

Ansari, R., Gholami, R., Faghih Shojaei, M., Mohammadi, V. and Darabi, M.A. (2016), "Coupled longitudinal-transverse-rotational free vibration of post-buckled functionally graded first-order shear deformable micro- and nano-beams based on the Mindlin's strain gradient Theory", *Appl. Math. Model.*, **40**(23), 9872-9891.

Asghari, M., Rahaeifard, M., Kahrobaiyan, M.H. and Ahmadian, M.T. (2011), "The modified couple stress functionally graded Timoshenko beam formulation", *Mater. Des.*, **32**(3), 1435-1443.

Bağdatlı, S.M. (2015), "Non-linear transverse vibrations of tensioned nanobeams using nonlocal beam theory", *Struct. Eng. Mech.*, **55**(2), 281-298.

Bayat, M., Pakar, I. and Emadi, A. (2013), "Vibration of electrostatically actuated microbeam by means of homotopy perturbation method", *Struct. Eng. Mech.*, **48**(6), 823-831.

Ebrahimi, F. and Shafiei, N. (2016), "Application of Eringen's nonlocal elasticity theory for vibration analysis of rotating functionally graded nanobeams", *Smart. Struct. Syst.*, **17**(5), 837-857.

Ehyaei, J., Ebrahimi, F. and Salari, E. (2016), "Nonlocal vibration analysis of FG nano beams with different boundary conditions", *Adv. Nano Res.*, **4**(2), 85-111.

Eringen, A.C. (1972), "Nonlocal polar elastic continua", *Int. J. Eng. Sci.*, **10**(1), 1-16.

Jafari-Talookolaei, R.A., Salarieh, H. and Kargarnovin, M.H. (2011), "Analysis of large amplitude free vibrations of unsymmetrically laminated composite beams on a nonlinear elastic foundation", *Acta Mech.*, **219**, 65-75.

Janghorban, M. and Zare, A. (2011), "Free vibration analysis of functionally graded carbon nanotubes with variable thickness by differential quadrature method", *Physica. E Low Dimens. Syst. Nanostruct.*, **43**(9), 1602-1604.

Jia, X.L., Ke, L.L., Feng, C.B., Yang, J. and Kitipornchai, S. (2015), "Size effect on the free vibration of geometrically nonlinear functionally graded micro-beams under electrical actuation and temperature change", *Compos. Struct.*, **133**, 1137-1148.

Kahrobaiyan, M.H., Asghari, M., Rahaeifard, M. and Ahmadian, M.T. (2010), "Investigation of the size-dependent dynamic characteristics of atomic force microscope microcantilevers based on the modified couple stress theory", *Int. J. Eng. Sci.*, **48**(12), 1985-1994.

Ke, L.L., Wang, Y.S., Yang, J. and Kitipornchai, S. (2012), "Nonlinear free vibration of size-dependent functionally graded microbeams", *Int. J. Eng. Sci.*, **50**(1), 256-267.

Ke, L.L., Yang, J., Kitipornchai, S. and Xiang, Y. (2009), "Flexural vibration and elastic buckling of a cracked Timoshenko beam made of functionally graded materials", *Mech. Adv. Mater. Struct.*, **16**(6), 488-502.

Lam, D.C.C., Yang, F., Chong, A.C.M., Wang, J. and Tong, P.

(2003), “Experiments and theory in strain gradient elasticity”, *J. Mech. Phys. Solid.*, **51**(8), 1477-1508.

Liao, S.J. (2004), *Beyond perturbation: introduction to homotopy analysis method*, CRC Press, Boca Raton.

Ma, H.M., Gao, X.L. and Reddy, J.N. (2008), “A microstructure-dependent Timoshenko beam model based on a modified couple stress theory”, *J. Mech. Phys. Solid.*, **56**(12), 3379-3391.

Malekzadeh, P. and Shojaee, M. (2013), “Surface and nonlocal effects on the nonlinear free vibration of non-uniform nanobeams”, *Compos. Part B-Eng.*, **52**, 84-92.

Malekzadeh, P. and Shojaee, M. (2015), “A two-variable first-order shear deformation theory coupled with surface and nonlocal effects for free vibration of nanoplates”, *J. Vib.. Control.*, **21**, 2755-2772.

Mindlin, R.D. and Tiersten, H.F. (1962), “Effects of couple-stresses in linear elasticity”, *Arch. Ration. Mech. Anal.*, **11**(1), 415-448.

Nateghi, A. and Salamat-talab, M. (2013), “Thermal effect on size dependent behavior of functionally graded micro beams based on modified couple stress theory”, *Compos. Struct.*, **96**, 97-110.

Rao, S.S. (2007), *Vibration of continuous systems*. John Wiley & Sons, New Jersi.

Schmid, S., Wagli, P. and Hierold, C. (2009, January), “Biosensor based on all-polymer resonant microbeams”, *Micro Electro Mechanical Systems, IEEE 22nd International Conference*, 300-303.

Sedighi, H.M., Chan-Gizian, M. and Noghreha-Badi, A. (2014), “Dynamic pull-in instability of geometrically nonlinear actuated micro-beams based on the modified couple stress theory”, *Lat. Am. J. Solid. Struct.*, **11**(5), 810-825.

Setoodeh, A.R. and Afrahim, S. (2014), “Nonlinear dynamic analysis of FG micro-pipes conveying fluid Based on strain gradient theory”, *Compos. Struct.*, **116**, 128-135.

Setoodeh, A.R., Derahaki, M. and Bavi, N. (2015), “DQ thermal buckling analysis of embedded curved carbon nanotubes based on nonlocal elasticity theory”, *Lat. Am. J. Solid. Struct.*, **12**(10), 1901-1917.

Setoodeh, A.R., Rezaei, M. and Zendehdel Shahri, M.R. (2016), “Linear and nonlinear torsional free vibration of functionally graded micro/nano-tubes based on modified couple stress theory”, *Appl. Math. Mech.*, English Edition, **37**, 1-16.

Shenas, A.G. and Malekzadeh, P. (2016), “Free vibration of functionally graded quadrilateral microplates in thermal environment”, *Thin. Wall. Struct.*, **106**, 294-315.

Taeprasartsit, S. (2013), “Nonlinear free vibration of thin functionally graded beams using the finite element method”, *J. Vib. Control.*, 1077546313484506.

Thai, H.T. and Choi, D.H. (2013), “Size-dependent functionally graded Kirchhoff and Mindlin plate models based on a modified couple stress theory”, *Compos. Struct.*, **95**, 142-153.

Younis, M.I., Abdel-Rahman, E.M. and Nayfeh, A. (2003), “A reduced-order model for electrically actuated microbeam-based MEMS”, *J. Microelectromech. Syst.*, **12**(5), 672-680.

CC

Appendix A

The coefficients of Eq. (48)

$$\bar{u}_1=\bar{u}_2=\frac{B_1\frac{I_2}{I_3}-A_1}{\frac{I_2^2}{I_3}-I_1},\quad \bar{u}_3=\frac{B_1-C_1\frac{I_2}{I_3}-\frac{D_1l^2}{4}\frac{I_2}{I_3}}{\frac{I_2^2}{I_3}-I_1},\quad \bar{u}_4=\frac{k_sD_1\frac{I_2}{I_3}}{\frac{I_2^2}{I_3}-I_1}$$

$$\bar{u}_5=-\frac{k_sD_1\frac{I_2}{I_3}}{\frac{I_2^2}{I_3}-I_1},\quad \bar{u}_6=-\frac{\frac{D_1l^2}{4}\frac{I_2}{I_3}}{\frac{I_2^2}{I_3}-I_1}$$

The coefficients of Eq. (49)

$$\bar{\psi}_1=-\frac{C_1+\frac{D_1l^2}{4}-B_1\frac{I_2}{I_1}}{\frac{I_2^2}{I_1}-I_3},\quad \bar{\psi}_2=-\frac{A_1\frac{I_2}{I_1}-B_1}{\frac{I_2^2}{I_1}-I_3},\quad \bar{\psi}_3=-\frac{A_1\frac{I_2}{I_1}-B_1}{\frac{I_2^2}{I_1}-I_3}$$

$$\bar{\psi}_4=\frac{k_sD_1}{\frac{I_2^2}{I_1}-I_3},\quad \bar{\psi}_5=-\frac{k_sD_1}{\frac{I_2^2}{I_1}-I_3},\quad \bar{\psi}_6=-\frac{D_1l^2/4}{\frac{I_2^2}{I_1}-I_3}$$

The coefficients of Eq. (50)

$$\bar{w}_1=\frac{3A_1}{2I_1},\ \bar{w}_2=\bar{w}_3=\frac{A_1}{I_1},\ \bar{w}_4=\bar{w}_5=-\frac{B_1}{I_1},$$

$$\bar{w}_6=\frac{1}{I_1}(k_sD_1+k_G),\ \bar{w}_7=-\frac{k_sD_1}{I_1},$$

$$\bar{w}_8=\bar{w}_9=-\frac{D_1l^2}{4I_1},\ \bar{w}_{10}=-\frac{k_l}{I_1}.$$

The coefficients of Eq. (60)

$$\alpha=\frac{\int_0^L\left[\left(-\frac{k_G}{m_0}\right)QQ''+\left(\frac{C_1}{m_0}+\frac{D_1l^2}{m_0}-\frac{B_1^2}{A_1m_0}\right)QQ''''\right]dx}{\int_0^L Q^2dx}+\frac{K_l}{m_0}$$

$$\beta=\frac{\frac{1}{m_0L}\int_0^L\left(\int_0^L B_1Q''dx\right)QQ''dx}{\int_0^L Q^2dx},$$

$$\gamma=\frac{-\frac{1}{m_0L}\int_0^L\left(\int_0^L\frac{A_1}{2}Q'^2dx\right)QQ''dx+\frac{1}{m_0}\int_0^L k_{nl}Q^4dx}{\int_0^L Q^2dx}$$

The coefficients of Eq. (64)

$$\alpha_1=-\frac{\int_0^L\bar{u}_1P(x)P''(x)dx}{\int_0^L P^2(x)dx},\ \alpha_2=-\frac{\int_0^L\bar{u}_2Q'(x)Q''(x)P(x)dx}{\int_0^L P^2(x)dx}$$

$$\alpha_3=-\frac{\int_0^L\left(\bar{u}_3S''(x)P(x)+\bar{u}_4S(x)P(x)\right)dx}{\int_0^L P^2(x)dx},$$

$$\alpha_4=-\frac{\int_0^L\left(\bar{u}_5Q'(x)+\bar{u}_6Q'''(x)\right)P(x)dx}{\int_0^L P^2(x)dx}$$

$$\beta_1 = \frac{\int_0^L \left(\bar{\psi}_1 S''(x) + S(x)\bar{\psi}_4\right) S(x)dx}{\int_0^L S^2(x)dx} \ ,$$

$$\beta_2 = \frac{\int_0^L \bar{\psi}_2 P''(x) S(x)dx}{\int_0^L S^2(x)dx} \ ,$$

$$\beta_3 = \frac{\int_0^L \bar{\psi}_3 Q'(x) Q''(x) S(x)dx}{\int_0^L S^2(x)dx} \ ,$$

$$\beta_4 = \frac{\int_0^L \left(\bar{\psi}_5 Q'(x) + \bar{\psi}_6 Q'''(x)\right) S(x)dx}{\int_0^L S^2(x)dx} \ ,$$

$$\gamma_1 = -\frac{\int_0^L \left(\bar{w}_6 Q''(x) + \bar{w}_8 Q''''(x) + \bar{w}_{10} Q(x)\right) Q(x)dx}{\int_0^L Q^2(x)dx},$$

$$\gamma_2 = -\frac{\int_0^L \left(\bar{w}_1 Q'^2(x) Q''(x) + \bar{w}_{11} Q^3(x)\right) Q(x)dx}{\int_0^L Q^2(x)dx},$$

$$\gamma_3 = -\frac{\int_0^L \left(\bar{w}_7 S'(x) + \bar{w}_9 S'''(x)\right) Q(x)dx}{\int_0^L Q^2(x)dx},$$

$$\gamma_4 = -\frac{\int_0^L \left(\bar{w}_2 P''(x) Q'(x) + \bar{w}_3 Q''(x) P'(x)\right) Q(x)dx}{\int_0^L Q^2(x)dx} \ ,$$

$$\gamma_5 = -\frac{\int_0^L \left(\bar{w}_4 S''(x) Q'(x) + \bar{w}_5 S'(x) Q''(x)\right) Q(x)dx}{\int_0^L Q^2(x)dx}$$

The coefficients of Eq. (65)

$$\bar{z}_1 = \beta_1 - \frac{\alpha_3 \beta_2}{\alpha_1} \ , \bar{z}_2 = \beta_3 - \frac{\beta_2 \alpha_2}{\alpha_1} \ , \bar{z}_3 = \beta_4 - \frac{\alpha_4 \beta_2}{\alpha_1} \ ,$$

$$\bar{z}_4 = \gamma_1, \ \bar{z}_5 = \gamma_2 - \frac{\gamma_4 \alpha_2}{\alpha_1} \ , \bar{z}_6 = \gamma_5 - \frac{\gamma_4 \alpha_3}{\alpha_1} \ ,$$

$$\bar{z}_7 = \gamma_3 \ , \bar{z}_8 = -\gamma_4 \frac{\alpha_4}{\alpha_1}$$

The coefficients of Eq. (81)

$$A_1 = -\frac{3}{8} \frac{\gamma^2 W_{\max}{}^5}{\omega_0{}^4}, \ A_2 = -\frac{5}{4} \frac{\gamma \beta W_{\max}{}^4}{\omega_0{}^4}$$

$$A_3 = \frac{\left(-3\alpha\gamma - 16\beta^2 + 51\gamma\omega_0{}^2\right) W_{\max}{}^3 - 24\beta\gamma W_{\max}{}^4 + 9\gamma^2 W_{\max}{}^5}{24\omega_0{}^4}$$

$$A_4 = \frac{\left(56\beta\omega_0{}^2 - 8\alpha\beta\right) W_{\max}{}^2 - 16\beta^2 W_{\max}{}^3 + 54\beta\gamma W_{\max}{}^4}{24\omega_0{}^4}$$

$$A_5 = \left(\frac{4\omega_0\omega_1 - \alpha - \omega_0{}^2}{2\omega_0{}^2}\right) W_{\max} + \left(\frac{\beta\omega_0{}^2 - \alpha\beta}{3\omega_0{}^4}\right) W_{\max}{}^2$$

$$+ \left(\frac{3\alpha\gamma + 32\beta^2 - 12\gamma\omega_0{}^2}{24\omega_0{}^4}\right) W_{\max}{}^3$$

$$A_6 = \left(\frac{2\alpha - 2\beta\omega_0{}^2}{3\omega_0{}^4}\right) W_{\max}{}^2$$