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Analysis of non-homogeneous orthotropic plates using EDQM

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Abstract. Element based differential quadrature method (EDQM) has been applied to analyze static, stability and free vibration of non-homogeneous orthotropic rectangular plates of variable or stepped thickness. The Young's modulus and the density are assumed to vary in exponential form in X-direction whereas the thickness is assumed to vary linear, parabolic or exponential variation in one or two directions. In-plane loading is assumed to vary linearly. Various combinations of clamped, simply supported and free edge conditions (regular and irregular boundary) have been considered . Continuous plates could also be handled with ease. In this paper, formulation for equilibrium, buckling and free vibration problems is discussed and several numerical examples are solved using EDQM and compared with the published results.

Keywords: EDQM; orthotropy; non-homogeneity; elastic foundation; buckling; vibration; mode shape

1. Introduction

Due to the advent of fiber reinforced plastics and other modern materials, non-homogeneous orthotropic plate finds important application in various branches of engineering such as aerospace industry and missile technology. The designers prefer such plates due to light weight and high strength and high compressive resistance. By adopting the plates with variable thickness or stepped plates, there will be greater efficiency in bending, buckling and vibration as compared to plates of uniform thickness thus leading to reduction in weight resulting in economy.

Leissa (1969, 1977, 1987) has given complete survey work up to 1985 on vibration of homogeneous isotropic and anisotropic plates of various geometrics. Vibration of orthotropic parallelogram plates with variable thickness are given by Dokainish and Kumar (1973). Bert and Malik (1996) applied Differential Quadrature method (DQM) for vibration analysis of tapered plates. A comparative study is presented by Kukraeti *et al.* (1996) for free vibration of tapered plate. Akiyama and Kuroda (1997) obtained fundamental frequencies of tapered plate. Free vibration and buckling analysis of rectangular plates with variable thickness are investigated by Ng and Araar (1989) using Galerkin method. Cheung and Zhou (1989) applied Rayleigh Ritz method for vibration analysis of tapered rectangular plates using new set of beam functions..

Effect of boundary constraints and thickness variation on the vibrating response of rectangular plates with abrupt variation in thickness is given by Lim and Liew (1993) and Liew and Wang (1993). After 1985, studies of homogeneous rectangular orthotropic plates have been carried out by several researchers (Gormman 1993, Lal *et al.* 1997, Lal *et al.* 2001, Civalek 2009) to mention a few prominent ones. Various types of DQ methods such as Generalized differential quadrature (GDQ), differential quadrature element (DQE) and Harmonic differential quadrature (HDQ) have been used to solve buckling and vibration problems of plates by Civalek (2004). Civalek and Ulker (2004) applied harmonic differential quadrature (HDQ) for axisymmetric bending analysis of thin isotropic circular plates and HDQ and Finite Difference (FD) integrated methodology for nonlinear static and dynamic response of doubly curved shallow shells (2005). Fares and Zenkour (1999)presented the free vibration analysis of non-homogeneous composite cross-ply laminated plated with various plate theories whereas Lal and Dhanpati (2007) discussed the vibration of non-homogeneous orthotropic rectangular plate of variable thickness using a spline technique. Other notable works dealing with the vibration of plates employing differential quarature method is reported by Malekzadeh and Shahpari (2005), Civalek (2006), Liu (2000). Other than Finite element method, there is no general method available in the literature for the static, stability and free vibration analysis of non-homogeneous orthotropic/isotropic plates with varying thickness. Rao *et al.* (1974) investigated the vibration of in-homogeneous plates using a high precision triangular element. Tomar *et al.* (1984) obtained the natural frequencies of free vibration of an isotropic non-homogeneous infinite plate of parabolically varying thickness.

Buckling problem of a plate subjected to in-plane loads was investigated by Timoshenko and Gere (1963) and Lekhnitskii (1968) obtained the solution for orthotropic plate. Many investigators have formulated the problem for orthotropic plates (Zhong and Gu 2010, Tang and Wang 2011, Call and Saini 2013) and obtained the solution and notable among them is by Kang and Leissa (2005). SFSF and CFCF and SFSC composite orthotropic plates due to in-plane moments were investigated by Lopatin and Morozov (2009, 2010, 2011, 2014).

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Table 1 Material properties of some orthotropic materials

Designation	Material	E_x GPa	E_y GPa	G_{xy} GPa	ν_{xy}	ρ kg/m^3
A1	*T*-Graphite/ Epoxy	185	10.5	7.3	0.28	1600
A2	*B*-Boron Epoxy	208	18.9	5.7	0.23	2000
A3	*K*=Aryl/ Epoxy	76	5.6	2.3	0.34	1460
A4	*E*-Glass/ Epoxy	60.7	24.8	12.0	0.23	1600

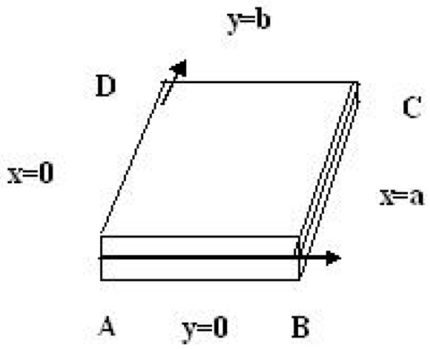


Fig. 1 Rectangular plate with coordinates

There are other methods such as Discrete Singular Convolution (DSC) applied for the buckling analysis of rectangular Kirchhoff plates subjected to compression loads based on the theory of distributions and wavelet analysis by Civalek *et al.* (2010). This method discretizes the spatial derivatives and therefore reduces the given partial differential equations into a standard eigen value problem. This method also requires 17×17 grid of plates for the buckling analysis to achieve reasonable accuracy. Each method has its own advantages and limitations and application areas. Finite element method (FEM) has its own advantage and applicable areas. FEM is still an effective way especially in systems with complex geometry and load and boundary conditions with nonlinear behaviour.

Consideration of non homogeneity, orthotropy, thickness variation, elastic foundation and aspect ratio for stability and vibration of plates subjected to variation in in-plane loads leads to a very complex problem involving several parameters. However with the choice of EDQM one can find an approximate solution to the present problem.

2. Orthotropic materials

Orthotropic material is one that has different material properties in different orthogonal directions (e.g., Glass reinforced plastic or wood). In bridge design orthotropic deck is made up of solid steel plate. Many composite plates may be modeled into orthotropic plates. Table 1 gives the material properties of some orthotropic materials. Orthotropic materials possess five properties viz:- E_x, E_y, G_{xy}, ν_{xy}, and ρ where E_x, E_y are the Young's moduli in x and y directions , G_{xy} modulus rigidity, ν_{xy} Poisson's ratios in x direction and ρ mass density of the material.

3. Governing equation

The five properties viz:- E_x, E_y, G_{xy}, ν_{xy}, and ρ are required to define stress strain relationship of orthotropic plates as (see Fig. 1)

$$\varepsilon_x = \frac{\sigma_x}{E_x} - \nu_{yx}\frac{\sigma_y}{E_y} \tag{1a}$$

$$\varepsilon_y = \frac{\sigma_y}{E_y} - \nu_{xy}\frac{\sigma_x}{E_x} \tag{1b}$$

A relationship between Poisson's ratios is given by

$$\frac{\nu_{xy}}{E_x} = \frac{\nu_{yx}}{E_y} \tag{2a}$$

If G_{xy} is not given it may be assumed as

$$G_{xy} \cong \frac{\sqrt{E_x E_y}}{2\left(1+\sqrt{\nu_{xy}\nu_{yx}}\right)} \tag{2b}$$

and for isotropic materials

$$G = \frac{E}{2(1+\nu)} \tag{2c}$$

Moment curvature relationships for a plate may be written as

$$M_x = -D_x\left(\frac{\partial^2 w}{\partial x^2} + \nu_{yx}\frac{\partial^2 w}{\partial y^2}\right) \tag{3a}$$

$$M_y = -D_y\left(\frac{\partial^2 w}{\partial y^2} + \nu_{xy}\frac{\partial^2 w}{\partial x^2}\right) \tag{3b}$$

and

$$M_{xy} = 2D_t\frac{\partial^2 w}{\partial x \partial y} \tag{3c}$$

where D_x, D_y are flexural rigidities in x and y directions, D_t, torsional rigidity and 'w' lateral deflection.

$$D_x = \frac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})};\ D_y = \frac{E_y h^3}{12(1-\nu_{xy}\nu_{yx})}\ ;\ D_t = \frac{G_{xy}h^3}{12}$$

if G_{xy} is given and if
G_{xy} is not given

$$2D_t = (1-\nu)D_{xy} \tag{4a}$$

where

$$\nu = \sqrt{\nu_{xy}\nu_{yx}};\quad D_{xy} = \sqrt{D_x D_y} \tag{4b}$$

In the above equations, h is the thickness of the plate.
B is defined as

$$B = \frac{1}{2}\left(\nu_{yx}D_x + \nu_{xy}D_y + 4D_t\right) = \left(\nu_{yx}D_x + 2D_t\right) \tag{4c}$$

Using the Love-Kirchhoff hypothesis the differential equation can be written as

$$D_x\frac{\partial^4 w}{\partial x^4} + 2B\frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y\frac{\partial^4 w}{\partial y^4} +$$

$$N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^w w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = q \tag{5}$$

where N_x, N_y are the axial forces per unit length in x and y directions respectively and N_{xy} is the shear force per unit length and 'q' is the lateral loading on the plate. Eq. (5) can be solved for static equilibrium and stability problems. For free vibration problems 'q' is given by inertia force as $q=\omega^2\rho hw$ and solving Eq. (5) with this substitution as an eigen-value problem we will be able to get the natural frequencies and mode shapes.

Assume

$$N_x = N\, t_x p_x\mathbf{;}\; N_y = Nt_y\, p_y;\;\; N_{xy} = N\; t_{xy} \tag{6}$$

where t_x, t_y and t_{xy} are the tracers for N_x, N_y and N_{xy} and they will have values of 1 if the corresponding force exists or zero otherwise.

For stability problems if q=0

$$D_x \frac{\partial^4 w}{\partial x^4} + 2B \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} =$$

$$-N\left(p_x \frac{\partial^2 w}{\partial x^2} + p_y \frac{\partial^w w}{\partial y^2} + 2 p_{xy} \frac{\partial^2 w}{\partial x \partial y} \right) \tag{7}$$

Solving Eq. (7) as an eigen-value problem, one will be able to solve for buckling load and can obtain buckled shape. In addition to uniform variation of N_x, N_y the following variation of axial forces can be considered:- p_x=(1−$\alpha_x y/b$) and p_y=(1−$\alpha_y x/a$) and the variation of the axial loads are shown in Fig. 2 depending the values of α. In x-direction if α_x=0, then it denotes uniform variation, α_x=1 triangular variation 1 at origin (y=0) and zero at the other end (y=b) and α_x=2 pure bending.

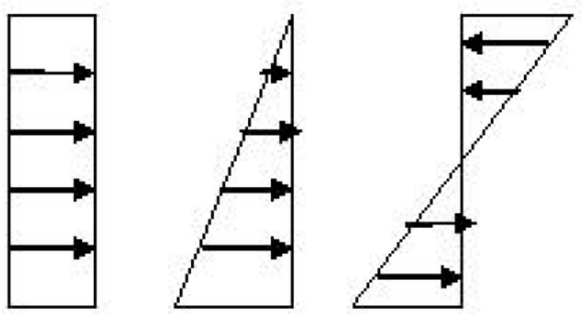
4. Boundary conditions

a) Fixed edge

On x=0 or x=a w=0 and $\frac{\partial w}{\partial x} = 0$ (8a)

On y=0 or y=b w=0 and $\frac{\partial w}{\partial y} = 0$ (8b)

b) Simply supported edge



α_x=0 α_x=1 α_x=2

Fig. 2 The variation of N_x

On x=0 or x=a w=0 and $M_x = -D_x\left(\frac{\partial^2 w}{\partial x^2} + \nu_{yx} \frac{\partial^2 w}{\partial y^2} \right) = 0$ (9a)

On y=0 or y=b w=0 and $M_y = -D_y\left(\frac{\partial^2 w}{\partial y^2} + \nu_{xy} \frac{\partial^2 w}{\partial x^2} \right) = 0$ (9b)

c) Free edge

On x=0 or x=a the shear force Q_x, bending moment M_x and twisting moment M_{xy} have to be zero according to Poisson. Later on Kirchhoff combined Q_x and M_{xy} and consider as one equation as $V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial y} \right) - N_x \frac{\partial w}{\partial x} = 0$. Hence two boundary conditions on free edges (x=0, or x=a) will be

$$V_x = \left(Q_x - \frac{\partial M_{xy}}{\partial y} \right) - N_x \frac{\partial w}{\partial x} = 0 \quad \text{or}$$

$$-D_x\left(\frac{\partial^3 w}{\partial x^3} + \left(4\frac{D_t}{D_x} + \nu_{yx} \right) \frac{\partial^3 w}{\partial x \partial y^2} \right) - N_x \frac{\partial w}{\partial x} = 0 \tag{10a}$$

$$M_x = -D_x\left(\frac{\partial^2 w}{\partial x^2} + \nu_{yx} \frac{\partial^2 w}{\partial y^2} \right) = 0 \tag{10b}$$

on y=0 or y=b free edge, the boundary conditions will be

$$V_y = -D_y\left(\frac{\partial^3 w}{\partial y^3} + \left(4\frac{D_t}{D_y} + \nu_{xy} \right) \frac{\partial^3 w}{\partial x^2 \partial y} \right) - N_y \frac{\partial w}{\partial y} = 0 \tag{11a}$$

$$M_y = -D_y\left(\frac{\partial^2 w}{\partial y^2} + \nu_{xy} \frac{\partial^2 w}{\partial x^2} \right) = 0 \tag{11b}$$

In addition for free corner

$$M_{xy} = 0 \tag{12}$$

5. Nonhomogeneous orthotropic plate with variable thickness-governing equation

The governing equation for an orthotropic rectangular plate (Fig. 3) with varying thickness in x and y directions is given by

$$\frac{\partial^2 M_x}{\partial x^2} + \frac{\partial^2 M_y}{\partial y^2} - \frac{\partial^2 M_{xy}}{\partial x \partial y} + N_x \frac{\partial^2 w}{\partial x^2}$$

$$+ N_y \frac{\partial^w w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} = q \tag{13}$$

When the thickness, E_x, E_y vary along x and y directions, one has to consider the variations of D_x, D_y and D_t in x and y directions as

$$D_x \frac{\partial^4 w}{\partial x^4} + 2B \frac{\partial^4 w}{\partial x^2 \partial y^2} + D_y \frac{\partial^4 w}{\partial y^4} + N_x \frac{\partial^2 w}{\partial x^2} + N_y \frac{\partial^w w}{\partial y^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} +$$

$$2\frac{\partial D_x}{\partial x}\left(\frac{\partial^3 w}{\partial x^3} + \nu_{yx} \frac{\partial^3 w}{\partial x \partial y^2} \right) + 2\frac{\partial D_y}{\partial y}\left(\frac{\partial^3 w}{\partial y^3} + \nu_{xy} \frac{\partial^3 w}{\partial x^2 \partial y} \right) + 4\frac{\partial D_t}{\partial x} \frac{\partial^3 w}{\partial x \partial y^2} +$$

$$4\frac{\partial D_t}{\partial y}\frac{\partial^3 w}{\partial x^2 \partial y}+\frac{\partial^2 D_x}{\partial x^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}\right)+\frac{\partial^2 D_y}{\partial y^2}\left(\frac{\partial^2 w}{\partial y^2}+\nu_{xy}\frac{\partial^2 w}{\partial x^2}\right)+ \\ 4\frac{\partial^2 D_t}{\partial x \partial y}\frac{\partial^2 w}{\partial x \partial y}=\omega^2 \rho h w \tag{14}$$

Accordingly one can derive the governing equation for stability and free vibration problems as given in section.3

6. Boundary conditions for variable thickness plate

The boundary conditions for fixed and simply supported edges are exactly same as orthotropic plate of uniform thickness. Now let us consider free edge boundary condition.

a) Free edge x=0 or $x=a$

$$M_x=0 \; or \; \left(\frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}\right)=0 \tag{15a}$$

and

$$V_x=Q_x-\frac{\partial M_{xy}}{\partial y}-N_x\frac{\partial w}{\partial x}=$$

$$D_x\left(\frac{\partial^3 w}{\partial x^3}+\nu_{yx}\frac{\partial^3 w}{\partial x \partial y^2}\right)+\frac{\partial D_X}{\partial x}\left(\frac{\partial^2 w}{\partial x^2}+\nu_{yx}\frac{\partial^2 w}{\partial y^2}\right)+$$

$$4D_t\frac{\partial^3 w}{\partial x \partial y^2}+4\frac{\partial D_t}{\partial y}\frac{\partial^2 w}{\partial x \partial y}-N_x\frac{\partial w}{\partial x}=0 \tag{15b}$$

b) Free edge y=0 or $y=b$

$$M_y=0 \; or \; \left(\frac{\partial^2 w}{\partial y^2}+\nu_{xy}\frac{\partial^2 w}{\partial x^2}\right)=0 \tag{16a}$$

and

$$V_y=Q_y-\frac{\partial M_{xy}}{\partial x}-N_y\frac{\partial w}{\partial y}=$$

$$D_y\left(\frac{\partial^3 w}{\partial y^3}+\nu_{xy}\frac{\partial^3 w}{\partial x^2 \partial y}\right)+\frac{\partial D_y}{\partial y}\left(\frac{\partial^2 w}{\partial y^2}+\nu_{xy}\frac{\partial^2 w}{\partial x^2}\right)+$$

$$4D_t\frac{\partial^3 w}{\partial x^2 \partial y}+4\frac{\partial D_t}{\partial x}\frac{\partial^2 w}{\partial x \partial y}-N_y\frac{\partial w}{\partial y}=0 \tag{16b}$$

If the plate is non-homogeneous such that

$$E_x=E_1 e^{\mu_1 X}; \; E_y=E_2 e^{\mu_2 X}; \quad \rho=\rho_0 e^{\beta X}$$

where

$$X=\frac{x}{a} \tag{17}$$

E_x, E_y Variation can be considered in the variation of D_x, D_y and D_t and the density variation can be considered in the inertia force.

7. Differential quadrature method

Differential quadrature method (DQM) is a useful technique to solve the governing equations directly. Early references on the DQM can be found in Bellman and Casti (1971), Bert and Malik (1996), Laura and Gutierrez (1993) and more recent development and applications can be found in (Shu 2000, Zong and Zhang 2009, Wang 2015, Rajasekaran 2013) among many others.

The DQM used the basis of the Gauss method for deriving derivatives of a function. It follows that the partial derivative of a function with respect to a space variable can be approximated by a weighted linear combination of function values at some intermediate points in that variety. A differential quadrature approximation at the i-th discrete point on a grid at the the x-axis is given by

$$\frac{\partial f(\xi_i,\eta)}{\partial \xi}=\sum_{j=1}^{N_x} a_{ij}^{(1)} f(\xi_j,\eta) \tag{18}$$

for i=1……N_x where N_x is the number of grid points on x-axis. Eq. (18) can be written as

$$[A_1][G]=[f(\xi)] \tag{19}$$

where $\xi=\frac{x}{a}$ a non-dimensional coordinate.

If the first column of G contains all 1s then the first column of $f(\xi)$ matrix will be zeros. If the second column contains ξ_i then the second column of $f(\xi)$ will be 1s. If the third column of G contains ξ_i^2 then the third column of $f(\xi)$ will be $2\xi_i$ as shown

$$[A_1]\begin{bmatrix} 1 & \xi_1 & \xi_1^2 & \cdots & \xi_1^{N_x-1} \\ 1 & \xi_2 & \xi_2^2 & \cdots & \xi_2^{N_x-1} \\ \vdots & \vdots & \vdots & \cdots & \vdots \\ 1 & \vdots & \vdots & \vdots & \vdots \\ 1 & \xi_{N_x} & \xi_{N_x}^2 & \cdots & \xi_{N_x}^{N_x-1} \end{bmatrix} = \begin{bmatrix} 0 & 1 & 2\xi_1 & 3\xi_1^2 & \cdots \\ 0 & 1 & 2\xi_2 & 3\xi_2^2 & \cdots \\ 0 & 1 & 2\xi_3 & 3\xi_3^2 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \cdots \\ 0 & 1 & \vdots & 3\xi_{Nx}^2 & \cdots \end{bmatrix} \tag{20}$$

Transposing Eq. (20) we get

$$[G]^T[A_1]^T=[f(\xi)]^T \tag{21}$$

Solving

$$[A_1]=[f_\xi][G]^{-1} \tag{22a}$$

Since $\xi=x/a$ is a non-dimensional variable, differential with respect to space variable x is given as

$$\frac{\partial f(x_i,y)}{\partial x}=\frac{1}{a}\sum_{j=1}^{N_x} A_{ij}^{(1)} f(\xi_j,\eta) \tag{22b}$$

where 'a' is the side of the plate in x-direction.

Similarly

$$\frac{\partial^m f(x_i,y)}{\partial x^m}=\frac{1}{a^m}\sum_{j=1}^{N_x} A_{ij}^{(m)} f(\xi_j,\eta)=\frac{1}{a^m}[A_m][f] \tag{23}$$

Denoting $[\bar{A}_1]=\frac{1}{a}[A_1]$

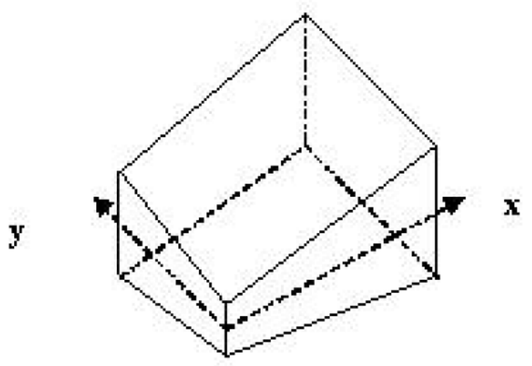


Fig. 3 Plate of variable thickness

Where

$$[\bar{A}_2] = [\bar{A}_1][\bar{A}_1]; \quad [\bar{A}_3] = [\bar{A}_1][\bar{A}_2] \ \ \textit{and so on} \tag{24a}$$

Or

$$[\bar{A}_m] = [\bar{A}_1][\bar{A}_{m-1}] \tag{24b}$$

Similarly considering the plate (shown in Fig. 3) the differential in y-direction can be written as

$$\frac{\partial f(x_i, y_i)}{\partial y} = \frac{1}{b}\sum_{j=1}^{N_x} B_{ij}^{(1)} f(x \ , y_j) \tag{25}$$

where i=1,2,3, N_y
or

$$\frac{\partial^n f(\xi, \eta_i)}{\partial \eta^n} = \sum_{j=1}^{N_y} B_{ij}^{(n)} f(\xi \ , \eta_j) = [B_n][f] \tag{26}$$

As discussed in x direction and defining $\eta = \frac{y}{b}$ a non-dimensional variable

And 'b' is the side of the plate in y direction.

$$[\bar{B}_1] = \frac{1}{b}[B_1] \tag{27}$$

or

$$[\bar{B}_2] = [\bar{B}_1][\bar{B}_1] \tag{28a}$$

$$[\bar{B}_3] = [\bar{B}_1][\bar{B}_2] \tag{28b}$$

$$[\bar{B}_n] = [\bar{B}_1][\bar{B}_{n-1}] \tag{28c}$$

The test functions for $f(x,y)=x^{\alpha-1}y^{\beta-1}$. When α=1,2,3…N_x, β=1,2,3…N_y, $[\bar{A}_m],[\bar{B}_n]$ give the higher order weighting coefficients. The above relations are now restricted to the choice of sampling points. Also calculation of weighting coefficients by these formulae contain a substantial reduction in numerical computations. This formulation of Eq. (19) to Eq. (28) can generally be used for $n\leq18$ after which round off errors arise. For most of the problems considered n=17 is used in the analysis. When actual material properties are specified, Eq. (28) can be used for $n\leq28$. Besides the formulations of Eqs. (19)-(28) explicit formulae are available to compute the weighting coefficients (Bert and Malik 1996).

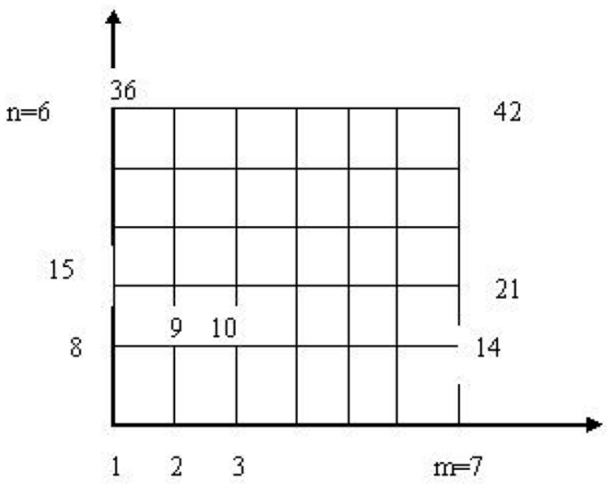


Fig. 4 Node numbering of a plate

8. Choice of sampling points and the relevant matrices

The section of location of the sampling points plays a significant role in the accuracy of the solution of differential equations. Using equally spaced points can be considered to be convenient and easy selection method. A more accurate solution by choosing a set of unequally spaced sampling points could be obtained. A simple and good choice can be roots of shifted Chebyshev and Legendre points. The points are

$$\xi_i = \frac{1}{2}\left[1 - \cos\left(\frac{(i-1)}{(n_x - 1)}\right)\pi\right] \quad i = 1,2,....n_x \ \text{ in the x direction and} \tag{29}$$

$$\xi_i = \frac{1}{2}\left[1 - \cos\left(\frac{(j-1)}{(n_y - 1)}\right)\pi\right] \quad i = 1,2,....n_y \ \text{ in the y direction.} \tag{30}$$

Assume in a rectangular plate, there are 'n_x=m' discrete points in x direction and 'n_y=n' points in y direction leading to total degrees of freedom (nt) as displacements at nt=m×n nodes which are numbered in order as shown in Fig. 4 for m=7 and n=6. $\frac{\partial w}{\partial x}$ for the plate is given by

$$\left\{\begin{matrix} \left\{\frac{\partial w}{\partial x}\right\}_{j=1} \\ \left\{\frac{\partial w}{\partial x}\right\}_{j=2} \\ \vdots \\ \left\{\frac{\partial w}{\partial x}\right\}_{j=n} \end{matrix}\right\} = \begin{bmatrix} [\bar{A}_1] & & & \\ & [\bar{A}_1] & & \\ & & \ddots & \\ & & & [\bar{A}_1] \end{bmatrix} \left\{\begin{matrix} \begin{Bmatrix} w_1 \\ w_2 \\ w_m \end{Bmatrix} \\ \begin{Bmatrix} w_{m+1} \\ w_{m+2} \\ w_{2m} \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} w_{m(n-1)+1} \\ w_{m(n-1)+2} \\ \vdots \\ w_{mn} \end{Bmatrix} \end{matrix}\right\} \tag{31}$$

where 'j' denotes the row numbers of the grid of the plate.

For the whole plate

$$\frac{\partial w}{\partial x} = [A_x]\{\underline{w}\} \tag{32}$$

where $\{\underline{w}\}$ are the nodal degrees of freedom in order.

Similarly

$$\begin{Bmatrix} \left\{\dfrac{\partial w}{\partial y}\right\}_{i=1} \\ \left\{\dfrac{\partial w}{\partial y}\right\}_{i=2} \\ \vdots \\ \left\{\dfrac{\partial w}{\partial y}\right\}_{i=m} \end{Bmatrix} = \begin{bmatrix} [\bar{B}_1] & & & \\ & [\bar{B}_1] & & \\ & & \ddots & \\ & & & [\bar{B}_1] \end{bmatrix} \begin{Bmatrix} \begin{Bmatrix} w_1 \\ w_{m+1} \\ \vdots \\ w_{m(n-1)+1} \end{Bmatrix} \\ \begin{Bmatrix} w_2 \\ w_{m+2} \\ \\ w_{m(n-1)+2} \end{Bmatrix} \\ \vdots \\ \begin{Bmatrix} w_m \\ w_{2m} \\ \vdots \\ w_{nm} \end{Bmatrix} \end{Bmatrix} \tag{33}$$

where 'i' denotes the column numbering of the grid of the plate.

$$\left\{\frac{\partial w}{\partial y}\right\}_{no} = [\bar{B}]\{\underline{w}\}_{no} \tag{34}$$

where the above equation shows the relationship between first differential of w with respect to y at any point and no-shows the degrees of freedom are not in order. Hence a relation can be written between not ordered degrees of freedom to the ordered degrees freedom given by

$$\{\underline{w}\}_{no} = [T]\{\underline{w}\} \tag{35}$$

Hence

$$\left\{\frac{\partial w}{\partial y}\right\} = [T]^T \left\{\frac{\partial w}{\partial y}\right\}_{no} = [T]^T[\bar{B}]\{\underline{w}\}_{no} = [T]^T[\bar{B}][T]\{\underline{w}\} \tag{36}$$

or

$$\frac{\partial w}{\partial y} = [B_y]\{\underline{w}\} \tag{37a}$$

where

$$[B_y] = [T]^T[\bar{B}][T] \tag{37b}$$

Other differentials can be derived as

$$\frac{\partial^2 w}{\partial x^2} = [A_x][A_x]\{\underline{w}\} = [A_{xx}]\{\underline{w}\}$$

$$\frac{\partial^3 w}{\partial x^3} = [A_x][A_{xx}]\{\underline{w}\} = [A_{xxx}]\{\underline{w}\}$$

$$\frac{\partial^4 w}{\partial x^4} = [A_{xx}][A_{xx}]\{\underline{w}\} = [A_{xxxx}]\{\underline{w}\}$$

$$\frac{\partial^2 w}{\partial y^2} = [\mathbf{B}_y][B_y]\{\underline{w}\} = [B_{yy}]\{\underline{w}\}$$

$$\frac{\partial^3 w}{\partial y^3} = [B_y][B_{yy}]\{\underline{w}\} = [B_{yyy}]\{\underline{w}\}$$

$$\frac{\partial^4 w}{\partial y^4} = [B_{yy}][B_{yy}]\{\underline{w}\} = [B_{yyyy}]\{\underline{w}\}$$

$$\frac{\partial^4 w}{\partial x^2 \partial y^2} = [A_{xx}][B_{yy}]\{\underline{w}\} = [C_{xxyy}]\{\underline{w}\} \tag{38}$$

$$\frac{\partial^4 w}{\partial x\, \partial y^3} = [A_x][B_{yyy}]\{\underline{w}\} = [C_{xyyy}]\{\underline{w}\}$$

$$\frac{\partial^4 w}{\partial x^3 \partial y} = [A_{xxx}][B_y]\{\underline{w}\} = [C_{xxxy}]\{\underline{w}\}$$

$$\frac{\partial^2 w}{\partial x\, \partial y} = [A_x][B_y]\{\underline{w}\} = [C_{xy}]\{\underline{w}\}$$

9. Formulation (single element)

9.1 Governing equation

Substituting differentials in Eq. (14) by the above matrices leads to the governing equation as

$$\underset{nt \times nt}{[DD]}\{\underline{w}\} + N[EE]\{\underline{w}\} - \omega^2[FF]\{\underline{w}\} = \underset{nt \times 1}{\{q\}} \tag{39}$$

where $nt = m \times n$.

$$[DD] = \begin{bmatrix} [D_{xi}][A_{xxxx}] + 2[B_i][C_{xxyy}] + [D_{yi}][B_{yyyy}] + 2[D_{x,xi}]\left\{[A_{xxx}] + \nu_{yx}[C_{xyy}]\right\} + \\ 2[D_{y,yi}]\left\{[B_{yyy}] + \nu_{xy}[C_{xxy}]\right\} + 4[D_{t,xi}][C_{xyy}] + 4[D_{t,yi}][C_{xxy}] + \\ [D_{x,xxi}]\left\{[A_{xx}] + \nu_{yx}[B_{yy}]\right\} + [D_{y,yyi}]\left\{\nu_{xy}[A_{xx}] + [B_{yy}]\right\} + 4[D_{t,xyi}[C_{xy}] \end{bmatrix} \tag{40a}$$

$[D_{xi}]$, $[B_i]$ etc are diagonal matrices. {EE] is given as

$$[EE] = t_x[p_{xi}][A_{xx}] + t_y[p_{yi}][B_{yy}] + 2t_{xy}[C_{xy}] \tag{40b}$$

where $[p_{xi}]$ …are diagonal matrices.
For vibration problems

$$[FF] = [\rho h_i] \tag{40c}$$

and [FF] is a diagonal matrix.

The matrices associated with 'i' are diagonal matrices.

9.2 Boundary conditions

9.2.1 At domain ends

1. Suppose if x=0 is a simply supported edge where w=0 and M_x=0, any node 'i' on this edge is designated by global node no as $node = nx \times (i-1) + 1$. The '$n$' th constraint equation for w=0 can be written as $B(n, \text{node})$=1.0 and M_x=0 constraint equation is written as

$$B(n, j) = \left\{A_{xx}(node, j) + \nu_{yx} \times B_{yy}(node, j)\right\}$$

where

$$j = 1,2,3\ldots\ldots nt \tag{41}$$

where $nt=m\times n$ degrees of freedom.

2. If y=0 is a clamped condition, global node number of 'i' th node on this edge is given as node=i and the boundary conditions w=o and $\frac{\partial w}{\partial y}=0$ can be written as

$$B(n,node)=1;\text{ and } B(n,j)=\{B_y\,(node,j)\}\quad \text{where } j=1,2,3\ldots nt \tag{42}$$

where 'n' is the constraint equation number.

3. Suppose if $x=a$ is a free edge condition, global node number of 'i'th node on the edge is given as $node=nx\times i$ where 'nx' is the number of nodes in x direction. The moment constraint equation can be written as stated in (1) as

$$B(n,j)=\{A_{xx}(node,j)+\nu_{yx}\times B_{yy}(node,j)\} \tag{43}$$

The modified shear V_x=0 can be written as

$$B[n,j)=\left\{A_{xxx}(node,j)+\left(4\left(\frac{D_t}{D_x}\right)_{node}+\nu_{yx}\right)\right.$$

$$\left.\left|\left(c_{xyy}(node,j)\right)-(N_x)_{node}A_x(node,j)\right.\right\} \tag{44}$$

Incorporating the boundary conditions at the domain ends, the governing equations with boundary conditions for one plate element can be written as

$$\begin{bmatrix}[DD]\\ [B]\end{bmatrix}\{\underline{w}\}+N\begin{bmatrix}[EE]\\ [F]\end{bmatrix}\{\underline{w}\}-\omega^2\begin{bmatrix}[FF]\\ [0]\end{bmatrix}\{\underline{w}\}=\begin{Bmatrix}\{q\}\\ \{0\}\end{Bmatrix} \tag{45}$$

where the size of [B] is $nc\times nt$ where nc is the number of constraint equations.

Applying Wilson's Lagrangian multiplier method (Wilson 2002), Eq. (45) is modified to

$$\begin{bmatrix}[DD] & [B]^T\\ [B] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}+N\begin{bmatrix}[EE] & [F]^T\\ [F] & [0]\end{bmatrix}$$

$$\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}-\omega^2\begin{bmatrix}[FF] & [0]\\ [0] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}=\begin{Bmatrix}\{q\}\\ \{0\}\end{Bmatrix} \tag{46}$$

For equilibrium problems

$$\begin{bmatrix}[DD] & [B]^T\\ [B] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}+N\begin{bmatrix}[EE] & [F]^T\\ [F] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}=\begin{Bmatrix}\{q\}\\ \{0\}\end{Bmatrix} \tag{47}$$

[B] has 'nr' (number of restraints) rows and 'nt' (no of displacement degrees of freedom) columns.

For Stability problems

$$\begin{bmatrix}[DD] & [B]^T\\ [B] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}=-N\begin{bmatrix}[EE] & [F]^T\\ [F] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix} \tag{48}$$

For free vibration problems (with or without axial forces)

$$\begin{bmatrix}[DD] & [B]^T\\ [B] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}+N\begin{bmatrix}[EE] & [F]^T\\ [F] & [0]\end{bmatrix}$$

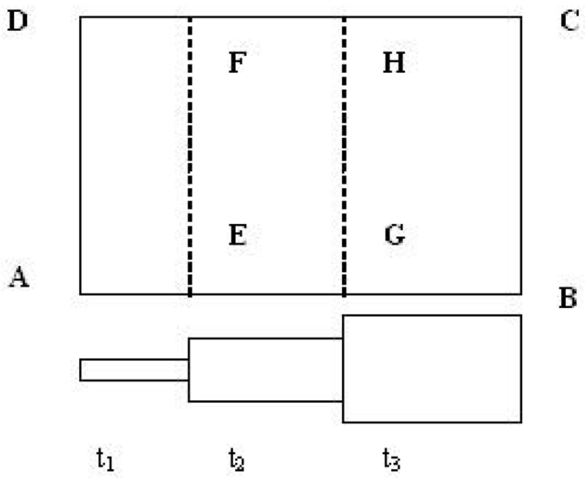


Fig. 5 Stepped plate

$$\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix}=\omega^2\begin{bmatrix}[FF] & [0]\\ [0] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}\\ \{\lambda\}\end{Bmatrix} \tag{49}$$

where λ denotes Lagrangian Multipliers.

Eq. (47), (48) and (49) are solved for equilibrium, stability and free vibration of a plate if it is idealized into one element.

10. Formulation (element based DQM)

Consider an orthotropic plate (stiffened plate) with stepped thickness as shown in Fig. 5. The three plate elements have thickness h_1,h_2,h_3 respectively. Assume the plate element 'i' has $nt_i=nx_i\times ny_i$ total degrees freedom where the 'i'th plate contains nx_i points in x direction and ny_i points in y direction. Let us consider for example free vibration problem. The governing equations with boundary conditions for the element 'i' can be written as

$$\begin{bmatrix}[DD]_i & [B]_i^T\\ [B]_i & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}_i\\ \{\lambda\}_i\end{Bmatrix}=\Omega\begin{bmatrix}[EE]_i & [F]_i^T\\ [F]_i & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}_i\\ \{\lambda\}_i\end{Bmatrix} \tag{50}$$

where Ω denotes the critical buckling load for stability problems or square of the natural frequency as $\Omega=\omega^2$ in case of free vibration problems and [EE]=[FF] and [F]=[0].

Where $[DD]_i$ is the matrix representing governing equilibrium equation and $[B]_i$ is the boundary condition matrix for the domain ends of an element 'i' and this excludes the compatibility and equilibrium conditions of the internal nodes on the common edges EF and GH connecting the elements 1-2 and 2-3. Combining the matrices given in Eq. (50) for all the three elements we get for stability problem

$$\begin{bmatrix}[DD]_1 & [0] & [0] & [B]_1^T & [0] & [0]\\ [0] & [DD]_2 & [0] & [0] & [B]_2^T & [0]\\ [0] & [0] & [DD]_3 & [0] & [0] & [B]_3^T\\ [B]_1 & [0] & [0] & [0] & [0] & [0]\\ [0] & [B]_2 & [0] & [0] & [0] & [0]\\ [0] & [0] & [B]_3 & [0] & [0] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}_{nt_1}\\ \{\underline{w}\}_{nt_2}\\ \{\underline{w}\}_{nt_3}\\ \{\lambda\}_{nr_1}\\ \{\lambda\}_{nr_2}\\ \{\lambda\}_{nr_3}\end{Bmatrix}=$$

$$\Omega\begin{bmatrix}[EE]_1 & [0] & [0] & [F]_1^T & [0] & [0]\\ [0] & [EE]_2 & [0] & [0] & [F]_2^T & [0]\\ [0] & [0] & [EE]_3 & [0] & [0] & [F]_3^T\\ [F_1] & [0] & [0] & [0] & [0] & [0]\\ [0] & [F_2] & [0] & [0] & [0] & [0]\\ [0] & [0] & [F]_3 & [0] & [0] & [0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}_{nt_1}\\ \{\underline{w}\}_{nt_2}\\ \{\underline{w}\}_{nt_3}\\ \{\lambda\}_{nr_1}\\ \{\lambda\}_{nr_2}\\ \{\lambda\}_{nr_3}\end{Bmatrix} \tag{51}$$

where $[DD]_i$ has nt_i rows and nt_i columns whereas $[B]_i$ has nr_i rows and nt_i columns. In general the edges of the plate (other than common edges) has one of the following conditions 1) free edge 2) clamped edge 3) simply supported edge 4)and the edge which is common to two elements. The common edges can be 4) simply attached edge between elements, 5) supported edge 6) edge with internal hinge and hence the boundary conditions of the edges are denoted by the numerals 1,2,3,4,5 and 6 depending the nature of support. Eq. (46) considered the boundary conditions for conditions, 1,2 and 3 and now we have to arrive at the boundary condition matrix for the internal nodes on the common edge for three conditions 4) simply attached 5) supported edge 6) edge with internal hinge.

11. Internal nodes on the attached EDGE

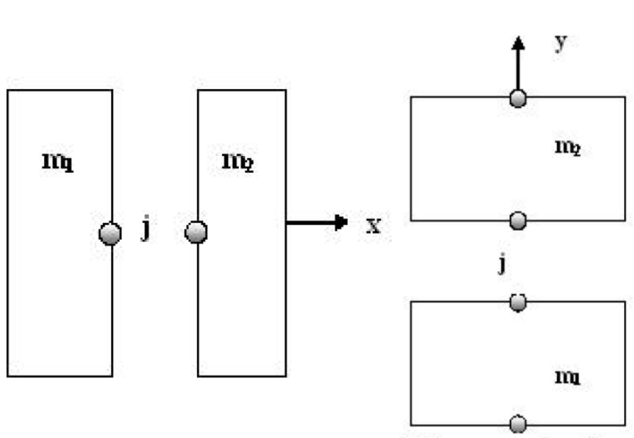
1) attached edge without support (nature of support-4) Consider two elements (m_1,m_2) attached in x direction as shown in Fig. 6(a)

The compatibility and equilibrium conditions at any node 'j' of the common edge may be given as

<u>Compatibility</u>

$$\begin{aligned}&\left(w^{m1}\right)_{nx,j}-\left(w^{m2}\right)_{1,j}=0\\ &\left(\left(\frac{\partial w}{\partial x}\right)^{m1}\right)=\left(\left(\frac{\partial w}{\partial x}\right)^{m2}\right)_{1,j}\end{aligned} \tag{52a}$$

<u>Equilibrium</u>



(a) (b)

Fig. 6 Edge connecting elements in X and Y directions

$$\begin{aligned}&\left(M_x^{m1}\right)_{nx,j}-\left(M_x^{m2}\right)_{1,j}=0\\ &\left(Q_x^{m1}\right)_{nx,j}-\left(Q_x^{m2}\right)_{1,j}=0\\ &\left(M_{xy}^{m1}\right)_{nx,j}-\left(M_{xy}^{m2}\right)_{1,j}\end{aligned} \tag{52b}$$

The three equilibrium conditions can be simplified to two equilibrium conditions as recommended by Kirchhoff, we get

$$\begin{aligned}&\left(M_x^{m1}\right)_{nx,j}-\left(M_x^{m2}\right)_{1,j}=0\\ &\left(V_x^{m1}\right)_{nx,j}-\left(V_x^{m2}\right)_{1,j}=0\end{aligned} \tag{52c}$$

Hence Eq. (47a) and Eq. (47c) are used as boundary conditions for the common unsupported edge.

2) Attached edge supported (nature of support-5)

<u>Compatibility</u>

$$\begin{aligned}&\left(w^{m1}\right)_{nx,j}=0\\ &\left(w^{m2}\right)_{1,j}=0\\ &\left(\left(\frac{\partial w}{\partial x}\right)^{m1}\right)_{nx,j}=\left(\left(\frac{\partial w}{\partial x}\right)^{m2}\right)_{1,j}\end{aligned} \tag{53a}$$

<u>Equilibrium</u>

$$\left(M_x^{m1}\right)_{nx,j}-\left(M_x^{m2}\right)_{1,j}=0 \tag{53b}$$

3) Attached edge with internal hinge (nature of support-6)

<u>Compatibility</u>

$$\left(w^{m1}\right)_{nx,j}-\left(w^{m2}\right)_{1,j}=0 \tag{54a}$$

<u>Equilibrium</u>

$$\begin{aligned}&\left(M_x^{m1}\right)_{nx,j}=0\\ &\left(M_x^{m2}\right)_{1,j}=0\\ &\left(V_x^{m1}\right)_{nx,j}-\left(V_x^{m2}\right)_{1,j}=0\end{aligned} \tag{54b}$$

In any case there will be four conditions at a node lying on the common edge.

Similarly if the conjunction of two elements m_1,m_2 is made in y direction as shown in Fig. 6(b) the conditions are written as

1) Attached edge without support (nature of support-4)

<u>Compatibility</u>

$$\begin{aligned}&\left(w^{m1}\right)_{i,ny}-\left(w^{m2}\right)_{i,1}=0\\ &\left(\left(\frac{\partial w}{\partial y}\right)^{m1}\right)_{i.ny}=\left(\left(\frac{\partial w}{\partial y}\right)^{m2}\right)_{i,1}\end{aligned} \tag{55a}$$

<u>Equilibrium</u>

$$\begin{aligned}&\left(M_y^{m1}\right)_{i.ny}-\left(M_y^{m2}\right)_{i,1}=0\\&\left(V_y^{m1}\right)_{i,ny}-\left(V_y^{m2}\right)_{i,1}=0\end{aligned}\tag{55b}$$

Similarly, four conditions are obtained for other two boundary conditions of the common edge.

In any case, the compatibility and equilibrium conditions for the common edge may be written as

$$[Cx]_1\begin{Bmatrix}\{\underline{w}\}_{nt_1}\\\{\underline{w}\}_{nt_2}\end{Bmatrix}=\{0\};\ [Cx]_2\begin{Bmatrix}\{\underline{w}\}_{nt_2}\\\{\underline{w}\}_{nt_2}\end{Bmatrix}=\{0\}\tag{56}$$

or in general, the conditions for the common edges may be given by

$$\begin{aligned}&[CO]_1\begin{Bmatrix}\{\underline{w}\}_{nt_1}\\\{\underline{w}\}_{nt_2}\\\{\underline{w}\}_{nt_3}\end{Bmatrix}=\{0\}\\&[CO]_2\begin{Bmatrix}\{\underline{w}\}_{nt_1}\\\{\underline{w}\}_{nt_2}\\\{\underline{w}\}_{nt_3}\end{Bmatrix}=\{0\}\end{aligned}\tag{57}$$

where $[CO]_1$, $[CO]_2$ contains ne_1 and ne_2 rows corresponding to constraint equations for the two common edges. Combining these conditions with Eq. (51) we get for the three element plate the equilibrium and compatibility matrix is given by

$$\begin{bmatrix}[DD]_1&[0]&[0]&[B]_1^T&[0]&[0]&\vdots&\vdots\\ [0]&[DD]_2&[0]&[0]&[B]_2^T&[0]&[CO]_1^T&[CO]_2^T\\ [0]&[0]&[DD]_3&[0]&[0]&[B]_3^T&\vdots&\vdots\\ [B]_1&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [0]&[B]_2&[0]&[0]&[0]&[0]&[0]&[0]\\ [0]&[0]&[B]_3&[0]&[0]&[0]&[0]&[0]\\ \cdots&[CO]_1&\cdots&[0]&[0]&[0]&[0]&[0]\\ \cdots&[CO]_2&\cdots&[0]&[0]&[0]&[0]&[0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}_{nt_1}\\\{\underline{w}\}_{nt_2}\\\{\underline{w}\}_{nt_3}\\\{\lambda\}_{nr_1}\\\{\lambda\}_{nr_2}\\\{\lambda\}_{nr_3}\\\{\lambda\}_{ne_1}\\\{\lambda\}_{ne_2}\end{Bmatrix}=$$

$$\Omega\begin{bmatrix}[EE]_1&[0]&[0]&[F]_1^T&[0]&[0]&[0]&[0]\\ [0]&[EE]_2&[0]&[0]&[F]_2^T&[0]&[0]&[0]\\ [0]&[0]&[EE]_3&[0]&[0]&[F]_3^T&[0]&[0]\\ [F]_1&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [0]&[F]_2&[0]&[0]&[0]&[0]&[0]&[0]\\ [0]&[0]&[F]_3&[0]&[0]&[0]&[0]&[0]\\ [0]&[0]&[0]&[0]&[0]&[0]&[0]&[0]\\ [0]&[0]&[0]&[0]&[0]&[0]&[0]&[0]\end{bmatrix}\begin{Bmatrix}\{\underline{w}\}_{nt_1}\\\{\underline{w}\}_{nt_2}\\\{\underline{w}\}_{nt_3}\\\{\lambda\}_{nr_1}\\\{\lambda\}_{nr_2}\\\{\lambda\}_{nr_3}\\\{\lambda\}_{ne_1}\\\{\lambda\}_{ne_2}\end{Bmatrix}\tag{58}$$

where $nt=nt_1+nt_2+nt_3$ denotes the total degrees of freedom, ne_1, ne_2 show the constraint equations on the two common edges connecting plate elements and $nr1$, $nr2$ and $nr3$ are the constraint equations for the three elements..

In case of free vibration problems instead of [EE] matrix we use [FF] matrix and [F]s are zero. Thus one will be able to solve orthotropic plate with stepped thickness or continuous orthotropic plate for stability, equilibrium and free vibration problems.

12. Designation of boundary conditions

For a single element plate we denote boundary conditions in the order of west-south-east-north edges. For example SCFC denotes x=0, simply supported, y=0, clamped, $x=a$, free edge and $y=b$, clamped edge. For a plate with three elements shown in Fig. 5, the boundary conditions can be denoted as SCAF-ASIF –ISFC denoting for the first element AEFD –(see Fig. 5) SCAF- west (AD)-simply supported, south (AE)- clamped, east–(EF) attached, north (FD)–free and for the second element west (EF)-attached, south (EG)–simply supported, east(GH)-supported, north–(HF) free and for the third element west(GH)–supported, south(GB)–simply supported, east (BC)-free, north(CH)-clamped If the attached is supported it designated as I as discussed above and if it is a hinge line, then it is designated by ‘M’.

13. For isotropic plate

We make the substitution for isotropic plate as

$$\begin{aligned}&E_x=E_y=E\\&\nu_{xy}=\nu_{yx}\\&G=\frac{E}{2(1+\nu)}\\&\rho=\textit{mass density (given)}\end{aligned}\tag{59}$$

Then one will be able to solve equilibrium, stability and free vibration of isotropic plates with continuously varying or stepped thickness resting on elastic foundation supported by any type of boundary conditions.

14. Equilibrium problems

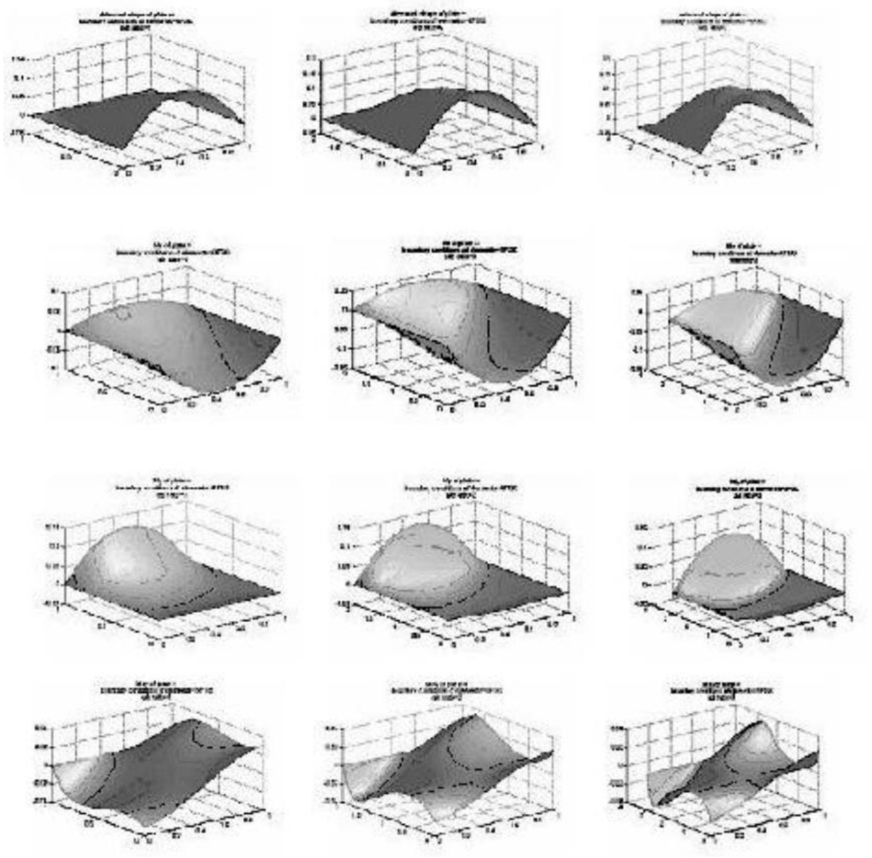
14.1 Static analysis of isotropic plate

An isotropic square plate with the edges built in is subjected to uniformly distributed load of ‘q’. By using EDQM, we obtain the maximum deflection at the centre of

Table 2 Deflections and bending moments for a uniformly loaded plate (SFSC) (E=1, v=0.3, b=1) ($m=n$=17)

b/a	analysis	$w_{\max}$	M_x ($x=a/2$, $y=b$)	M_y ($x=a/2$, y=0)
0.33	present	1.02632	0.0859	0.428
	Ref	1.02648	0.0702	0.428
0.5	present	0.63587	0.1172	0.319
	Ref	0.63554	0.1172	0.319
1	present	0.12271	0.0971	0.118
	Ref	0.12339	0.0972	0.119
1.5	present	0.15449	0.123	0.123
	Ref	0.15397	0.123	0.124
2	present	0.16381	0.13	0.124
	Ref	0.16381	0.131	0.125
3	present	0.16604	0.132	0.124
	Ref	0.16598	0.133	0.125

(Ref-Timoshenko and Kreiger 1959)



a b c

Fig. 7 Static analysis –deflection, bending and twisting moments of SFSC plate

the plate as 0.0138 as against Timoshenko and Krieger (1959), $0.0137572=\dfrac{0.00126qa^4}{D}$ for side a=1, h=1, E=1; v=0.3, q=1 (m=n=17). Table 2 shows the static analysis of uniformly loaded SFSC isotropic plate (b=1; E=1, v=0.3, h=1,for various b/a ratios and Fig. 7(a), 7(b), 7(c) show the deflection 'w', M_x, M_y and M_{xy} for b/a ratios of 1, 2 and 3 and compared with Timoshenko and Kreiger (1959) and the comparison is quite good. Timoshenko and Kreiger (1959) give the values of $w_{\max}=k_1qb^4/D$; M_x ($x=a/2$, $y=b$)=k_2qa^2, $M_y(x=a/2,\ y=0)=\ k_3qb^2$ and for example b/a=1/3, k_1,k_2,k_3 values are 0.094, 0.0078 and 0.428 respectively (Table 39 of Timoshenko and Krieger 1959). It is seen that the deflection decreases up to b/a ratio=1 and then increases.

14.2 Static analysis of orthotropic plate clamped at all four edges

A simply supported orthotropic square plate of side=1 with unit thickness and E_x=25; E_y=1;G_{xy}=0.5;v_{xy}=0.25 is subjected to lateral load 'q=1'. The maximum deflection occurs at the centre of the plate and its value is 0.0065 which is subjected to uniformly distributed load agrees with Reddy's (2004) value of 0.006497. The maximum M_x moment of 0.1311 occurs at the centre of the plate and the maximum moment M_y=0.007519 occurs at x=0.5 y=0.8535 and the maximum twisting moment M_{xy} is 0.00774 which occurs at four corners.

14.3 Continuous orthotropic plate

A rectangular plate with a single step change in thickness (two spans of length of each=1 m) and width 1 m simply supported on all sides and supported at centre with E_x=25 GPa; E_y=1 GPa; G_{xy}=0.5 GPa; v_{xy}=0.25 is subjected to uniformly distributed load of unit value (q=1 N/sq.m) in both the spans. The right span of the plate has twice the thickness of left span of the plate. h_1=0.001 m; h_2=0.002 m is assumed. Maximum deflection 0.00277426 m occurs at y=0.5 m and x=0.4025375 m and maximum moment M_x =0.13199 N.m occurring at x=1 m y=0.5 m and maximum moment of M_y=0.0049858 N.m occurring at x=0.4025375 m and y=0.8535 m and maximum twisting moment of 0.0049507 N.m occurring at x=2 m and y=1 m. Figs. 8, 9, 10 and 11 show the deformed shape, M_x, M_y and M_{xy} diagrams for the continuous orthotropic plate. Two elements are used with m=n=17.

15. Stability problems

15.1 Orthotropic plates with 'CFCF', 'SFSF' and 'SFSC' boundary conditions

Consider an orthotropic rectangular plate with thickness h=2 mm dimensions b=1 m and a=1,2 and 4 m for first two cases and a=1,3 5 m for SFSC condition plate. In the first

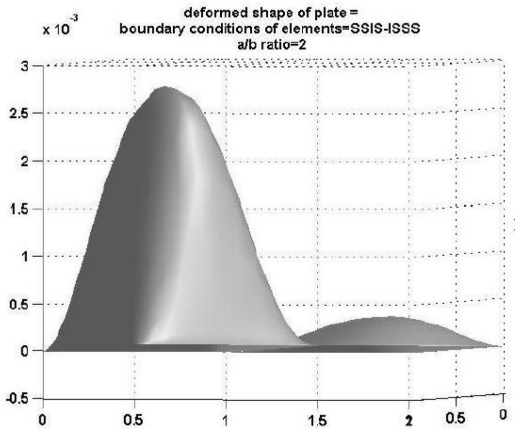


Fig. 8 Deformed shape of continuous orthotropic plate *a/b*=2

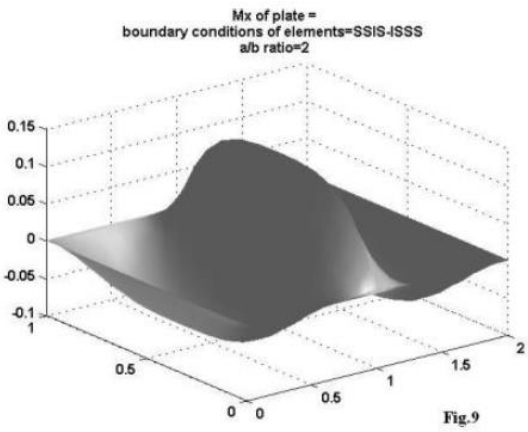


Fig. 9 Bending moment M_x of continuous orthotropic plate *a/b*=2

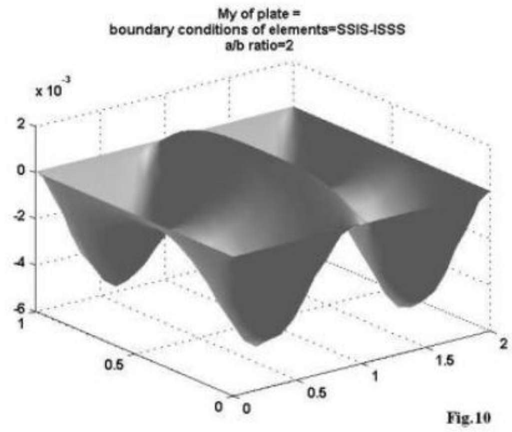


Fig. 10 Bending moment M_y of continuous orthotropic plate *a/b*=2

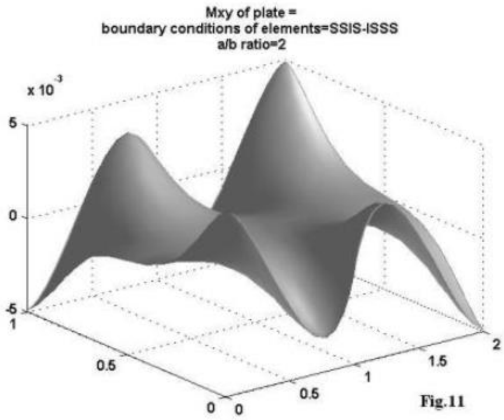


Fig. 11 Torsional moment M_{xy} of continuous orthotropic plate *a/b*=2

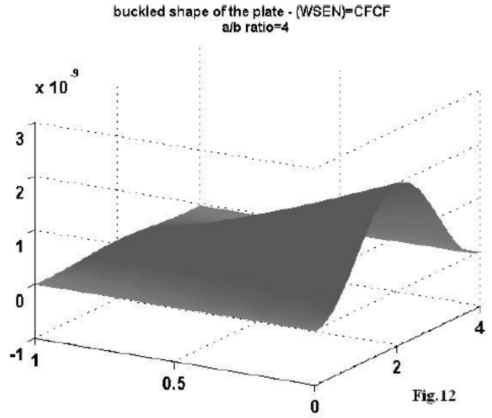


Fig. 12 Buckled shape of 4×1 m CFCF orthotropic plate for material=3 α_x=2

case the plate is clamped at x=0 and x=a and free at y=0 and y=b. The three variant orthotropic elastic characteristics that differ from each other by the values of E_x, E_y are considered as Mat 1, 2 and 3 as shown in Table 3. The plate is subjected to axial compressive force at x=0 and x=a. and we will consider the three variations in p_x expression of the axial load as (α_x=0−*uniform*; α_x=1 *triangle* (max at y=0 and 0 at y=b); α_x=2−pure bending). The material properties expressed in Table 3 are typical of modern advanced composite materials normally used in industry. The critical buckling loads are calculated and presented in the Table 3 for (SFSF) and (CFCF) and compared with Lopatin and Morozov (2009, 2010, 2014). Lopatin and Morozov employed Kantorovich procedure to obtain the governing equation and boundary conditions and the buckling problem for SFSF and CFCF is solved using the generalized Galerkin method. They also verified the results by finite element model by idealizing the plate with 2000, 5000, and 10000 elements of 1×1 m , 2×1 m and 4×1 m orthotropic plates. In Table 3 the results of present analysis are compared with Finite element method of Lopatin and Morozov. The buckling mode shapes for 4×1 m CFCF orthotropic plate with material properties Material=3 for α_x=2 is shown in Fig. 12.

An orthotropic plate (SFSC) (Lopoatin and Morozov 2011) with h=1 mm is analyzed for buckling for various distributions of axial force. The material properties are shown in the Table 4. Table 4 presents the results of present analysis and their comparison with Lopatin and Morozov (2014). The buckled shape for 5×1 m (a×b) for orthotropic plate with material properties (material 1) for α_x=2 and a=5 is shown in Fig. 13. The buckled mode shape for 5×1 m

Table 3 Critical buckling load $N_x(cri)$ (Newtons) for orthotropic rectangular plate for various axial force distributions for CCFF and SSFF plates (b=1)
(bracketed values are from Lopatin and Morozov (2010) by FEM

Material	α_x	a	CFCF	SFSF
			This Analysis	This analysis
Mat-1 (*orthotropic*) E_x=144 GPa E_y=9.65 GPa G_{xy}=4.16 GPa v_{xy}=0.3 v_{yx}=0.020104	0	1	3821.42	950.42
		2	953.30	237.22
		4	238.17	59.245
	1	1	4533.95	1227.688
		2	1250.75	357.187
		4	360.2272	105.114
	2	1	5073.33 (5117.32)	1460.946
		2	1530.915 (1525.16)	485.629
		4	499.327 (497.75)	182.945
Mat-2 (*orthotropic*) E_x=54.55 GPa E_y=54.55 GPa G_{xy}=20.67 GPa v_{xy}=0.32 v_{yx}=0.32	0	1	1568.653	377.875
		2	385.447	91.835
		4	94.4116	22.5981
	1	1	2261.389	649.9043
		2	663.7412	175.7009
		4	180.5089	44.6878
	2	1	2886.740 (2872.69)	1025.925
		2	1060.69 (1063.53)	447.2879
		4	457.302 (461.67)	214.2456
Mat-3 (*orthotropic*) E_x=9.65 GPa E_y=144 GPa G_{xy}=4.16 GPa v_{xy}=0.020104 v_{yx}=0.3	0	1	255.3767	63.528
		2	63.8823	15.8757
		4	15.9616	3.9489
	1	1	383.0393	111.9597
		2	112.580	30.576
		4	30.7374	7.8196
	2	1	527.4673 (526.13)	191.3804
		2	192.821 (192.12)	83.9052
		4	84.2213 (83.93)	40.1712

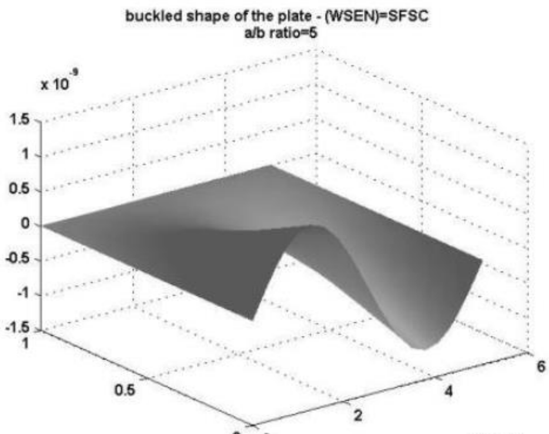


Fig..13

Fig. 13 Buckled shape of 5×1 m SFSC orthotropic plate for material=1 α_x=2

Table 4 Critical buckling load $N_x(cri)$ (Newtons for orthotropic rectangular plate for
Various axial force distributions for SFSC plates (b=1)

Material	α_x	a	This Analysis	Lopatin and Morozov(2014)
Mat-1 (*orthotropic*) E_x=142.8 GPa E_y=9.13 GPa G_{xy}=5.13 GPa v_{xy}=0.32 v_{yx}=0.02	0	1	125.407	127.26
		2	29.98	29.86
		4	33.0	33
	1	1	155.1051	157.78
		2	37.51	37.02
		4	40.82	40.92
	2	1	229.293	207.58
		2	49.48	48.70
		4	53.687	53.83
Mat-2 (*isotorpic*) E_x=54.55 GPa E_y=54.55 GPa G_{xy}=20.67 GPa v_{xy}=0.32 v_{yx}=0.32	0	1	86.24	
		2	63.06	
		4	62.69	
	1	1	110.25	
		2	80.08	
		4	78.66	
	2	1	147.63	
		2	106.4	
		4	104.32	
Mat-3 (*orthotropic*) E_x=9.13 GPa E_y=142.8 GPa G_{xy}=5.49 GPa v_{xy}=0.02 v_{yx}=0.32	0	1	30.48	30.64
		2	28.986	29.96
		4	29.22	29.39
	1	1	37.98	37.99
		2	36.14	37.15
		4	36.49	36.44
	2	1	49.996	49.98
		2	47.452	48.87
		4	48.07	47.94

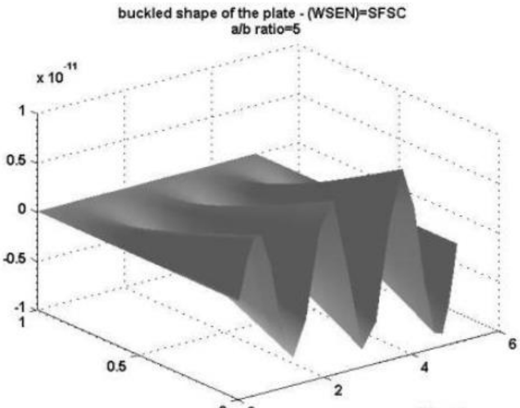


Fig.14

Fig. 14 Buckled shape of 5×1 m SFSC orthotropic plate for material=3 α_x=2

(a×b) orthotropic plate with material properties =material 3 as shown in Table 4 for α_x=2 is shown in Fig. 14.

15.2 Buckling of non-homogeneous orthotropic plates with varying thickness

First the method is applied to simply supported isotropic

Table 5(a) Effect of taper ratio on buckling load parameter $p=\frac{N_x L_y^2}{\pi^2 D_{av}}$ for different modes of buckling. (h_0=1; E=1; ν=.33) for a square plate
(*values are from Reference Eisenberger and Alexandrev 2003)

Mode \ h_1/h_0	1	1.125	1.25	1.5	1.75	2.0
1	4.0000	3.9660 3.966*	3.8811 3.882*	3.6355 3.638*	3.3607 3.364*	3.0954 3.1000*
2	6.2499	6.2346	6.2346	6.1740	6.0580	5.8950
3	11.1111	11.0723	10.9773	10.6689	10.2979	8.9399
4	15.9998	15.4375	14.3635	12.1892	10.3835	9.9084
5	18.0626	17.9916	17.8100	17.2497	16.5604	15.2002

Table 5(b) Non-homogeneous orthotropic plate with varying thickness
(Critical load in Newtons)

BCS	a/b	μ	β	$m=n$=13	$m=n$=17	$m=n$=21	$m=n$=25	$m=n$=29
SCSF	2.0	0.5	0.5	27.1888	23.0286	21.5280	20.7136	20.6072
SCSS	1.0	0	0	32.2174	32.2174	32.2174	32.2174	32.2174
SCSC	0.5	-0.5	-0.5	18.0138	18.0139	18.0139	18.0139	18.0139

plate with varying thickness $h=h_0\left(1+\frac{\beta_x x}{a}\right)$ in x direction where $\beta=\frac{(h_1-h_0)}{h_0}$, and h_1 and h_0 represent the thickness of the plate at x=0 and $x=a$ respectively. Nondimensional buckling load parameter is givn by $p=\frac{N_x L_y^2}{\pi^2 D_{av}}$ where N_x is the axial compressive load in x-direction, L_y, length of the plate in y-direction and D_{av} is the average stiffness of the plate. $D_{av}=\frac{E h_{av}^3}{12(1-\nu^2)}$. h_{av} is he average thickness of the plate given by $h_{av}=\frac{(h_0+h_1)}{2}$ and Poisson's ratio ν=0.33. Table 5(a) gives the nondimensional buckling loads for various modes for different thickness ratios. The fundamental mode buckling load is compared with Eisenberger and Alexandrov (2003) in Table 5(a) and there is a good comparison. It is seen that non dimensional buckling load parameter decreases for all the modes as the thickness ratio increases.

Consider a non-homogeneous orthotropic plate of variable thickness is subjected to axial force in x-direction. The Youngs modulii E_x,E_y thickness vary as

$$E_x=E_1 e^{\mu X};\ E_y=E_2 e^{\mu X};\ \ h=h_0 e^{\beta x X} \qquad (60)$$

where E_1,E_2 and h_0 are the values of E_x,E_y and h at the origin x=0 and $X=x/a$ a non-dimensional variable. The elastic constants for the material are taken as $E_1=1\times10^{10}$ MPa; $E_2=5\times10^9$ MPa; ν_{xy}=0.2; ν_{yx}=0.1, $G_{xy}=\frac{\sqrt{E_x E_y}}{2\left(1+\sqrt{\nu_{xy}\nu_{yx}}\right)}$ and h_0 is taken as 0.001 m. The boundary condition x=0 and $x=a$ can be taken as simply supported and we will consider three boundary conditions at y=0 and $y=b$ as CC, CS and CF and the aspect ratio is taken as $c=a/b$. μ, β are varied for each problem and the buckling load is calculated as shown in Table 5(b) and no comparison is available. Results are available for $m=n$=13, 17, 21, 25 and one can see the monotonic convergence of the critical load.

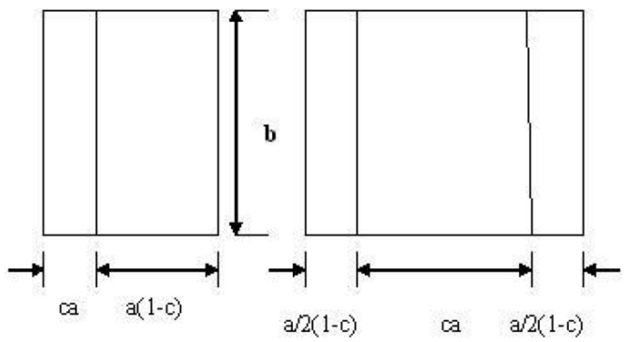


Fig. 15 Layout of Levy square plate having two and three spans

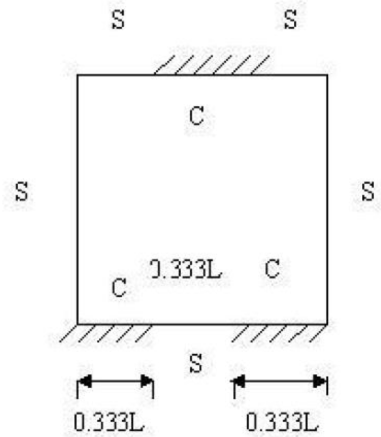


Fig. 16 Isotropic square plate with irregular boundary

15.3 Buckling of continuous isotropic plates of uniform thickness

Fig. 15 shows the layout of Levy square plate having two or three spans. The location of interior line supports are denoted by location parameter c as shown in Fig. 15. The plate may be loaded with a uni-axial in-plane loading in the x-direction (t_x=1, p_x=1; t_y=0) or y direction (t_x=0, t_y=1, p_y=1) or biaxial in-plane loading (t_x=1, p_x=1; t_y=1, p_y=1) respectively. The buckling load in terms of non-dimensional buckling parameter is given by $n_{cr}=\frac{N_{cr}b^2}{\pi^2 D}$. AD and BC are simply supported ends. The two span plate will have boundary condition denoted by SSIS-ISSS denoting AB and DC also simply supported. 'I' denotes the intermediate continuous support. Table 6 shows the comparison of buckling load parameter with those of Xiang (2003) for symmetric boundary conditions and Table 7 to un-symmetric boundary conditions. Xiang used Levy's solution procedure for the calculation of buckling load. Two or three elements for one or two spans with $m=n$=17 are used.

Table 6 Exact buckling factor $n_{cr}=\dfrac{N_{cr}b^2}{\pi^2 D}$ for symmetric Levy plate with SS, FF and CC boundary conditions for two and three unequal spans
Values shown by * are those obtained by Xiang (2003)

Span case	c	p_x=1; p_y=0			p_x=0; p_y=1			p_x=1; p_y=1		
		SS	FF	CC	SS	FF	CC	SS	FF	CC
Two Unequal spans	1/5	5.3165	2.3966	8.0315	8.4838	1.7611	10.209	3.3631	1.1076	5.2776
		5.3165*	2.3966*	8.0315*	8.4838*	1.7611*	10.209*	3.3631*	1.1076*	5.2776*
	2/5	6.0482	2.0936	9.8920	12.8672	2.4128	17.3611	4.5774	1.1746	7.7621
		6.0482*	2.0936*	9.8921*	12.8672*	2.4128*	17.3611*	4.5774*	1.1746*	7.7621*
	1/2	6.2499	2.0429	10.386	15.999	2.6726	22.422	5.0	1.1899	8.6205
		6.25*	2.0429*	10.386*	16.0*	2.6726*	22.422*	5.0*	1.1899*	8.6205*
Three Unequal spans	1/3	11.111	2.5935	15.6715	35.998	8.6897	43.1573	9.9998	2.0215	14.2329
		11.111*	2.5935*	15.665*	36.0*	8.6974*	43.115*	10.0*	2.0218*	14.225*
	1/2	10.3865	2.6265	12.114	22.420	12.389	22.800	8.6205	2.2039	9.9270
		10.386*	2.6277*	12.114*	22.422*	12.389*	22.801*	8.6205*	2.2040*	9.9270*
	7/10	8.6119	2.8741	9.0573	12.0356	10.1904	12.3421	6.0147	2.3133	6.3532
		8.6119*	2.8740*	9.0524*	12.036*	10.191*	12.342*	6.0147*	2.3133*	6.3532*

Table 7 Exact buckling factor $n_{cr}=\dfrac{N_{cr}b^2}{\pi^2 D}$ for symmetric Levy plate with SF, CF and CS boundary conditions for two and three unequal spans
Values shown by * are those obtained by Xiang (2003)

Span case	c	p_x=1; p_y=0			p_x=0; p_y=1			p_x=1; p_y=1		
		SF	CF	CS	SF	CF	CS	SF	CF	CS
Two Unequal spans	3/10	2.3928	2.3959	5.8880	2.3220	2.3693	10.223	1.2857	1.3005	4.0610
		2.3928*	2.3959*	5.8880*	2.3221*	2.3693*	10.223*	1.2857*	1.3005*	4.0610*
	1/2	2.3658	2.4078	7.4163	3.8046	3.9455	17.7883	1.5125	1.5551	5.9943
		2.3658*	2.4078*	7.4163*	3.8046*	3.9455*	17.788*	1.5125*	1.5551*	5.9943*
	7/10	2.5264	2.6811	8.2579	7.8082	9.7427	12.6566	1.9557	2.1346	6.0121
		2.5264*	2.6811*	8.2579*	7.8082*	9.7427*	12.657*	1.9557*	2.1346*	6.0121*
Three Unequal spans	1/3	2.8629	2.8748	12.3028	9.4466	9.4534	37.7275	2.2616	2.2719	11.0969
		2.8657*	2.8777*	12.317*	9.4716*	9.4785*	37.794*	2.2655*	2.2758*	11.112*
	1/2	3.1204	3.1415	11.124	13.5268	13.5731	22.6084	2.6544	2.6745	9.2001
		3.1203*	3.1415*	11.124*	13.527*	13.573*	22.609*	2.6543*	2.6745*	9.2001*
	7/10	3.6601	3.6681	8.8269	11.0139	11.1458	12.1869	3.0543	3.0751	6.1793
		3.6597*	3.6850*	8.8269*	11.014*	11.137*	12.187*	3.0540*	3.0876*	6.1793*

Table 8 Buckling coefficient factor for an isotropic plate with irregular boundary

	p_x=1; p_y=0	p_x=0; p_y=0	p_x=1; p_y=10	p_x=0; p_y=0, t_{xy}=1
n_{cr}	7.3032	6.4293	3.5206	11.7765

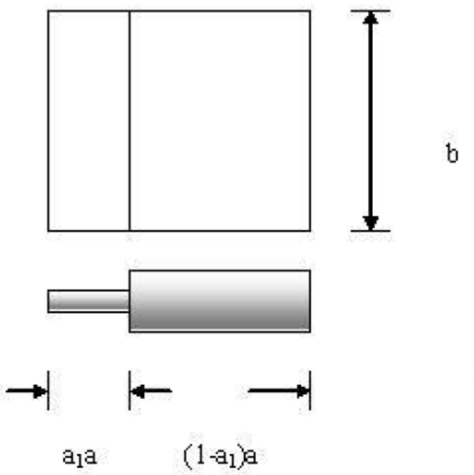


Fig. 17 One step Levy rectangular isotropic stiffened plate

15.4 Buckling of isotropic square plate with irregular boundary

Consider an isotropic square plate with irregular boundary as shown in Fig. 16 is subjected to uni-axial, biaxial compressive forces and shear force on all edges. The critical buckling load parameters are calculated and tabulated in Table 8 and no comparison is available. Three EDQM elements with m=n=17 are used.

15.5 Buckling of isotropic rectangular plate with one step change in thickness

A one step square Levy plate subjected in in-plane load as shown in Fig. 17 is considered. For the buckling analysis of thin plate, we consider three in-plane loading cases namely 1) uni-axial in-plane compressive load in x-direction 2) uni-axial in-plane compressive load in y-direction 3) eqi-biaxial in-plane compressive load. Table 9 presents the buckling coefficient ($n_{cr}=\dfrac{N_{CR}B^2}{\pi^2 d}$) generated

Table 9 Comparison of buckling factor for a thin iso-tropic square plate with one step subjected to uni-axial, biaxial and shear loadings
Values denoted by * are from Xiang and Wei (2004)

(p_x, p_y, p_{xy})	t_1/t_0	C	SS	CC	SF	CS
(1,0,0)	1.2	0.3	5.7331 5.7389*	10.1841 101929*	4.2068 3.9616*	7.7198 7.7310*
		0.5	4.9827 4.9616*	8.4421 8.3862*	3.9781 3.97668*	6.8243 6.8186*
		0.7	4.5346 4.5093*	7.6326 7.5966*	3.6487 3.4945	5.8510 5.8171*
	2	0.3	10.6025 10.4296*	19.8860 19.6097*	10.5777 10.3339*	18.9232 18.7209*
		0.5	7.8921 7.7698*	14.0292 13.8128*	7.8882 7.7133*	12.6111 12.2933*
		0.7	5.9374 5.8981*	9.8776 9.8258*	5.9309 5.8587*	9.1842 9.0179*
(0,1,0)	1.2	0.3	5.96645 5.9772*	11.7770 11.7662*	2.3053 2.2603*	8.6345 8.6478*
		0.5	5.2107 5.1961*	9.9643 9.9132*	2.1258 2.0894*	7.6638 7.6546*
		0.7	4.6289 4.6011*	8.6090 8.5567*	1.8846 1.8654*	6.7218 6.6832*
	2	0.3	16.5408 16.4352*	34.0522 33.8744*	8.6284 8.5261*	25.1581 25.1480*
		0.5	11.1327 10.8789*	19.1717 18.8222*	6.4215 6.3249*	16.2563 15.9189*
		0.7	7.7557 7.5921	11.623 11.4514*	4.4907 4.3931*	11.1794 10.9912*
(1,1,0)	1.2	0.3	2.9472 2.9524*	5.7281 5.7385*	1.7902 1.7328*	4.1079 4.1147*
		0.5	2.5670 2.5584*	4.9931 4.8650*	1.6678 1.6246*	3.6307 3.6270*
		0.7	2.2984 2.2848*	4.4124 4.3902*	1.4905 1.4652*	3.155 3.1367*
	2	0.3	6.7742 6.6699*	13.1434 13.0149*	5.9192 5.8610*	11.2688 11.2214*
		0.5	4.8327 4.7072*	8.9332 8.7214*	4.2660 4.1886*	7.2562 7.0850*
		0.7	3.4962 3.4286*	6.4851 6.4258*	3.0492 3.0002*	5.2243 5.0902*
(0,0,1)	1.2	0.3	13.850	18.8697	13.817	16.2719
		0.5	11.9393	16.0808	11.6648	14.0703
		0.7	10.5767	14.1354	9.7945	12.2303
	2	0.3	33.7097	47.6975	32.4006	43.8622
		0.5	19.5812	25.9741	18.1113	23.9167
		0.7	14.2918	18.4366	12.8108	16.4361

the present analysis and compared with Xiang and Wei (2004) for the four symmetric Levy plates (SS, CS, SF, CS) for x=0 and x=a and simply supported at y=0 and y=b. The step parameter 'c' varies as 0.3,0.5 and 0.7. The stepped thickness ratios of the plates are set to t_1/t_0=1.2 and 2.0 for thin plates. Two elements with m=n=17 are used to idealize the plate. It is observed that the buckling factor decreases as the step length parameter 'c' increases in all cases. The ratio of decrease is more pronounced for plates subjected to uni-axial in-plane load in y-direction

The buckling coefficient increases as the step thickness ratio changes from 1.2 to 2. Even in this case, it is observed that the ratio is more significant for plates subjected to uni-axial in-plane load in y-direction. Table 10 shows the critical buckling coefficient values for two and three steps isotropic stiffened rectangular plate and compared with Rajasekaran and Wilson (2013) and most of the values are in good agreement.

Table 10 Comparison of buckling coefficients $n_{cr} = \dfrac{N_{cr}L^2}{\pi^2 D}$ for thin isotropic rectangular plate having one, two and three steps (longer edges simply supported) (t_1/t_0=1.1; t_2/t_0=1.1; t_3/t_0=1.3)
Values indicated by * are from Rajasekaran and Wilson (2013)

case	(p_x, p_y, p_{xy})	SS	CC	SF	CS
One step	(1,0,0)	4.4176 4.3694*	5.2392 5.3294*	3.0604 3.0597*	5.2253 5.0288*
	(0,1,0)	1.7912 1.7886*	2.2075 2.1434*	1.3618 1.3503*	2.0055 1.9872*
	(1,1,0)	1.3762 1.4173*	1.6906 1.6070*	1.2593 1.2069*	1.4844 1.5075*
	(0,0,1)	7.1064	7.3144	7.0923	7.3018
Two steps	(1,0,0)	4.4164 4.3743*	5.2381 5.1655*	3.971 4.3742*	5.2174 5.1655*
	(0,1,0)	1.5793 1.5415*	1.7142 1.6277*	1.5010 1.5313*	1.6003 1.6247*
	(1,1,0)	1.3747 1.3623*	1.4964 1.4654*	1.3674 1.3616*	1.3802 1.4649*
	(0,0,1)	7.0893	7.3138	7.0880	7.0893
Three steps	(1,0,0)	4.4176 4.3743*	5.2267 5.1655*	4.4175 4.3742*	5.2267 5.3655*
	(0,1,0)	1.5537 1.5415*	1.643 1.6277*	1.5478 1.5313*	1.6394 1.6247*
	(1,1,0)	1.3740 1.3623*	1.4788 1.4654*	1.3738 1.3616*	1.4782 1.4649*
	(0,0,1)	7.0910	7.2970	7.0907	7.2965

Table 11 Frequency parameter $\gamma = a\sqrt[4]{\dfrac{\omega^2 \rho h}{D_1}}$ for orthotropic plate a×b=1×1.2 m h=0.002 for material 1 of Table 1 $\left(D_1 = \dfrac{E_x h^3}{12(1-\nu_{xy}\nu_{yx})}\right)$

BCS	$\gamma = a\sqrt[4]{\dfrac{\omega^2 \rho h}{D_1}}$					
	1	2	2	3	4	5
SSCC Xiang and Liu (2009)	4.02 4.02	4.39 4.39	5.10 5.10	6.06 6.07	7.12 7.12	7.17 7.17
SCCC Xiang and Liu (2009)	4.05 4.05	4.5 4.5	5.293 5.29	6.31 6.31	7.123 7.12	7.307 7.31
CCCC Xiang and Liu (2009)	4.806 4.80	5.1023 5.10	5.704 5.70	6.573 6.57	7.612 7.62	7.80 7.90
SFSF	3.1404	3.2368	3.5839	4.2495	5.1774	6.1774
SSCF	3.9451	4.0923	4.4877	5.1923	6.1376	7.0851
SCCF	3.9510	4.1373	4.6060	5.3849	6.3793	7.0860
CCCF	4.7531	4.8738	5.1921	5.7902	6.6511	7.6665

Table 12 Frequency parameter $\gamma = a\sqrt[4]{\dfrac{\omega^2 \rho h}{D_2(1-\nu_{xy}\nu_{yx})}}$ for orthotropic plate with material 4 of Table 1 of Part I h=0.002 m (values denoted by * are from Xiang and Liu (2009) ($D_2 = \dfrac{E_y h^3}{12(1-\nu_{xy}\nu_{yx}}$) (Boundary conditions correspond to WSEN)

BCS	b/a	$\gamma = a\sqrt[4]{\dfrac{\omega^2 \rho h}{D_2(1-\nu_{xy}\nu_{yx})}}$ 1	2	3	4	5	6
SSSC	1	5.821	8.098	9.343	10.7123	10.9157	12.8133
		5.803*	8.087*	9.339*	10.703*	10.911*	12.806*
	2	5.1173	5.4869	6.6262	7.8187	8.990	9.157
		5.113*	5.679*	6.618*	7.813*	8.99*	9.299*
SCCC	1	6.357	8.427	10.234	11.072	11.373	13.252
		6.119*	8.696*	9.437*	11.007*	11.599*	13.189*
	2	5.1581	5.8206	6.8471	8.0984	9.0020	9.3438
		5.149*	5.803*	6.831*	8.087*	9.0*	9.339*
CCCC	1	6.8058	8.966	10.316	11.641	11.74	13.694
		6.714*	8.921*	10.277*	11.605*	11.72*	13.663*
	2	6.103	6.537	7.34	8.367	9.696	9.964
		6.073*	6.503*	7.308*	8.401*	9.678*	9.962*
SFSF	1	3.9411	4.649	6.457	7.891	.308	8.945
	2	3.9457	4.1543	4.7606	5.6801	6.8251	7.8965
CFCF	1	5.9553	6.2848	7.4054	9.4365	9.8897	10.1769
	2	5.9680	6.0838	6.3124	6.8913	7.6783	8.7128
SSCF	1	5.068	6.2	8.366	8.98	9.697	11.076
	2	4.9803	5.2703	5.8859	6.8108	7.967	8.925
SCCF	1	5.1278	6.5485	8.9372	9.0011	9.8480	11.4679
	2	4.988	5.3316	6.0279	7.0287	8.2401	8.9284
	1	6.0782	7.1172	9.2124	9.9777	10.6645	11.8926
	2	5.9971	6.225	6.7212	7.518	8.5742	9.8113

16. Free vibration problems

16.1 Free vibration of orthotropic plate of uniform thickness

Xiang and Liu (2009) obtained the exact solutions for free vibrations of thin orthotropic rectangular plates by using the method of novel separation of variables for the three cases SSCC, SCCC, CCCC boundary conditions and computed with Finite element Method. The thickness of the plate is considered as 0.002 m. Free vibration analysis is carried out using EDQM for the plate $a\times b$=1×1.2 m with material 1 as given in Table 1. The frequency parameter $\gamma = a\sqrt[4]{\dfrac{\omega^2 \rho h}{D_1}}$ is calculated and compared with Xiang and Liu only for some boundary condition in Table 11. In Table 12 the frequency parameters compared with the method of Green function method of Xiang and Liu (2009) for SSCC, CSCC, CCCC plates and the results for SFSF, CFCF, SSCF. SCCF and CCCF are given for completeness. ρ value is assumed as 1600 kg/m^3. Most of the results agree with Xiang and Liu (2009) except fifth and sixth natural frequencies for CSCC plate with b/a=2 are in error as 4% and 6% respectively.

Table 13 Natural frequency parameter $\gamma = \sqrt[4]{\dfrac{\rho_0 h_0 \omega^2 a^4}{D_0(1-\nu_{xy}\nu_{yx})}}$ for orthotropic plate with variable thickness varying in one direction for various boundary conditions- Material 4 of Table 1) (the values indicated by * are from Huang et.al (2005)

BCS	b/a	β_x	$\gamma = \sqrt[4]{\dfrac{\rho_0 h_0 \omega^2 a^4}{D_0(1-\nu_{xy}\nu_{yx})}}$ 1	2	3	4	5	6
CSCS	0.5	0	7.9481	11.147	13.2865	14.7659	15.1067	17.7536
			7.943*	11.124*	13.268*	14.792*	14.956*	17.664*
		0.4	8.68	12.176	14.464	16.122	16.519	19.4
			8.672*	12.141*	14.437*	16.01*	16.472*	19.2821*
		0.8	9.3178	13.071	15.432	17.3	17.774	20.849
			9.307*	13.026*	15.397*	17.344*	17.528*	20.71*
	1.0	0	6.366	7.948	10.172	10.433	11.149	12.851
			6.361*	9.961*	10.149*	10.408*	11.125*	12.814*
		0.4	6.9522	8.68	11.1058	11.3817	12.1755	14.0413
			6.945*	8.67*	11.35*	11.35*	12.141*	13.993*
		0.8	7.464	9.318	11.916	12.192	13.071	15.089
			7.454*	9.305*	11.874*	12.155*	13.026*	15.026*
	2.0	0	6.0406	6.3697	7.00	7.95	9.119	9.9476
			6.036*	6.361*	6.992*	7.905*	8.972*	9.925*
		0.4	6.596	6.952	7.669	8.680	9.955	10.86
			6.589*	6.945*	7.635*	8.631*	9.743*	10.83*
		0.8	7.079	7.464	8.213	9.318	10.679	11.653
			7.078*	7.454*	8.196*	9.263*	10.804*	11.611*
SSFS	0.5	0	6.519	8.133	10.877	12.734	13.628	14.211
			6.515*	8.126*	10.853*	12.719*	13.608*	14.148*
		0.4	7.318	8.936	10.877	12.734	13.628	14.211
			7.292*	8,.126*	10.853*	12.719*	13.608*	14.148*
		0.8	8.004	9.671	12.821	15.094	16..549	16.684
			7.935*	9.753*	12.714*	15.086*	16.433*	16.611*
	1.0	0	3.517	5.95	6.52	8.13	9.42	9.61
			3.533*	5.945*	6.509*	8.129*	9.41*	9.571*
		0.4	3.924	6.484	7.318	8.937	10.27	10.76
			3.916*	6.485*	7.310*	8.917*	10.243*	10.712*
		0.8	4.285	6.981	8.003	9.671	11.016	11.642
			4.280*	6.967*	7.994*	9.647*	10.982*	11.59*
	2.0	0	2.123	3.528	4.999	5.222	5.957	6.52
			2.126*	3.523*	4.994*	5.219*	5.952*	6.474*
		0.4	2.344	3.929	5.598	5.626	6.495	7.31
			2.323*	3.918*	5.62*	5.587*	6.482*	7.267*
		0.8	2.613	4.309	5.997	6.141	6.989	8.007
			2.519*	4.281*	5.982*	6,.12*	6.965*	7.95*

16.2 Free vibration of orthotropic plate of variable thickness

The described method is used to obtain the frequency parameter of orthotropic plate with variable thickness and various boundary conditions. E-Glass/Epoxy (material 4 of Table 1) is used. The thickness function is chosen as $h = h_0\left(1+\dfrac{\beta_x x}{a}\right)$ if it varies in x direction or $h = h_0\left(1+\dfrac{\beta_x x}{a}\right)\left(1+\dfrac{\beta_y y}{b}\right)$ if the thickness varies in both x and y directions respectively. The thickness h_0 is assumed as a/100 (a=1 m) is adopted in all the calculations of Table 13. The results of the frequency parameter

Table 14 Natural frequency parameter $\gamma = \sqrt[4]{\dfrac{\rho_0 h_0 \omega^2 a^4}{D_0(1-\nu_{xy}\nu_{yx})}}$ for orthotropic square plate with variable thickness varying in two direction for various boundary conditions- Material 4 of Table 1 of Part I) (the values indicated by * are from Huang *et al*. (2005)

BCS	β_x	β_y	$\gamma = \sqrt[4]{\frac{\rho_0 h_0 \omega^2 a^4}{D_0(1-\nu_{xy}\nu_{yx})}}$ 1	2	3	4	5	6
CCCC	-0.5	-0.5	4.956	6.549	7.443	8.47	8.6063	9.984
			4.955*	6.548*	7.440*	8.52*	8.533*	9.989*
	-0.5	0.5	6.457	8.521	9.767	11.044	11.149	13.085
			6.453*	8.510*	9.748*	11.070*	11.056*	13.031*
	0.5	-0.5	6.451	8.536	9.689	11.097	11.189	13.014
			6.447*	8.510*	9.671*	11.103*	11.108*	12.993*
	0.5	0.5	8.403	11.106	12.717	14.438	14.523	17.003
			8.390*	11.076*	12.666*	14.389*	14.418*	16.887*
SSSC	-0.5	-0.5	3.871	5.722	6.255	7.479	7.798	8.805
			3.872*	5.716*	6.252*	7.483*	7.778*	8.786*
	-0.5	0.5	5.034	7.509	8.027	9.667	10.175	11.460
			5.038*	7.499*	8.015*	9.678*	10.148*	11.433*
	0.5	-0.5	5.016	7.447	8.121	9.726	10.200	11.454
			5.016*	7.442*	8.115*	9.717*	10.169*	11.423*
	0.5	0.5	6.523	9.772	10.429	12.554	13.345	14.911
			6.536*	9.729*	10.318*	12.580*	13.311*	14.855*
SSSS	-0.5	-0.5	3.631	5.338	6.089	7.212	7.377	8.634
			3.635*	5.335*	6.086*	7.221*	7.358*	8.614*
	-0.5	0.5	4.706	6.937	7.97	9.379	9.561	11.439
			4.704*	6.937*	7.904*	9.372*	9.536*	11.425*
	0.5	-0.5	4.707	6.937	7.91	9.4	9.613	11.235
			4.708*	6.9338	7.904*	9.397*	9.59*	11.207*
	0.5	0.5	6.100	9.011	10.357	12.193	12.490	14.856
			6.086*	9.022*	10.35*	12.136*	12.439*	14.858*
SCFC	-0.5	0.5	3.428	4.818	5.448	6.708	7.061	7.441
			3.431*	4.815*	5.475*	6.730*	7.064*	7.410*
	-0.5	0.5	4.458	6.272	7.093	8.734	9.233	9.694
			4.467*	6.303*	6.989*	8.628*	9.206*	9.717*
	0.5	-0.5	4.883	6.267	7.87	8.893	9.126	10.894
			4.831*	6.231*	7.879*	8.771*	9.069*	10.936*
	0.5	0.5	6.285	8.066	10.285	11.488	11.896	14.084
			6.301*	8.084*	10.235*	11.429*	11.849*	14.20*
CCCS	-0.5	-0.5	4.734	6.187	7.265	8.064	8.352	9.721
			4.734*	6.184*	7.258*	8.047*	8.336*	9.706*
	-0.5	0.5	6.262	8.016	9.739	10.45	10.916	12.665
			6.259*	8.009*	9.72*	10.421*	10.891*	12.629*
	0.5	-0.5	6.161	8.06	9.46	10.548	10.861	12.651
			6.158*	8.052*	9.444*	10.522*	10.832*	12.613*
	0.5	0.5	8.151	10.439	12.682	13.663	14.198	16.474
			8.137*	10.417*	12.63*	13.605*	14.132*	16.382*
SSCC	-0.5	-0.5	4.245	5.961	6.784	7.864	8.047	9.353
			4.248*	5.959*	6.779*	7.875*	8.015*	9.329*
	-0.5	0.5	5.481	7.801	8.725	10.198	10.446	12.223
			5.481*	7.795*	8.715*	10.187*	10.412*	12.198*
	0.5	-0.5	5.639	7.659	8.933	10.229	10.362	12.22
			5.637*	7.653*	8.922*	10.270*	10.274*	12.191*
	0.5	0.5	7.253	10.01	11.507	13.287	13.415	15.872
			7.248*	9.991*	11.475*	13.252*	13.359*	15.781*

$\gamma = \sqrt[4]{\dfrac{\rho_0 h_0 \omega^2 a^4}{D_0(1-\nu_{xy}\nu_{yx})}}$ where $D_0 = \dfrac{E_y h_0^3}{12(1-\nu_{xy}\nu_{yx})}$ are compared with Huang *et al.* (2005) and there is very good agreement.

Table 15 Natural frequency parameter $\lambda = \sqrt{\dfrac{\rho_0 h_0 \omega^2 a^4}{D_{x0}}}$ for the stepped plate $\left(\dfrac{h_1}{h_0}\right)=1.1$; $\left(\dfrac{h_2}{h_0}\right)=1.2$; $\left(\dfrac{h_3}{h_0}\right)=1.3$ (width of the plate=1 and each step of length=1) (m=n=12)

No of steps	mode	SS	CC	SF	CS
One step	λ_1	16.8846	27.3235	11.0925	21.2722
	λ_2	44.5814	50.0608	23.5039	47.1479
	λ_3	46.2912	68.3553	41.5453	56.4755
	λ_4	67.6968	85.2163	51.0003	75.6674
	λ_5	92.9429	96.2865	58.5132	95.1742
Two steps	λ_1	29.277	37.2850	28.3562	32.6867
	λ_2	56.5911	77.5105	30.3977	68.4618
	λ_3	96.8455	100.6072	69.6352	99.5708
	λ_4	109.0106	130.0388	96.2299	123.733
	λ_5	117.5422	142.1167	106.1503	124.412
Three steps	λ_1	48.6207	54.6359	46.5387	51.6732
	λ_2	73.8599	92.9032	56.6947	82.7206
	λ_3	124.8373	156.9582	88.0259	140.513
	λ_4	171.8243	176.3608	144.772	176.219
	λ_5	196.988	205.5754	171.8103	202.4145

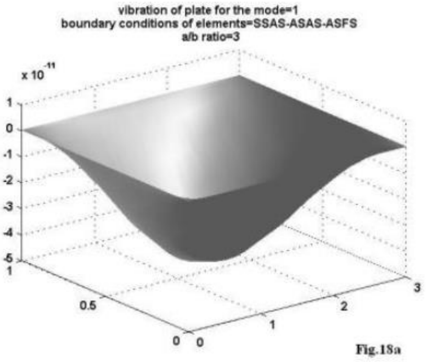


Fig. 18(a) First mode shape of two stepped orthotropic plate

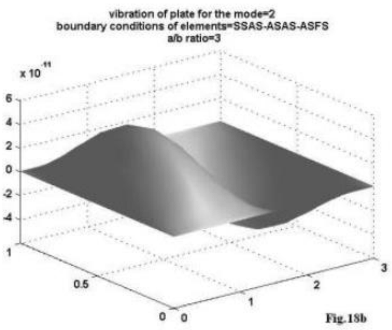


Fig. 18(b) Second mode shape of two stepped orthotropic plate

As another application of the present method, the numerical results are given for the orthotropic plate with linearly varying thickness in two directions. Table 14 presents the results for the plate with six kinds of boundary conditions and four kinds of thickness variation and compared with the results of Huang *et al.* (2005) that solved the problem by using Green function.

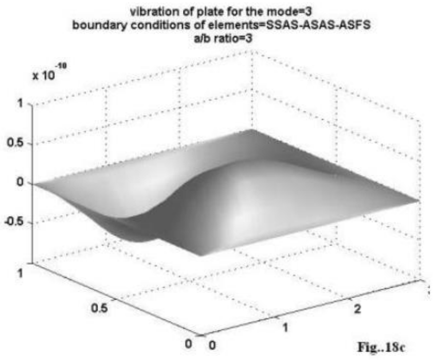


Fig. 18(c) Third mode shape of two stepped orthotropic plate

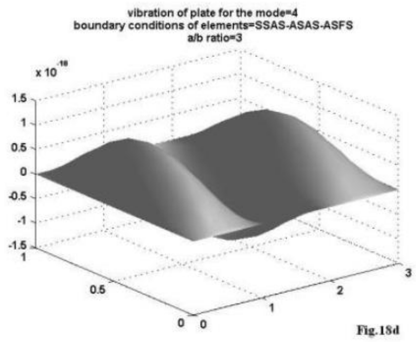


Fig. 18(d) Fourth mode shape of two stepped orthotropic plate

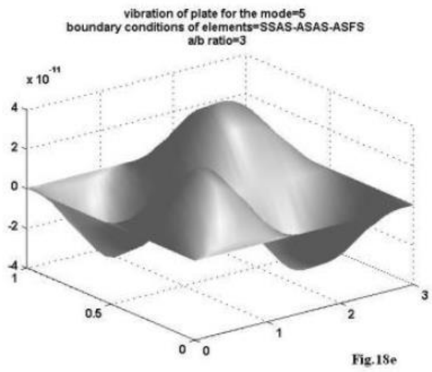


Fig. 18(e) Fifth mode shape of two stepped orthotropic plate

16.3 Free vibration of stepped orthotropic plate

A one, two and three steps rectangular Levy orthotropic plate is considered for four boundary conditions (SS, CS, SF, CS) and simply supported boundary conditions at y=0 and $y=b$. The step thickness is considered as (1+.1n)*0.002 where n is the number of steps. The material properties of material 1 from Table 1 is considered. The natural frequencies are tabulated as shown in Table 15. Fig. 18 shows the all the five mode shapes for SSSS two stepped orthotropic plate. Because of memory limitations $m=n$=13 is adopted for all the plates and no comparison is available.

16.4 Free vibration of non-homogeneous orthotropic plate with varying thickness

Consider a non-homogeneous plate of variable thickness

Table 16 Six frequencies for Non-homogeneous orthotropic plate with Variable Thickness for various m and n

BCS	a/b	μ	β	γ	λ				
					$m=n$=13	$m=n$=17	$m=n$=21	$m=n$=25	$m=n$=29
SCSC	0.5	-0.5	-0.5	0	8.4897	8.4897	8.4897	8.4897	8.4897
					13.3925	13.3923	13.3923	13.3925	13.3925
					20.5719	20.5720	20.5720	20.5719	20.5720
					28.3796	28.3796	28.3796	28.3796	28.3794
					29.8438	29.8345	29.8346	29.8346	29.8346
					32.7822	32.7826	32.7826	32.7822	32.7823
SCSS	1	0.0	0.0	-0.5	19.3374	19.3374	19.3374	19.3374	
					44.0804	44.0804	44.0804	44.0804	
					47.8554	47.8554	47.8554	47.8554	
					71.9630	71.9630	71.9633	71.9633	
					82.7234	82.7234	82.7230	82.7230	
					96.7380	96.7380	96.7394	96.7394	
SCSF	2	0.5	0.5	0.5	36.6401	32.6324	31.2893	30.5519	30.7176
					79.2453	74.4003	72.4976	71.8859	71.0671
					113.6747	109.82222	108.2584	107.4324	106.9445
					153.3554	145.3066	143.0301	142.0359	141.3087
					174.5363	165.4132	162.1788	160.6868	158.8697
					242.5662	239.4490	236.3347	234.4580	232.7721

Table 17 Frequency parameter for isotropic plate $a\times b$ as 1×1, t=1 for different boundary conditions

Boundary condition		mode 1	mode 2	mode 3	mode 4	mode 5	mode 6
CCCC	present	35.98	73.39	73.39	108.21	131.58	132.21
	Leissa (1969)	35.98	73.39	73.39	108.21	131.58	132.21
SSSS	present	19.73	49.34	49.34	78.95	98.69	98.69
	Leissa (1969)	19.73	49.34	49.34	78.95	98.69	98.69
CCCS	present	31.82	63.33	71.07	100.79	116.35	130.35
	Leissa (1969)	31.82	63.34	71.08	100.83	116.41	130.37
CCSS	present	27.05	60.53	60.78	92.83	114.55	114.71
	Leissa (1969)	27.05	60.54	60.79	92.86	114.57	114.72
SCSS	present	23.64	51.67	58.64	86.13	100.26	113.22
	Leissa (1969)	23.64	51.67	58.64	86.13	100.26	113.22
CSCS	present	28.95	54.74	69.32	94.58	102.21	129.09
	Leissa (1969)	28.95	54.74	69.32	94.58	102.21	129.09
SCSF	present	12.68	33.06	41.71	63.01	72.39	90.61
	Leissa (1969)	12.68	33.06	41.71	63.01	72.39	90.61
CFCF	present	22.71	27.92	44.81	62.75	71.01	80.68
	Leissa (1969)	22.27	26.52	43.66	61.46	67.54	79.94
SSSF	present	11.68	27.75	41.19	59.06	61.86	90.29
	Leissa (1969)	11.68	27.75	41.19	59.06	61.86	90.29
SFSF	present	9.63	16.13	36.72	38.94	46.73	70.73
	Leissa (1969)	9.63	16.13	36.72	38.94	46.73	70.74
CCSF	present	17.79	36.26	52.69	71.92	74.48	106.89
	Leissa (1969)	17.61	36.04	52.06	72.194	74.35	106.28

considered in section 2.2 in which $E_x=E_{x0}e^{\mu X}$; $E_y=E_{y0}e^{\mu X}$; $h=h_0e^{\beta X}$; $\rho=\rho_0e^{\gamma X}$; where $X=x/a$, $E_{x0}=1\times10^{10}$ MPa; $E_{y0}=5\times10^9$

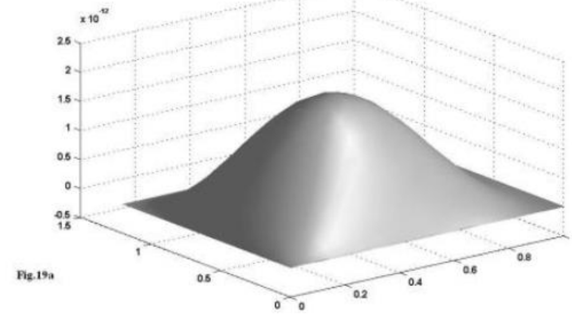


Fig. 19(a) First mode shape of isotropic plate *a/b*=0.8333 of uniform thickness

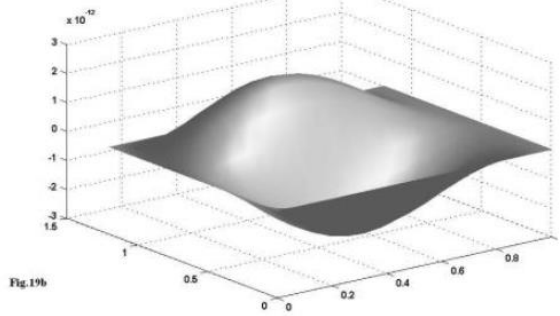


Fig. 19(b) Second mode shape of isotropic plate *a/b*=0.8333 of uniform thickness

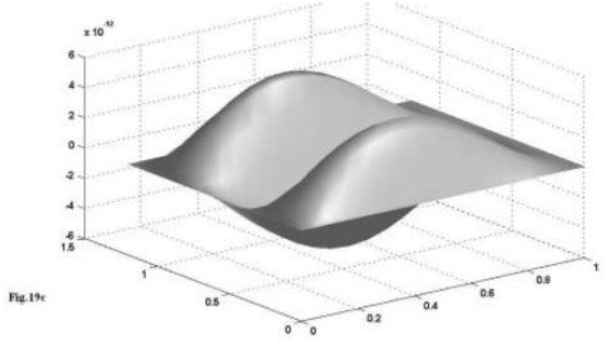


Fig. 19(c) Third mode shape of isotropic plate *a/b*=0.8333 of uniform thickness

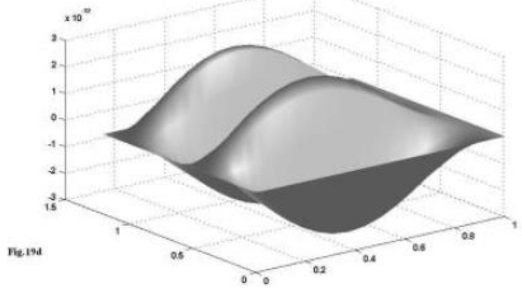


Fig. 19(d) Fourth mode shape of isotropic plate *a/b*=0.8333 of uniform thickness

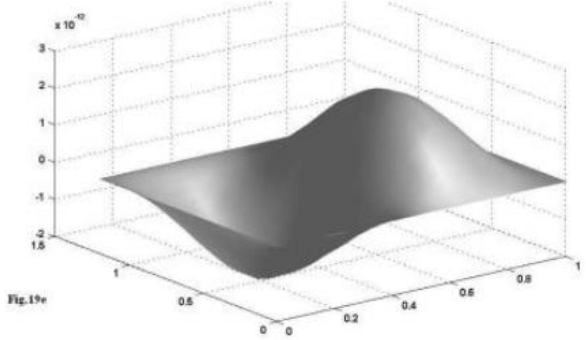


Fig. 19(e) Fifth mode shape of isotropic plate *a/b*=0.8333 of uniform thickness

MPa; ν_{xy}=0.2; ν_{xy}=0.1 and $G_{xy}=\dfrac{\sqrt{E_xE_y}}{2\left(1+\sqrt{\nu_{xy}\nu_{yx}}\right)}$ and h_0 is taken as 0.001 m and ρ_0=1600 kg/m^3. The boundary conditions x=0 and $x=a$ are assumed as simply supported and y=0 and $y=b$ as CC, CS, and CF boundary conditions. The dimensionless frequency parameter $\lambda=\sqrt{\dfrac{\rho_0 h_0}{D_{x0}}}\omega a^2$ (where $D_{x0}=\dfrac{E_{x0}h^3}{12(1-\nu_{xy}\nu_{yx})}$ is calculated and tabulated in Table 16 and no previous results is available for comparison.

16.5 Free vibration of isotropic plate with uniform thickness

Table 17 shows the results of free vibration of isotropic square plate of uniform thickness with various boundary conditions and compared with Leissa (1969) and the comparison is quite good. Fig. 19 shows all the five modes shapes for *a/b*=0.8333 with SSCC boundary conditions. Fig. 20 shows the mode shapes for square plate for SSSS, SCSS and SCSF boundary conditions. For all the examples considered $m=n$=16 is adopted.

17. Conclusions

In this paper, general formulation for non-homogeneous, orthotropic and isotropic plate with continuously varying or stepped thickness is discussed and the Element Based Differential Quadrature (EBDQ) has been explained and the relevant matrices are derived to solve equilibrium, stability and free vibration problems. Numerical examples are solved and compared with the published results and the efficiency of the method is discussed.

The buckling problem has been solved for the orthotropic non-homogeneous plate. The critical buckling coefficients have been determined as a result of the numerical solution of the corresponding eigen-value problem. The buckling problem was solved for a CFCF, SFSF and SFSC orthotropic plate subjected to linearly varying axial load that can be of uniform compression, combination uniform compression and in-plane bending or pure in-plane bending. The buckling problem has been solved for orthotropic plate having various aspect ratios.

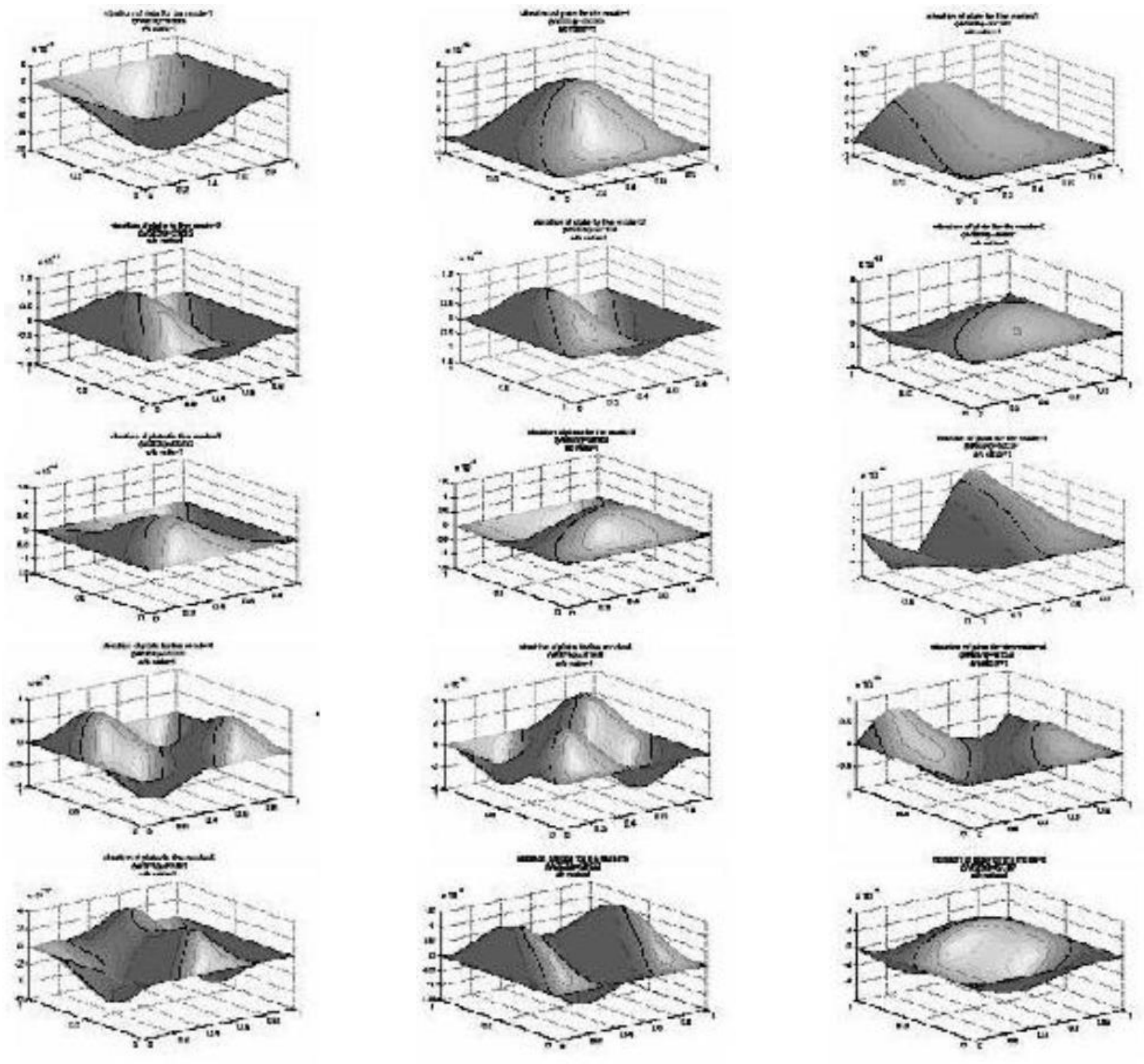


Fig. 20 Mode shapes of isotropic square plate for SSSS, SCSS and SCSF boundary condition

The effect of the stiffness characteristics on the buckling loads and buckled shapes were demonstrated for orthotropic plate with various aspect ratios. Results of EDQM were validated for isotropic and orthotropic plates and compared with the values found in the literature. The comparisons have shown that the calculation of buckling coefficients for the plate under study can be performed with sufficient accuracy using the analytical approach developed in the paper.

Vibration of thin orthotropic plate of uniform thickness has been carried out using EDQM for SSCC, SCCC, CCCC plates and compared with the results obtained by using exact Method of Xiang and Liu (2009) and there is very good agreement. The formulation is applied to find the natural frequency of orthotropic plate where two kinds of thickness variation are compared with Huang (2005) who used Green functions.

Free vibration analysis is also carried out for stepped orthotropic plate for one, two and three steps. Non-homogeneous orthotropic pate with varying thickness is analyzed for the vibration for SCSC, SCSS and SCSF boundary conditions. Lastly the free vibration analysis is carried out for isotropic plate with uniform thickness for various boundary conditions and compared with the values obtained by Leissa (1969).

Besides advantages of numerical solution technique, the EDQM method is recognized for analytical simplicity. The author believes that the paper presents EDQM to ortho/isotopic plates and it may be very well be employed to other problems with the same advantages as exemplified by the plate problems of this paper.

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PL

Symbols

a	length of the plate in x-direction
a/b	aspect ratio
b	length of the plate in y-direction
[B]	boundary condition matrix ($nr\times nt$)
D_x, D_y	flexural rigidities of the plate in x and y directions
D_t	torsional rigidity
[D]	equilibrium matrix ($nt\times nt$)
E_x, E_y	Youngs moduli in x and y directions
[EE]	stability matrix
[FF]	dynamic matrix
G	shear modulus
h	thickness of the plate
[K]	flexural stiffness matrix
k	buckling coefficient
$[K_G]$	geometric stiffness matrix
M_x,M_y	bending moments in x and y directions
M_{xy}	twisting moment
[M]	mass matrix
nx,ny	number of grid points in x and y directions
N_x,N_y	in-plane forces per unit length in x and y directions
N_{xy}	in-plane shearing force per unit length
nt_i	total degrees of freedom for element 'i'
Q_x,Q_y	vertical shearing forces per unit length in x and y planes
t_x,t_y,t_{xy}	Tracers for N_x,N_y and N_{xy}
V_x,V_y	modified shearing forces per unit length in x and y planes
W	lateral deflection
α_x, α_y	non-dimensional parameter to denote the variation of N_x,N_y
β_x, β_y	non-dimensional parameter to denote the variation of thickness in x and y directions
ε_x, ε_y	strain in x and y directions respectively
η	y/b
λ	Lagrangian multipliers
μ_1, μ_2	Exponential operator for variation of E_x,E_y along x direction
ν_{xy}, ν_{yx}	Poisson's ratios in x and y directions
ρ	density of plate material
ω	natural frequency
ξ	non-dimensional variable X/a