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A probabilistic analysis of Miner's law for different loading conditions

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Abstract. In this paper, the normalized variable *V*=(*log N*-*B*)(*log* $\Delta\sigma$-*C*), as derived from the probabilistic *S-N* field of *Castillo and Canteli*, is taken as a reference for calculation of damage accumulation and probability of failure using the Miner number in scenarios of variable amplitude loading. Alternative damage measures, such as the classical Miner and logarithmic Miner, are also considered for comparison between theoretical lifetime prediction and experimental data. The suitability of this approach is confirmed for it provides safe lifetime prediction when applied to fatigue data obtained for riveted joints made of a puddle iron original from the Fão bridge, as well as for data from experimental programs published elsewhere carried out for different materials (aluminium and concrete specimens) under distinct variable loading histories.

Keywords: fatigue; cumulative damage; Miner's rule; statistical analysis

1. Introduction

The cumulative damage concept proposed by Palmgren and Miner (Miner 1945) maintains that the fatigue damage (*M*) can be calculated in terms of the number of cycles applied at a given stress range (n_i) divided by the corresponding number of cycles required to produce failure at the same stress range (N_{fi})

$$M = \sum_{i=1}^{n} \frac{n_i}{N_{fi}} \tag{1}$$

Failure occurs when the summation of these damage increments at the intervening stress ranges becomes unity (*M*=1). After its formulation, this rule is repeatedly applied to different materials under multi-step and variable amplitude loading programs with sundry outcomes, depending on the material and load history (Pereira 2006, Pereira *et al.* 2008, Pereira *et al.* 2009). Though its applicability has been often questioned, it has been practically adopted and/or recommended by all international codes related to structural and mechanical fatigue design (CEN-TC 250, EN13445,

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ASME B&PVC) in a deterministic form.

Though Birnbaum and Saunders (Birnbaum *et al.* 1968) seems to be the first in searching a probabilistic distribution for the Miner number, in this case from the crack growth information of the material, van Leeuwen and Siemes (Van Leeuwen *et al.* 1977, 1979) suggested another way of interpreting the variability of the Miner number by emphasizing the relation between the scatter of the Miner number and that of the Wöhler curve for constant amplitude cycling. Assuming log-normal distribution for the lifetime at a given constant stress range, they derived theoretical expressions for the mean and standard deviation values directly from those of the *S-N* field, initially for constant amplitude load. Later on, Fernández-Canteli (Canteli 1982) justifies a generalization of the Van Leeuwen and Siemes proposal by providing the statistical distribution of the Miner number from a normalized *S-N* field assuming a log-normal distribution. The approach was successfully applied to the extensive research program on concrete cylindrical specimens under compression performed by Holmen (Holmen 1979).

Over the recent past, new proposals were presented by Liu and Mahadevan (2007) and Imam *et al.* (2008). Liu and Mahadevan (2007) suggested a general methodology for stochastic fatigue damage modeling under variable amplitude loading, which is based on a new stochastic *S-N* curve approach and Palmgren-Miner's rule (1945). Also, Imam *et al.* (2008) proposed a probabilistic fatigue assessment methodology applied to riveted railway bridges, which uses the *S-N* curves and cumulative damage model with probabilistic treatment based on a Monte Carlo simulation.

Rathod *et al.* (2011) proposed a methodology for probabilistic modeling of fatigue damage accumulation for single stress level and multistress level loading, which uses the following steps:

i) Linear damage accumulation model by Palmgren-Miner (Miner 1945);

ii) A probabilistic *S-N* curve using a normal distribution of the fatigue lives are depicted at different stress levels;

iii) An approach for a one-to-one transformation of probabilistic density functions of fatigue damage accumulation, *D*, as follows

$$f_d(D) = \frac{1}{m' \cdot \sigma_{N_f} \cdot \sqrt{2\pi}} exp\left(-\frac{1}{2}\left(\frac{(D/m') - \mu_{N_f}}{\sigma_{N_f}}\right)^2\right) \tag{2}$$

The relation between standard deviations before and after transformation is the following

$$\sigma_D = m' \cdot \sigma_{N_f} \tag{3}$$

where, σ_D represents standard deviation of damage accumulation, σ_{Nf} denotes standard deviation of failure life or usage cycle, and m' represents slope of damage accumulation trend line.

Tawil and Jaoude (2013) proposed an analytic prognostic model based on nonlinear damage laws. This model can be written as follows

$$\widetilde{D}(N) = 1 - \left[\left(1-\widetilde{D}_0\right)^{\alpha+1} - \frac{N-N_0}{N_c}\left(1-\frac{\sigma_0}{\Delta\tilde{\sigma}_j/2}\right)^m (\alpha+1)\right]^{1/(\alpha+1)}, \qquad \widetilde{D}_0 = \frac{\tilde{a}_0}{a_c - \tilde{a}_0} \tag{4}$$

where, m and α are constants depending on the material and the loading condition, N_0 is the number of cycles corresponding to an initial crack length, a_0, N_c is the number of cycles to failure that corresponds to a critical crack length, a_c, σ_0 is the fatigue limit (i.e., the endurance limit stress of material), $\Delta\tilde{\sigma}_j$ is the random variable of stress range in a loading cycle, $\tilde{a}_0$ is the random variable of the initial crack length, and $\widetilde{D}_0$ is the random variable of the damage at $N=N_0$ cycles corresponding to an initial crack length a_0.

A statistically consistent fatigue damage model under constant and variable amplitude loadings based on linear Miner's rule and supported by the Monte Carlo Method was proposed by Sun (Sun *et al.* 2014). This statistical fatigue damage model was proposed for the calculation of the damage caused by one loading block, D_B, and was formulated by the following equations

$$D_B = \sum_{i=1}^{n} \left(\frac{1}{N_i}\right)^a - \frac{n}{N_i} \cdot \frac{N_i - \mu}{\mu}, for\ constant\ amplitude\ loading \tag{5}$$

where N_i is a random fatigue life from some distribution, μ is the mean value of the life distribution, and a is the consistent index (determined by Monte Carlo method), and n is number of cycles in one loading block.

$$\begin{aligned} D_B &= \sum_{i=1}^{m} \left(\sum_{j=1}^{n_i} \left(\frac{1}{N_{ij}}\right)^{a_i} - \frac{n_i}{N_i} \cdot \frac{N_i - \mu_i}{\mu_i} \right), N_f \\ &= \frac{1}{D_B} \cdot \sum_{i=1}^{m} n_i\ , for\ variable\ amplitude\ loading \end{aligned} \tag{6}$$

where D_B is fatigue damage caused by one loading block, N_f is the predicted whole fatigue life, m is the number of load levels in one loading block, n_i is the cycle number under i load level, N_{ij} and N_i are both random fatigue lives under i load level, a_i is the consistent index of i load level, and l_i is the mean value of fatigue life under the i load level.

More recently, Liang and Chen (2015) and Beretta and Regazzi (2015) proposed new developments to estimate the fatigue damage accumulation using statistical approaches. Liang and Chen (2015) proposed a regularized Miner's rule in conjunction with the Mittag-Leffler Monte Carlo Method to assess the fatigue reliability of composites. In the study of Liang and Chen, a regularized Miner's rule was developed, which is given by

$$D_r = \sum_{i=1}^{k} \frac{n_i^{\lambda_i}}{N_i} \tag{7}$$

where D_r is the regularized fatigue damage accumulation, and λ_i is a material parameter, also called regularized parameter. This regularized parameter, λ_i, is expressed as

$$\lambda_i = \frac{M - A \cdot logS_i}{logn_i} \tag{8}$$

where S_i is the stress amplitude, A and M are constants (if D>1, then λ<1, otherwise D=1, λ=1). Then, according to the regularized Miner's rule and using the Mittag-Leffler distribution, the fatigue damage accumulation distribution can be obtained as follows

$$F(D_r) = P(D_r(n) \le D_r) = P\left(n^\lambda / N_f \le D_r\right) = 1 - F_{N_f}\left(n^\lambda / D_r\right) \tag{9}$$

Beretta and Regazzi (2015) described a procedure for the determination of railway axle risk of fatigue failure under service loading for a simple fatigue assessment in terms of a safety factor for fatigue damage calculations. This probabilistic approach is presented using the Gaussian distribution with parameters (μ_{logS_D}, σ_{logS_D}) that allows us to suggest D as a Gaussian random variable of the fatigue damage accumulation.

Other approaches and applications, using similar statistical assumptions to the above references, may be found in Nallasivam *et al.* (2008), Kim *et al.* (2011), Derbanne *et al.* (2011), Changfeng *et al.* (2012), Park and Kang (2013), Zhu *et al.* (2014), Shokrieh *et al.* (2014), Zhang (2015), Tee *et al.* (2013), Ye *et al.* (2016), Mechab *et al.* (2016), Tian *et al.* (2016).

According to these presented developments and approaches, reinforced by some further theoretical advances as displayed in (Castillo *et al.* 2009, Castillo *et al.* 2011), it follows that the Miner number may be used as a basis for a consistent fatigue design to ascertain lifetime of a component under variable loading, in accordance with the concept of fatigue failure as a limit state, rather than used as a measure of an abstract “degree of damage” of problematic interpretation.

In this paper, a damage accumulation approach based on the probabilistic interpretation of the Miner-Palmgren rule related to the Weibull *S-N* field model, as proposed by Castillo and Canteli (Castillo *et al.* 2009) is presented. The approach is applied to fatigue data obtained for riveted joints made of a puddle iron original from the Fão bridge (De Jesus *et al*. 2014, 2015) as well as for data from experimental programs published elsewhere carried out for different materials (aluminium and concrete specimens) under distinct variable loading histories (Hardrath *et al.* 1952, Holmen 1979). The main advantage of the proposed approach consists in the feasibility of associating the classical Miner number to the failure probability as a result of the successive loads applied without the need of performing extensive variable amplitude testing. In this way, the derivation of the cumulative distribution of Miner number, *M*, can be obtained simply and directly from the *S-N* field data assessment (Canteli *et al.* 2014).

The following proposal is limited to loading data histories showing extensive varying amplitudes, for which sequential effects are not noticeable in fatigue damage accumulation. This discards, for instance, data arising from test programs consisting in two-blocks loading (e.g., high-low or low-high sequences) for which the classical Miner damage approach is not well-suited due to the strong sequential effect implied. Neither the effect of single or periodic overloads superimposed to constant amplitude loading is considered. In such cases, the mean of the Miner number can strongly differ from the expected value, i.e., 1, what could be attributed not only to the expected scatter of the fatigue data but also to such sequential or overloading effects.

2. Probabilistic approach of Miner’s law for different loading conditions

The proposed probabilistic approach, relating failure probabilities to the Miner number, rests on the Weibull regression model of Castillo and Canteli (2009), which implies a probabilistic definition of the *S-N* or *ε-N* fields, in the sense of representing the relation lifetime-stress range by percentile curves, i.e., iso-damage curves associated with a constant probability of failure. The definition of the normalized variable $V=(logN-B)(log\Delta\sigma-C)$, as the product of the applied stress or strain ranges by the lifetime *N*, allows the *S-N* field to be reduced to a simple Weibull or Gumbel cumulative distribution function (see Fig. 1). The probability of failure represents a monotonic increasing function of the normalizing variable, *V*, and, consequently, of the number of cycles and stress/strain ranges or amplitudes. For a fixed stress/strain range/amplitude, the probability of failure increases with the number of cycles; in the same way, for a fixed number of cycles, the probability of failure grows up with increasing stress/strain ranges/amplitudes. By considering the normalized variable, *V*, the equivalent loading conditions are established as those leading to an

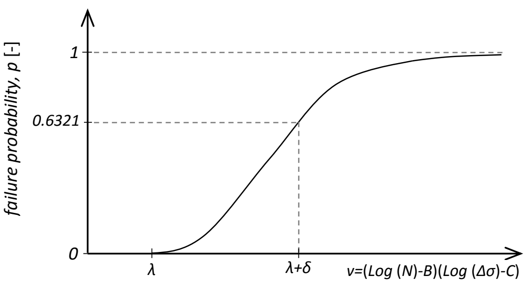

Fig. 1 Weibull cumulative distribution function of the normalized variable *V* (Castillo *et al*. 2009)

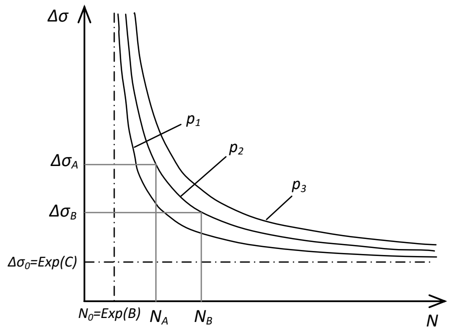

Fig. 2 Probabilistic Weibull *S-N* percentile curves representing equivalent damage or loading conditions: same probability of failure, *p*, and normalized variable, *V* (Castillo *et al*. 2009)

identical damage represented by the same probability of failure. Considering the *S-N* field of Fig. 2, both loading conditions ($\Delta\sigma_A$, N_A) and ($\Delta\sigma_B$, N_B) are equivalent since they exhibit the same damage, that is, the same probability of failure as a result of representing the same value of the normalizing variable, $V_A=V_B$

$$V_A = V_B \Rightarrow (\log N_A - B)(\log \Delta\sigma_A - C) = (\log N_B - B)(\log \Delta\sigma_B - C) \tag{10}$$

from which the number of cycles N_B can be determined for given $\Delta\sigma_B$ as:

$$N_B = \exp\left[\frac{(\log N_A - B)(\log \Delta\sigma_A - C)}{(\log \Delta\sigma_B - C)} + B\right] = \exp\left[\frac{V_A}{(\log \Delta\sigma_B - C)} + B\right] \tag{11}$$

or, alternatively, $\Delta\sigma_B$ can be determined for given N_B:

$$\Delta\sigma_B = \exp\left[\frac{(\log N_A - B)(\log \Delta\sigma_A - C)}{(\log N_B - B)} + C\right] = \exp\left[\frac{V_A}{(\log N_B - B)} + C\right] \tag{12}$$

This property can be understood as the requirement implied by a real damage accumulation conversion, since the final loading condition given by ($\Delta\sigma_A$, N_A) allows the start point for the next applied stress range $\Delta\sigma_B$ to be identified as ($\Delta\sigma_B$, N_B) both representing equivalent damage states, i.e., the same probability of failure. This condition is verified for the same percentile curve. Accordingly, the percentile (iso-probability) curves can be interpreted as iso-damage curves, since they depict equivalent loading conditions, i.e., the same value of the normalizing variable, *V*, whereas the probability of failure, *p*, for a particular percentile curve may be alternatively understood as a damage measure (Castillo *et al.* 2009). In this way, a probability of failure is assigned to any value of the until now supposedly deterministic Miner number, *M*, to which a new concept of failure hazard and safety margin can be related according to a consequent structural safety design.

For a given varying history of the loading range, it is possible to compute the evolution of the normalizing variable, *V*, cycle-by-cycle, or block-by-block in case of a sequence of constant amplitude loading blocks. At the same time, the probability of failure may be computed from the Weibull distribution of the *S-N* field for the material or mechanical/structural component. With this procedure, the failure probability associated to any loading history of variable range can be calculated using exclusively the fatigue characteristics provided by the probabilistic *S-N* field as assessed from constant stress range tests. This information enhances and complements that furnished when computing the classical Miner number for a material or mechanical/structural component, which exemplary results from the consideration of the median *S-N* curve, i.e., that representing 50% probability of failure, which incidentally can also be easily obtained from the probabilistic *S-N* field with the aid of the ProFatigue software (Canteli *et al.* 2014). Note that the consideration of a reference percentile curve, say for p=1% or 5%, different as the median one, i.e., for p=50%, usually considered, leads apparently to distinct values for the Miner number. Nevertheless, the same probability of failure is obtained for whatever load history provided a consistent corrective statistical interpretation is applied based on the Miner number distribution so obtained, see (Sendín *et al.* 2011). The same methodology, as that already exposed, can also be applied to derive the cumulative distribution function for the Log-Miner number resulting in an equivalent probabilistic failure prediction for the assumed damage states proving the feasibility of relating any damage rule to probability of failure and the equivalence among them provided a consequent probabilistic analysis is performed.

The probabilistic approach of Miner's law for different loading conditions can be summarized as follows:

i) A probabilistic *S-N* or *ε-N* field must be derived for the material or mechanical/structural component under consideration from constant range stress- or strain-based fatigue data. For this purpose the probabilistic model by (Castillo *et al.* 2009) is proposed. Both Weibull and Gumbel distributions are possible candidates (Castillo *et al.* 1988). The procedure proposed is sufficiently generic and can be extended to other damage parameters (e.g., *SWT* (Smith *et al.* 1970)). The probabilistic *S-N* field according to the Weibull model is given by

$$F(logN; log\Delta\sigma) = p = 1 - exp\left\{-exp\left[\frac{(logN - B)(log\Delta\sigma - C) - \lambda}{\delta}\right]\right\} \qquad (13)$$
$$(logN - B)(log\Delta\sigma - C) \geq \lambda$$

where: N is the lifetime; $\Delta\sigma$ is the stress level; $F()$ is the cumulative probability distribution function (*cfd*) of N for given $\Delta\sigma$; $B=log(N_0)$, N_0 being a threshold value of lifetime; $C=\log(\Delta\sigma_0)$, $\Delta\sigma_0$ being the endurance fatigue limit; and λ, β and δ are nondimensional model parameters (β: Weibull shape parameter; δ: Weibull scale parameter; λ: Weibull location parameter defining the position of the zero-percentile curve).

ii) For available variable amplitude data, the Miner number (M) or Logarithmic Miner (LM) numbers are computed using the experimental observed lifetime data and the median *S-N* or *ε-N* curves (p=0.5) derived using constant amplitude data in the previous step i). Both damage measures are respectively defined as follows

$$M = \sum_{i=1}^{n} \frac{N_i}{N_{fi}} \qquad (14)$$

$$LM = \sum_{i=1}^{n} \frac{log\, N_i}{log\, N_{fi}} \qquad (15)$$

where N_i corresponds to the number of cycles applied at any stress range $\Delta\sigma_i$, and N_{fi} represents the number of cycles to failure at the stress range $\Delta\sigma_i$, referred to the 50% percentile of the *P-S-N* field evaluated in step i). In this step, a probabilistic *P-S-N* field is supposed, but the process would be similar when using other probabilistic fields, such as the *P-ε-N* fields.

iii) For the same variable amplitude loading of previous step ii), the development of the normalized variable, V, along the applied loading history is computed. This process can be performed for a block loading $\{(\Delta\sigma_i, N_i), i=1,n\}$ as follows

$$\begin{gathered} First\ Block\ (\Delta\sigma_1; N_1) \\ V_1 = (logN_1 - B)(log\Delta\sigma_1 - C) \end{gathered} \tag{16}$$

$$\begin{gathered} Second\ Block\ (\Delta\sigma_2; N_2) \\ N_{2,t} = N_2 + exp\left(\frac{V_1}{(log\Delta\sigma_2 - C)} + B\right) \\ V_2 = \left(logN_{2,t} - B\right)(log\Delta\sigma_2 - C) \end{gathered} \tag{17}$$

$$\begin{gathered} i^{th}\ Block\ (\Delta\sigma_i; N_i) \\ N_{i,t} = N_i + exp\left(\frac{V_{i-1}}{(log\Delta\sigma_i - C)} + B\right) \\ V_i = \left(logN_{i,t} - B\right)(log\Delta\sigma_i - C) \end{gathered} \tag{18}$$

$$\begin{gathered} Last\ Block\ (\Delta\sigma_n; N_n) \\ N_{n,t} = N_n + exp\left(\frac{V_{n-1}}{(log\Delta\sigma_n - C)} + B\right) \\ V_n = \left(logN_{n,t} - B\right)(log\Delta\sigma_n - C) \rightarrow p_n = 1 - exp\left(-\left(\frac{V_n - \lambda}{\delta}\right)^{\beta}\right) \end{gathered} \tag{19}$$

iv) A direct relation between the Miner (*M*) or Logarithmic Miner (*LM*) numbers computed in step ii) and the probability of failures computed in step iii) is established. This step may be implemented in distinct phases starting with the determination of the *V* vs. *p* relation; then the *M* vs. *V* or *LM* vs. *V* relations followed by the *M* vs. *p* or *LM* vs. *p* relations may be assessed.

v) Finally, the experimental cumulative distribution functions of the Miner number may be computed by assigning probabilities to the experimental data using a plotting point position rule. In this way, the theoretical and the experimental distribution, derived in step iv), are compared. When only limited amount of experimental data under variable amplitude loading are available, the experimental distribution function obtained could be far from the expected one.

3. Applications

3.1 Stress controlled test data for a structural component

In this section, the methodology described above is used to illustrate the relation between Miner or Log-Miner numbers and probability of failure. The experimental fatigue test results are

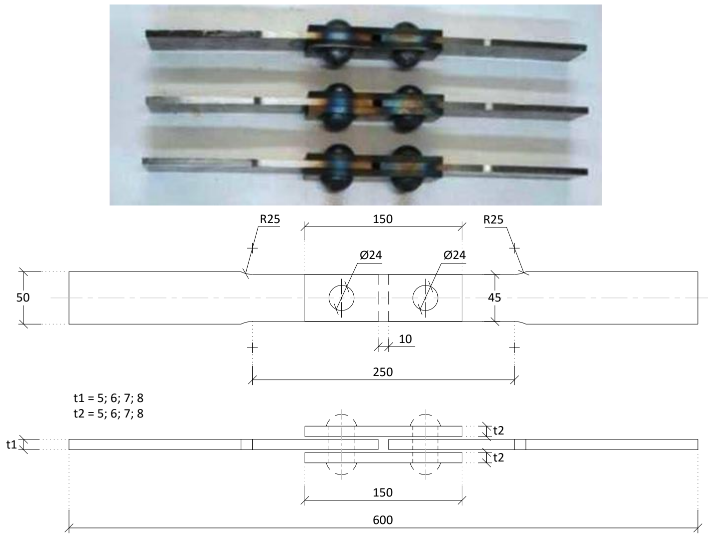

Fig. 3 Riveted joints made of puddle iron, tested under constant and variable amplitude fatigue loading (dimensions in mm) (De Jesus *et al*. 2014, 2015)

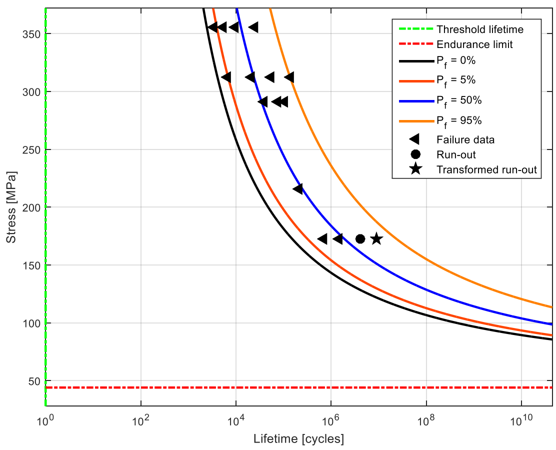

Fig. 4 Probabilistic *S*-*N* field obtained for the riveted joints using the Weibull probabilistic model

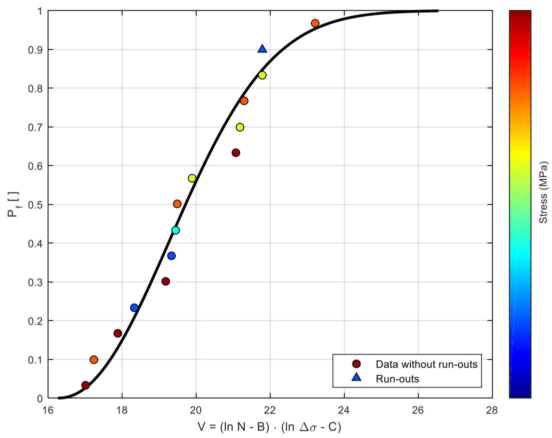

Fig. 5 Probabilistic *S*-*N* field obtained for the riveted joints using the Weibull probabilistic model

available from (De Jesus *et al*. 2014, 2015). The tests were performed under constant and variable stress range for null stress R-ratio (R_σ=0) on simple riveted joints, made of puddle iron extracted from the centenary Fão riveted bridge, the geometry details of which are given in Fig. 3. First, the probabilistic *P*-*S*-*N* field for the riveted joint is determined by applying the Weibull model proposed in (Castillo *et al*. 2009), from constant stress range data, as illustrated in Fig. 4. Thereafter, the cumulative distribution function resulting for the normalized variable *V* from constant amplitude fatigue data is calculated, as shown in Fig. 5.

It has to be noted that a significant scatter band is observed when the *S*-*N* field was obtained considering all original data given in (De Jesus *et al*. 2014, 2015), which could be due to the presence of the two particular data points at $\Delta\sigma$=400 MPa, possibly implied in the plastic regime

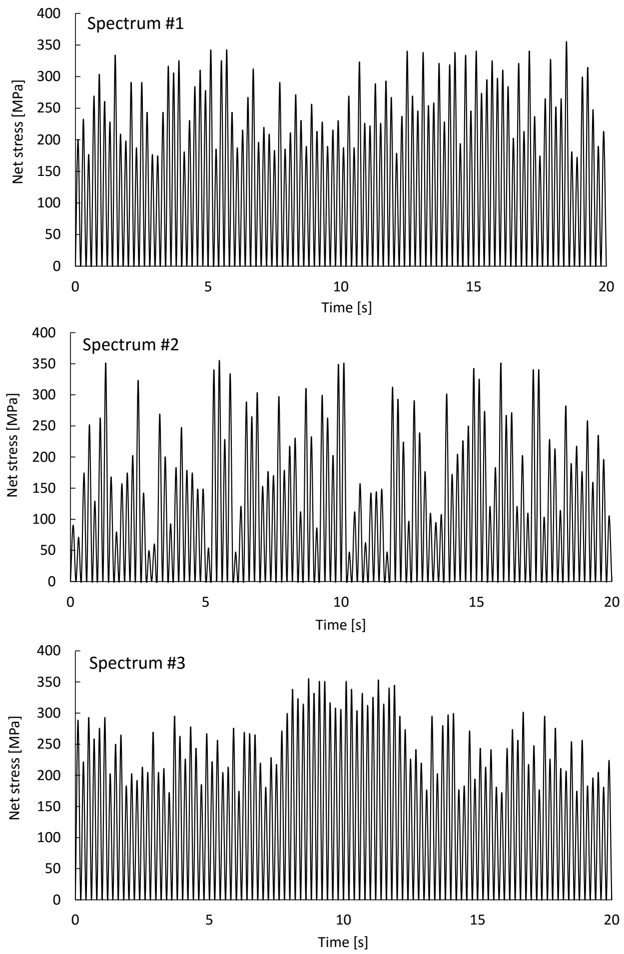

Fig. 6 Variable amplitude stress range blocks applied to the riveted joints made of puddle iron from the Fão bridge (De Jesus *et al.* 2014, 2015)

(low cycle fatigue region). For that reason these two results have been omitted to evaluate the *S*-*N* field leading to a narrower and more realistic probabilistic field as represented in Fig. 4.

Variable stress range blocks (load spectra) were repeatedly applied to the riveted joints corresponding to three distinct stress spectra, as illustrated in Fig. 6. The first block (spectrum 1) includes individual stress cycles with stress ranges varying in the interval 170 to 360 MPa. The second block (spectrum 2) includes individual stress cycles with stresses fluctuating between 45 to 360 MPa. Finally, in the third block (spectrum 3), the stresses range between 170 and 360 MPa, showing a central region with higher stresses. Each block consists of 100 cycles at null stress ratio. A total of 8 specimens were tested: 2 specimens under spectrum 1, 3 specimens under spectrum 2 and 3 specimens under spectrum 3. Table 1 summarizes the experimental results for the tested riveted joints, in terms of both blocks to failure and number of cycles to failure.

Assuming homogeneity among the test results, despite the different stress range spectra applied, Fig. 7 illustrates the cumulative distribution function of the normalized variable *V*, as computed for any of the variable amplitude fatigue tests. Only a total of eight data points were

Table 1 Results of the variable stress range tests (De Jesus *et al.* 2014, 2015)

Specimens	Block Type	Interval of stress ranges (MPa)	No. of Blocks to Filure	Cycles to Failure
57	Spectrum #1	170-360	601	60100
58	Spectrum #1	170-360	701	70100
59	Spectrum #2	45-360	1309	130900
60	Spectrum #2	45-360	574	57400
61	Spectrum #2	45-360	1301	130100
62	Spectrum #3	170-360	282	28200
63	Spectrum #3	170-360	894	89400
64	Spectrum #3	170-360	660	66000

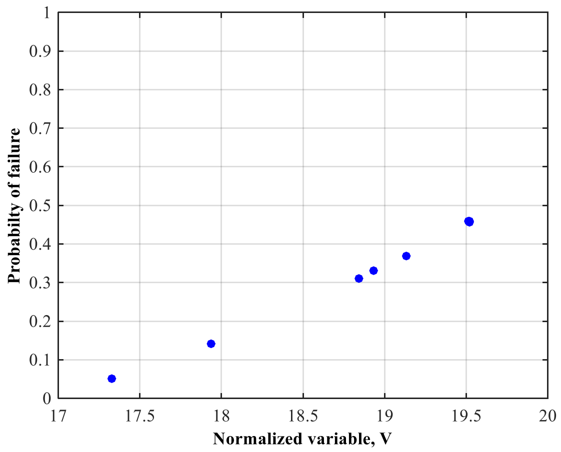

Fig. 7 Normalized variable V vs. probability of failure for the different variable amplitude tests as computed using the Weibull distribution

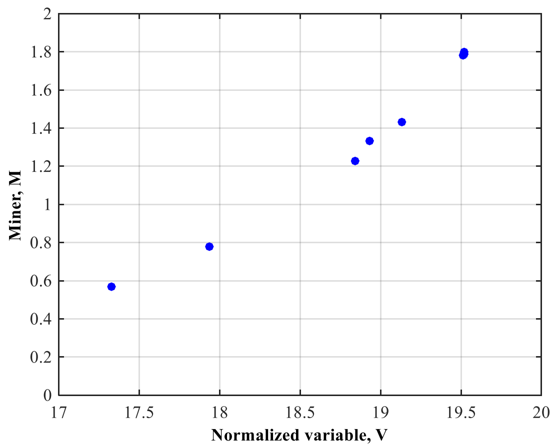

Fig. 8 Normalized variable V vs. Miner number for riveted joints

available comprising the specimens 57 to 64, as given in Table 1. The probabilities of failure are computed by applying the procedure depicted in the step iii) of the Section 3 to the probabilistic Weibull *S-N* field obtained from the test results of the riveted joints, as shown in Fig. 4. Due to the limited number of test results available, the probabilities of failure implied range only from 5% to 15% and 30% to 55%, as seen in Fig. 7, where the cumulative distribution function seems to be almost linear, impeding a reliable definition of lower failure probabilities. The corresponding Miner and Log-Miner numbers are computed from Eqs. (14) and (15), respectively using the 50% percentile (median *S-N* curve) of the *P-S-N* field for the riveted joints in the denominator of these equations. The results are plotted in Figs. 8 and 9, for the Miner and Log-Miner numbers, respectively, versus the normalized variable, V, both representations being approximately linear. It is also worth mentioning that though the Miner numbers range from 0.97 to 3.11 and the Log-Miner numbers range between 0.93 and 1.05, both damage rules yield equivalent results concerning failure prediction. Replacing the normalized variable V in Figs. 8 and 9 by the corresponding probability of failure, resulting from the cumulative distribution function of Fig. 7, provides the probabilistic representations of M and $logM$ given by Figs. 10 and 11, which also

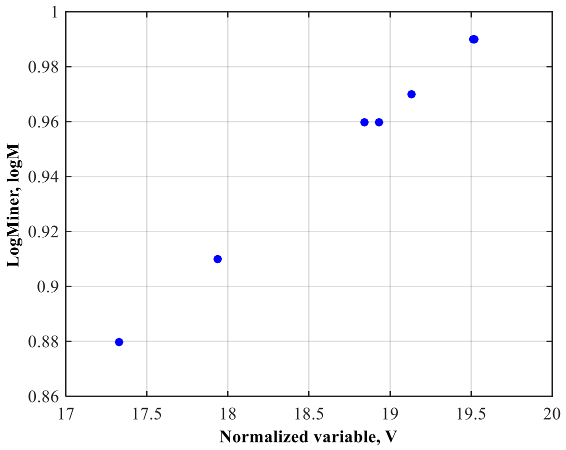

Fig. 9 Normalized variable V vs. Logarithmic Miner number for riveted joints.

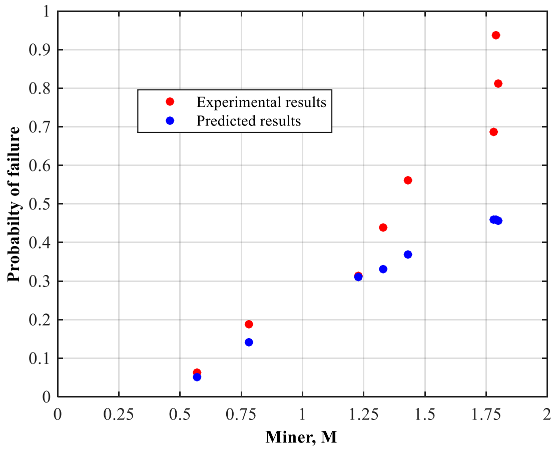

Fig. 10 Predicted and experimental Miner numbers as a function of failure probability for riveted joints

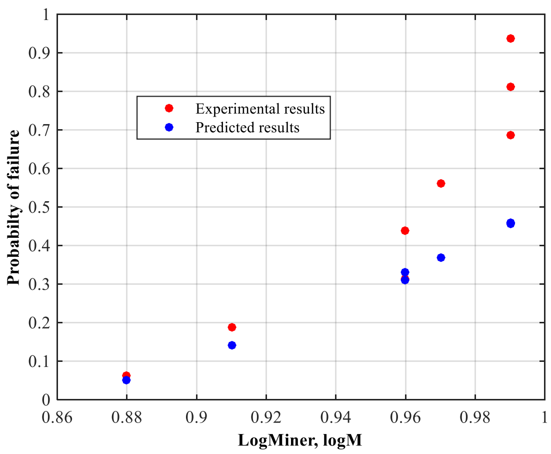

Fig. 11 Predicted and experimental Log-Miner numbers as a function of failure probability for riveted joints

include the experimental cumulative distribution for the Miner and Log-Miner numbers, as computed using the Hazen plotting position rule. Discrepancies arise between the predicted and experimental Miner and Log-Miner numbers, which could be assigned to the heterogeneity in the loading histories of the sample and mostly to the lower number of experimental results available which does not allows an accurate definition of the experimental cumulative distributions for the Miner and Log-Miner numbers.

3.2 Rotating-beam 24S-T4 aluminium alloy specimens

In this Section an analogous methodology, as described above, is applied to experimental data available from the literature (Hardrath *et al.* 1952) for rotating-beam specimens made of 24S-T4 aluminium alloy. Such a material is typically used on aircraft structural components subjected to

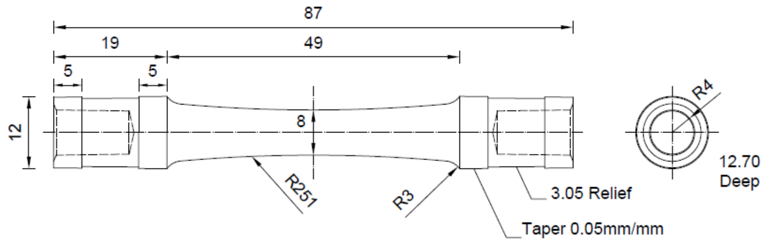

Fig. 12 Specimen geometry used for the rotating-beam tests in (Hardrath *et al.* 1952)

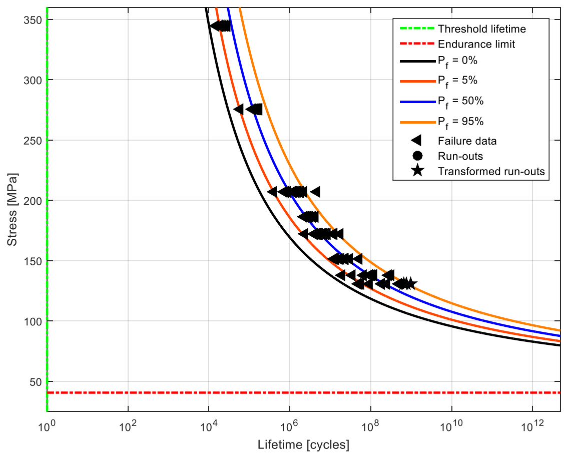

Fig. 13 Probabilistic *S-N* field obtained for rotating-beam tested aluminium specimens using the Weibull probabilistic model

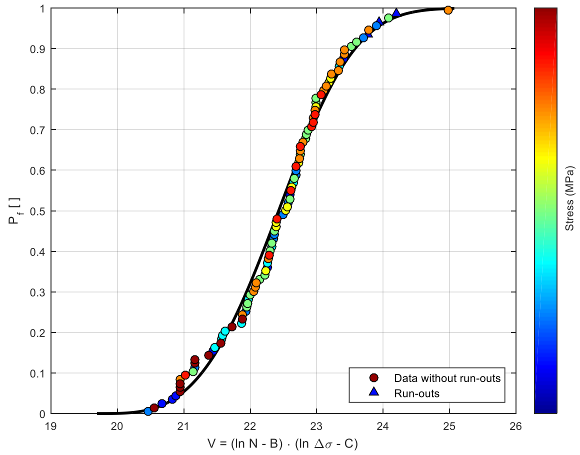

Fig. 14 Cumulative distribution function for normalized variable *V* associated to the *S-N* data of 24S-T4 aluminium alloy rotating-beam specimens

loads varying in time. In this case, both constant and variable amplitude fatigue data are available for this material.

Fig. 12 shows the geometry of the machined aluminium specimens. All samples were subjected to a finishing process in both longitudinal and circumferential direction using different polishing agents. The fatigue tests were run in a R.R. Moore rotating-beam fatigue testing machine. The results of 100 specimens, tested under constant stress range conditions, were used to obtain the probabilistic *S-N* field. Fig. 13 illustrates the Wöhler field based on the Weibull model (Castillo *et al.* 2009). The cumulative distribution function of the *V* variable obtained from *S-N* field information is represented in Fig. 14.

The fatigue tests under variable stress range were performed employing two different stress spectra: "Cam A", to achieve stress amplitudes varying in a sinusoidal way, and "Cam B" to provide stress amplitudes varying according to an exponential function, see Fig. 15. Each block corresponds to 10000 stress cycles. Limit values of each load spectrum were modified so that 13 different stress histories combinations were generated according to reference (Hardrath *et al.* 1952), in which the details concerning stress ranges and number of cycles to failure, for each specimen subjected to variable-amplitude fatigue tests are given.

As in the previous section, the Miner and Log-Miner numbers are computed in parallel with the accumulated normalized variable *V* at each load step, so that three values (corresponding to *M*, *LM*

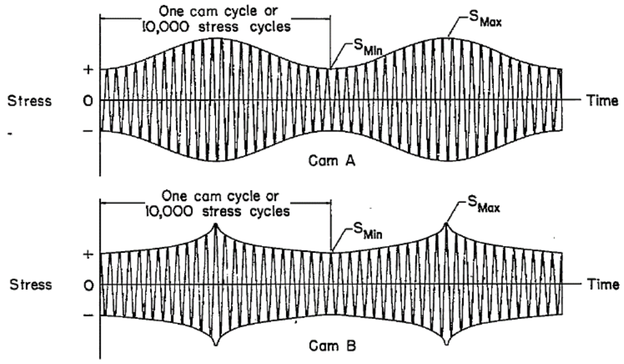

Fig. 15 Load spectrum for "Cam A" and "Cam B" applied to rotating specimens of aluminium alloy (Hardrath *et al.* 1952)

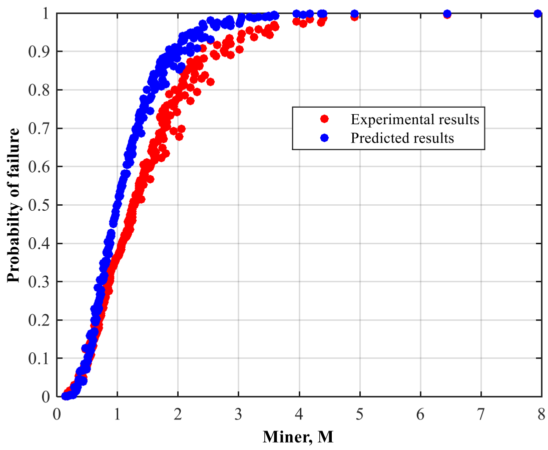

Fig. 16 Predicted and experimental cdfs of the Miner numbers for aluminium rotating-beam specimens (Hardrath *et al.* 1952)

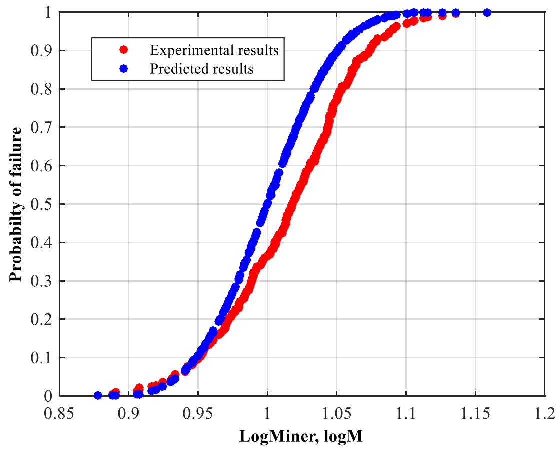

Fig. 17 Predicted and experimental cdf of the Log-Miner numbers for aluminium rotating-beam specimens (Hardrath *et al.* 1952)

and V) are calculated by each test for the number of cycles at which the failure of the specimens were observed. In this way, a relationship between these variables can be determined so that, once established the probability of failure associated with the V value, it is possible to infer the probability associated to the current values of the Miner and Log-Miner numbers.

Figs. 16 and 17 show the computed cumulative distribution functions for Miner and Log-Miner numbers that allows the experimental distributions (using the Hazen plotting position rule (Castillo 1988)) to be compared with the predicted results. A good agreement between experimental and predicted data is observed in the range below 0.2 probability of failure (note that a safe design of components subjected to fatigue loads is usually referred to probabilities of failure lower than 10%.

3.3 Fatigue analysis of plain concrete subjected to fatigue loading under compression

The distribution of the Miner and Log-Miner numbers resulting for a fatigue test program on

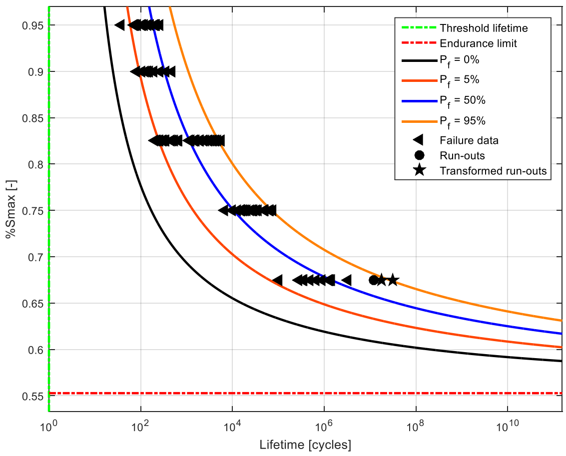

Fig. 18 Probabilistic *S-N* field obtained for plain concrete specimens under compression using the Weibull probabilistic model

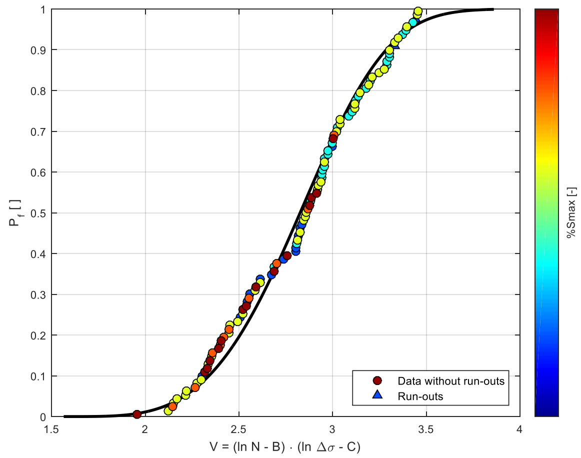

Fig. 19 Cumulative distribution function for the normalizing variable *V* for plain concrete specimens tested under compression

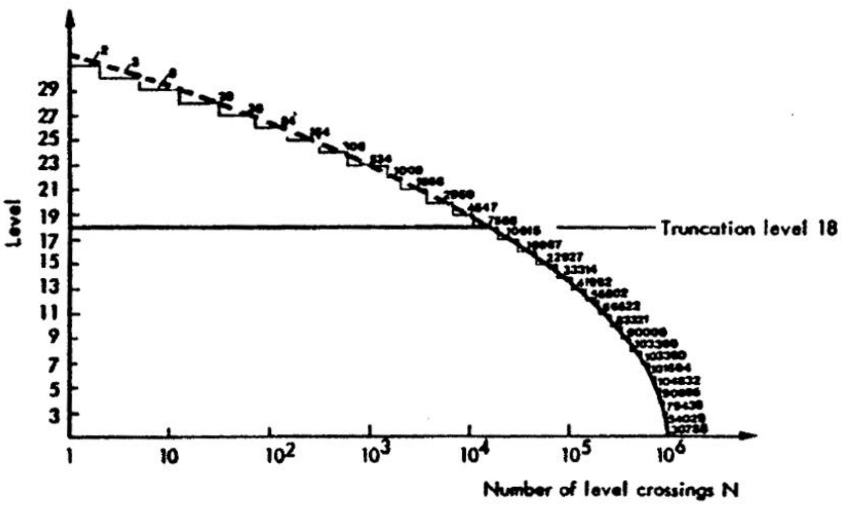

Fig. 20 Load spectrum applied to plain concrete specimens under compression (Holmen 1979)

concrete specimens, subjected to compressive stresses is evaluated, using experimental data available in the literature (Holmen 1979). More than a hundred of specimens were tested with constant-amplitude load to obtain the probabilistic *S-N* field (Fig. 18). The results of the varying amplitude fatigue tests are used to obtain the values of the normalizing variable *V*, and the Miner and Log-Miner numbers corresponding to each test. Three different loading histograms are run according to three different models, from which only the third model, as depicted in Fig. 20 is considered for the assessment. It consists in a stress succession of constant amplitude steps. This load histogram is divided into 31 levels, the 18^{th} one being chosen as truncation level so that all levels beyond that one are handled as 18^{th} level. Further, all levels below the 9^{th} level are omitted because they did do not presumably contribute to the damage process.

Table 2 Results of the compression tests under varying loading applied to plain concrete specimens

Test ref.	$\%S_{max}$	N_f	Number of basic load blocks	Test ref.	$\%S_{max}$	N_f	Number of basic load blocks
1	0,775	294670	201	30	0,825	4416	8
2	0,775	291911	199	31	0,825	30583	53
3	0,775	687112	468	32	0,836	8724	15
4	0,775	76328	52	33	0,836	34492	60
5	0,775	143992	98	34	0,836	20299	35
6	0,775	548075	373	35	0,836	47683	82
7	0,775	809727	551	36	0,836	27093	47
8	0,8	49698	34	37	0,836	12300	22
9	0,8	64508	44	38	0,743	34490	60
10	0,825	25537	18	39	0,743	178077	305
11	0,825	55710	38	40	0,743	79856	137
12	0,825	92798	64	41	0,743	153162	263
13	0,825	102330	70	42	0,743	75187	129
14	0,836	31890	22	43	0,743	85593	147
15	0,836	31971	22	44	0,743	96517	166
16	0,836	20388	14	45	0,743	192318	330
17	0,836	83638	57	46	0,756	120790	207
18	0,775	41038	71	47	0,756	22476	39
19	0,775	189486	325	48	0,779	49445	85
20	0,775	213384	366	49	0,779	15298	27
21	0,775	161450	277	50	0,779	66704	115
22	0,775	28714	50	51	0,779	38960	67
23	0,775	43345	75	52	0,79	55944	96
24	0,775	72929	125	53	0,79	106438	183
25	0,775	32715	57	54	0,79	58068	100
26	0,8	28717	50	55	0,79	40810	70
27	0,8	39697	68	56	0,79	51615	89
28	0,825	33856	58	57	0,79	21427	37
29	0,825	9615	17				

The influence of the sequence is reduced by considering a proportional fraction of the number of cycles relative to the original histogram. In this way, a "basic loading block" is defined to be repeatedly applied until failure of each specimen on the basis of values of Fig. 19 and Table 2, which summarize the results of 57 tested specimens.

Based on the probabilistic model in (Castillo *et al.* 2009), the cumulative distribution function of the normalizing variable V is obtained as represented in Fig. 19.

Once again, the relationship between the normalizing variable V and the Miner and Log-Miner numbers is determined in order to assign a probability of failure to any Miner or Log-Miner value. Figs. 21 and 22 show the correspondence of Miner and Log-Miner values with probability of

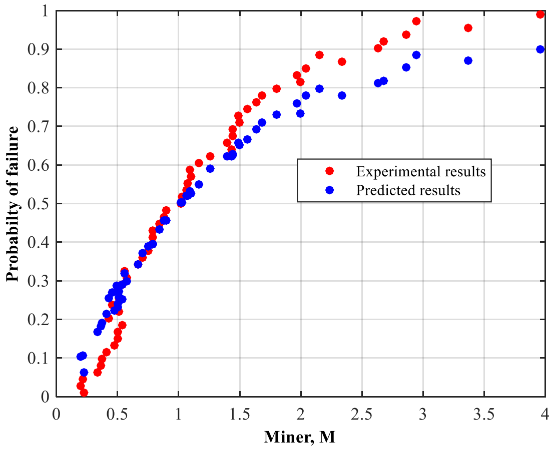

Fig. 21 Predicted and experimental Miner numbers as a function of failure probability for plain concrete specimens under compression fatigue

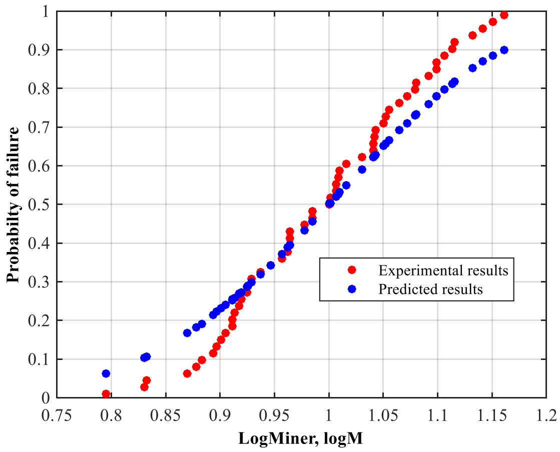

Fig. 22 Predicted and experimental Log-Miner numbers as a function of failure probability for plain concrete specimens under compression fatigue

failure, respectively.

A satisfactory correspondence between the predicted and experimental Miner and Log-Miner numbers is obtained, mainly in the range of probabilities of failure below 35%, that is, those of interest in the practical fatigue design.

3.4 Results discussion

The probabilistic model, as proposed by Castillo and Fernández-Canteli (Castillo *et al.* 2009), allows a probabilistic interpretation for the fatigue damage to be made by taking the normalizing variable, V, as a damage indicator in particular under varying stress range histories. In this way, the experimental results obtained under variable stress range data can be reliably predicted using the basic information provided by the probabilistic S-N field (see Figs. 5, 14 and 19).

The cumulative distribution function (*cdf*) of the normalized variable, V, is derived from the Weibull field, which on its turn is calculated from constant stress range tests. It provides the failure probability for any loading history that can be associated to classical damage parameters such as the Miner number or the Log-Miner number, allowing in this way the cdfs for these damage parameters to be obtained. This requires a monotonic relation between the normalized variable V and the damage parameter as observed for both Miner and Log-Miner numbers in the experimental campaigns analysed (see Figs. 9, 10). The comparison between the predicted and experimental cdfs (based on experimental data ranking) exhibits some level of disagreement for the riveted joints (see Figs. 10 and 11). This discrepancy may be attributed to the low number of experimental data points which results in lower confidence for the experimentally-based cumulative distribution but also due to the dissimilar loading histories applied to the different test cases. On the contrary, the results analyzed for the aluminium and concrete campaigns are fully satisfactory, especially for low probabilities of failure, i.e., the ones usually involved in the practical fatigue design.

Generally, the experimental Log-Miner number spreads over a narrower band than the conventional Miner number, what could be interpreted as representing in a more reliable way the traditional limit state concept for failure design. Nevertheless, a consequent statistical analysis and its interpretation prove the fully equivalence between the probabilistic prediction of failure resulting from both approaches.

Taking advance of the use of the probabilistic model proposed in (Castillo *et al.* 2009) the distribution of the normalized variable, *V*, can be used to determine the distribution functions for the classical Miner and Log-Miner numbers thus allowing a probabilistic prediction of the fatigue damage and failure of mechanical components under variable amplitude loading to be achieved what represents an evident enhancement of the classical approach for both. The application of the Miner or Log-Miner numbers accounts for fatigue scatter into separate steps. Firstly, the Miner or Log-Miner numbers are thereby computed using the available constant fatigue stress range data. Usually, the 50% percentile curve is chosen as a reference *S-N* curve though any other failure probability could be also specified for this task. In a second step, the computed Miner or Log-Miner numbers are correlated with the corresponding values of the normalized variable *V*, which on its turn may be identified as a Weibull cumulative distribution function. Accordingly, a probability of failure is assigned to any value of the Miner or Log-Miner numbers.

In the case of the experimental program performed by Holmen on concrete specimens under compression, a better agreement has been found between the predicted lifetimes and experimental data in the current study than in the former work (Canteli *et al.* 2014, Correia *et al.* 2015), co-authored by the authors of this paper. The reason for this improvement lies simply in keeping the decimal points for the normalized variable by the successive stress range conversions along the varying loading process.

The Miner rule is not able to reproduce either sequence effects, i.e., the influence of loading interaction or shifting downwards of the fatigue limit when damage occurs along the fatigue test, the latter being caused by crack growth due to loads overpassing the fatigue limit, inherent to any variable loading process. Consequently, some non-conservative results should be expected in the probabilistic prediction. Nevertheless, this is not generally the case in the present study, what could be attributed to retardation effects due to overloads compensating the above mentioned limitations of the Miner's rule. In random type loading spectra, retardation and acceleration effects from loading sequential events tend to cancel each other, justifying the use of the Miner model.

As a last comment: Interpreting the Miner rule as "linear damage", the same as denoting any other possible more complex proposal as "non-linear damage" rules, is wrong besides unimportant, because assigning linear damage with a linear increasing Miner number is fully gratuitous. In fact, it has been proved here that the Miner number does not represent at all a linear damage in the sense of probability of failure. Seemingly, an attempt is intended for reproducing the physical damage process being experienced by the material at a microscale, what from the design point of view is fully irrelevant for the practitioner engineer. In no case, probabilistic information concerning a possible fatigue failure is given. On the contrary, the key point consists in how to relate the reference magnitude, this being the Miner, the Log-Miner, the Marco-Starkey or any else damage model (Schijve 2005, Dowling 1993) etc., to the probability of failure, which is the crucial point contemplated from the limit states concept what, in principle, is not an obvious task. In our work, a methodology is proposed for relating the Miner, or Log-Miner rule, to the probability of failure associated to the normalized variable *V* by the Castillo-Canteli model. Other alternatives are certainly possible but they require resorting, in any case, to a probabilistic *S-N* model, as for instance that proposed here (see Castillo *et al.* 2009) or like those proposed by (Pascual *et al.* 1999, Meeker *et al.* 1998) or any others.

4. Conclusions

The main conclusions from this work are:

- A statistical interpretation of the Miner and Log-Miner numbers is possible, even maintaining the simplicity of their calculation in the conventional approach. This allows enhancement of the reliability by lifetime prediction of structural and mechanical components against conventional approaches.
- A probabilistic definition of the *S-N* field is required to proceed to the adequate probabilistic assessment of the Miner and Log-Miner numbers.
- For the riveted joints made of puddle iron, the low number of experimental data points does not guarantee general confident agreement for prediction of the failure cumulative distribution function.
- The approach proposed is also applied to fatigue programs extracted from the literature implying other materials subjected to different fatigue test loading types, as aluminium under rotative-bending and plain concrete under compression, and distinct load spectra. The results verify the versatility and validity of the methodology proposed. For these cases a very satisfactory agreement is found between the predicted and experimental based cdfs for the Miner or Log-Miner numbers, in particular for low probabilities of failure which focusses the interest of the engineering practice.
- The probabilistic results for the Miner number are fully comparable to those using Log-Miner model. This proves that the Miner rule does not properly suit to a “linear cumulative damage hypothesis” as generally believed.

Acknowledgments

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