Structural Engineering and Mechanics, Vol. 6, No. 7 (1998) 747-756
DOI: http://dx.doi.org/10.12989/sem.1998.6.7.747

On Beck's column with shear and compressibility

L.J. Cveticanin and T.M. Atanackovic†

Department of Technical Sciences, Trg D. Obradovica 6, 21000 Novi Sad, Yugoslavia

Abstract. In this paper the influence of rotary inertia, shear and compressibility on the value of the critical force for the Beck's column is analyzed. The constitutive equation is of Engesser's type. As a result, the critical load parameter for which instability of flutter type occurs is calculated for several values of the column's parameters.

Key words: beck's column; elastic stability; column with shear and compressibility

1. Introduction

In his classical paper, Beck (1952) ananlyzed stability of an elastic column subjected to constant follower type force at the free end. He showed, by the eigen modal analyzes, that the column will loose stability, by oscillating with increasing amplitude, when the magnitude of the compressive follower force reaches the critical value

$$F_{cr} = 20.05 \frac{k_M}{L^2} \tag{1}$$

where k_M=EI=bending rigidity of the column; L=length of the column. In deriving Eq. (1) the classical Bernoulli-Euler theory was used. Later many additions and generalizations to the Beck's problem were proposed. For example, Carr and Malhardeen (1979) showed that eigen modal analysis gives indeed the stability boundary for the Beck column. Smith and Herrmann (1972) analyzed the influence of an elastic foundation of Winkler type on the stability, while Hauger (1975) used a model of compressible elastic rod and determined the critical force. Becker, Hauger and Winzen (1977) generalized the problem by introducing external and internal damping. Beck's column with variable cross section was treated by Matsuda, Sakiyama and Morita (1993). Critical review of some results for Beck column is given by Panovko and Gubanova (1987).

The main objective of this paper is to investigate the effects of rotary inertia, shear and compressibility on the critical load for the Beck's column. Thus, in deriving the equations of motion, the constitutive equations for the column in the form given by Schmidt and DaDeppo

† Professor

(1971) are taken which belong to the so-called Engesser type of constitutive equations (see Gjelsvik 1991). The use of Schmidt and DaDeppo type of constitutive equations for the study of Beck's column is the main contribution of this paper. Other possibility would be to use Haringx's type of constitutive equations, as presented, for example, by Libai (1992).

2. Formulation

Consider a thin, heavy elastic column, fixed at one and free at the other end. The column is loaded at the free end C by a tangential force P. We assume that the rod is constrained to move in a fixed plane II (see Fig. 1) to which its axis belongs at the initial moment (t=0).

Let $\bar{x}$ and $\bar{y}$ be Cartesian coordinate system with the origin at B. Assume that the axis of the column is extensible and in that unloaded straight state has a length L. A differential element of the column has a length dS at time t=0 (rod is undeformed) and in the deformed state at time t has length ds. Then, the strain of the rod axis is

$$\varepsilon = \frac{ds - dS}{dS} \tag{2}$$

From the D'Alembert's principle we obtain (see Fig. 1b)

$$dH = -\, q_x \, dS \tag{3a}$$

$$dV = -\, q_y \, dS \tag{3b}$$

$$dM = V\, dx - H dy + m\, dS \tag{3c}$$

where H=component of contact force in $\bar{x}$ -direction; V=$\bar{y}$ -component of the contact force; M=bending moment; x and y=coordinates of and arbitrary point; q_x, q_y=intensities of distributed forces in $\bar{x}$ and $\bar{y}$ direction per unit length of the undeformed column axis, respectively; m=couple per unit length of the undeformed column axis. The positive signs for moment and forces are shown in Fig. 1b. It is assumed that the only distributed forces and couples come from the inertia so that

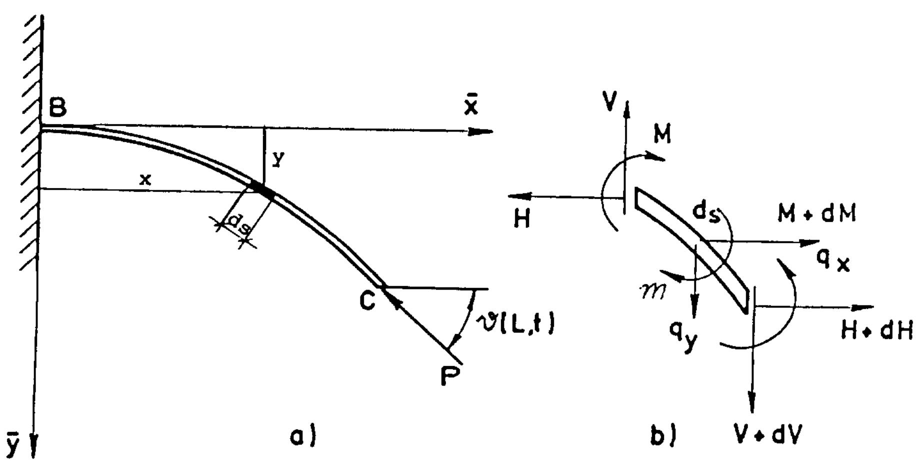

Fig. 1 Coordinate system and load configuration

$$q_x = -\rho\frac{\partial^2 x}{\partial t^2} \tag{4a}$$

$$q_y = -\rho\frac{\partial^2 y}{\partial t^2} \tag{4b}$$

$$m = -\rho J\frac{\partial^2 \vartheta}{\partial t^2} \tag{4c}$$

where ρ=mass per unit length in the undeformed state; ϑ=angle between the tangent to the column axis and $\bar{x}$ axis; J=a constant describing the rotary inertia of the column cross section. The following geometrical relations are added to (3a)-(3c)

$$\frac{\partial x}{\partial S} = (1+\varepsilon)\cos\vartheta \tag{5a}$$

$$\frac{\partial y}{\partial S} = (1+\varepsilon)\sin\vartheta \tag{5b}$$

The constitutive equations in the form given by Schmidt and DaDeppo (1971) are as follows

$$\varepsilon = \frac{N}{k_E} \tag{6a}$$

$$\gamma = \frac{Q}{k_G} \tag{6b}$$

$$\frac{\partial\vartheta}{\partial S} - \frac{\partial\gamma}{\partial S} = -\frac{M}{k_M\cos\gamma} \tag{6c}$$

where k_E=the constant describing extensional rigidity of column; k_G=the constant describing the shear rigidity of the column; k_M=the constant describing the bending rigidity; N=the component of the contact force in the direction of the tangent to the column axis; Q=the component of the contact force in the direction of the sheared planes.

Q makes angle $(\pi/2)+\gamma$ with the tangent to the rod's axis. In writing (4c) an $\partial^2(\vartheta-\gamma)/\partial t^2$ can be substituted for $\partial^2\vartheta/\partial t^2$. This would lead to Timoshenko's beam theory. The expression $\partial^2(\vartheta-\gamma)/\partial t^2$ represents the angular acceleration of the column cross section while $\partial^2\vartheta/\partial t^2$ represents angular acceleration of an element of the colume axis. The choice of angular acceleration is that proposed by Schmidt and DaDeppo (1971). Physically $\partial^2\vartheta/\partial t^2$ corresponds to the column that consists of "light" discs mounted on a "heavy" wire.

Q and N are expressed in terms of H and V. From the definition of Q and N it follows that (see Atanackovic and Spasic 1991).

$$Q = V\frac{\cos\vartheta}{\cos\gamma} - H\frac{\sin\vartheta}{\cos\gamma} \tag{7a}$$

$$N = H\frac{\cos(\vartheta-\gamma)}{\cos\gamma} + V\frac{\sin(\vartheta-\gamma)}{\cos\gamma} \tag{7b}$$

Eliminating ε from the systems (3)-(7) we obtain

$$\frac{\partial H}{\partial S} - \rho\frac{\partial^2 x}{\partial t^2} = 0 \tag{8a}$$

$$\frac{\partial V}{\partial S}-\rho\frac{\partial^2 y}{\partial t^2}=0 \tag{8b}$$

$$\frac{\partial M}{\partial S}-V\frac{\partial x}{\partial S}+H\frac{\partial y}{\partial S}+\rho J\frac{\partial^2 \vartheta}{\partial t^2}=0 \tag{8c}$$

$$\frac{\partial x}{\partial S}-(1+\frac{1}{k_E}\frac{H\cos(\vartheta-\gamma)+V\sin(\vartheta-\gamma)}{\cos\gamma})\cos\vartheta=0 \tag{8d}$$

$$\frac{\partial y}{\partial S}-(1+\frac{1}{k_E}\frac{H\cos(\vartheta-\gamma)+V\sin(\vartheta-\gamma)}{\cos\gamma})\sin\vartheta=0 \tag{8e}$$

$$\frac{\partial v}{\partial S}+\frac{\dfrac{M}{k_M\cos\gamma}-\dfrac{1}{k_G}\dfrac{\rho[(\partial^2 y/\partial t^2)\cos v-(\partial^2 x/\partial t^2)\sin\vartheta]}{\cos\gamma-\gamma\sin\gamma}}{1+\dfrac{V\sin\vartheta+H\cos\vartheta}{\cos\gamma-\gamma\sin\gamma}\dfrac{1}{k_G}}=0 \tag{8f}$$

$$\frac{\partial\gamma}{\partial S}+\frac{1}{k_G}\frac{V\sin\vartheta-H\cos\vartheta}{\cos\gamma-\gamma\sin\gamma}\frac{\partial\vartheta}{\partial S}-\frac{1}{k_G}\frac{\rho[(\partial^2 y/\partial t^2)\cos\vartheta-(\partial^2 x/\partial t^2)\sin\vartheta]}{(\cos\gamma-\gamma\sin\gamma)}=0 \tag{8g}$$

In particular (8g) follows from (6b) and (6c). The following boundary conditions must be applied to Eqs. (8a)-(8g) at the base (B)

$$x(0,t)=0 \tag{9a}$$

$$y(0,t)=0 \tag{9b}$$

$$\vartheta(0,t)=0 \tag{9c}$$

at the free end (C)

$$H(L,t)=-P\cos\vartheta(L,t) \tag{9d}$$

$$\gamma(L,t)=0 \tag{9e}$$

$$V(L,t)=-P\sin\vartheta(L,t) \tag{9f}$$

$$M(L,t)=0 \tag{9g}$$

Note that (9c) corresponds to the case were the axis of the rod has fixed direction or clamped conditions at (B). Instead of $\vartheta(0, t)=0$, $\vartheta(0, t)-\gamma(0, t)=0$ can be used. This condition corresponds to the so called welded end. This is in agreement with our choice of angular acceleration term.

The trivial solution to (8a)-(8g) i.e., the solution in which the rod axis remains straight reads

$$V^0=0 \tag{10a}$$

$$M^0=0 \tag{10b}$$

$$y^0=0 \tag{10c}$$

$$\vartheta^0=0 \tag{10d}$$

$$\gamma^0=0 \tag{10e}$$

$$x^0=S(1-\frac{P}{EA}) \tag{10f}$$

$$H^0=-P \tag{10g}$$

The solution to (8a)-(8g) is as follows

$$H = H^0 + \overline{H} \tag{11a}$$

$$V = V^0 + \overline{V} \tag{11b}$$

$$M = M^0 + \overline{M} \tag{11c}$$

$$x = x^0 + \overline{u} \tag{11d}$$

$$y = y^0 + \overline{v} \tag{11e}$$

$$\vartheta = \vartheta^0 + \overline{\vartheta} \tag{11f}$$

$$\gamma = \gamma^0 + \overline{\gamma} \tag{11g}$$

and introducing the non-dimensional quantities

$$H = \frac{\overline{H}L^2}{k_M} \tag{12a}$$

$$V = \frac{\overline{V}L^2}{k_M} \tag{12b}$$

$$M = \frac{\overline{M}L}{k_M} \tag{12c}$$

$$u = \frac{\overline{u}}{L} \tag{12d}$$

$$v = \frac{\overline{v}}{L} \tag{12e}$$

$$\xi = \frac{S}{L} \tag{12f}$$

$$\tau = \Omega_1 t \tag{12g}$$

$$\gamma_1 = \frac{\sqrt{J}}{L} \tag{12h}$$

$$\lambda = \frac{PL^2}{k_M} \tag{12i}$$

$$\mu = L\frac{\sqrt{k_E}}{k_M} \tag{12j}$$

$$\beta = L^2 \frac{k_G}{k_M} \tag{12k}$$

$$\Omega_1 = \sqrt{\frac{k_M}{\rho L^4}} \tag{12l}$$

Then, the linearization of (8a)-(8g) about trivial solution (10a)-(10g) reads

$$\frac{\partial H}{\partial \xi} = \frac{\partial^2 u}{\partial \tau^2} \tag{13a}$$

$$\frac{\partial V}{\partial \xi} = \frac{\partial^2 v}{\partial \tau^2} \tag{13b}$$

$$\frac{\partial M}{\partial \xi} = -\gamma_1^2 \frac{\partial \vartheta}{\partial \tau^2} + V\left(1 - \frac{\lambda}{\mu^2}\right) + \lambda \frac{\partial v}{\partial \xi} \tag{13c}$$

$$\frac{\partial u}{\partial \xi} = \frac{H}{\mu^2} \tag{13d}$$

$$\frac{\partial v}{\partial \xi} = \left(1 - \frac{\lambda}{\mu^2}\right)\vartheta \tag{13e}$$

$$\frac{\partial \vartheta}{\partial \xi} = -\frac{M - \frac{1}{\beta}\frac{\partial^2 v}{\partial \tau^2}}{1 - \frac{\lambda}{\beta}} \tag{13f}$$

$$\frac{\partial \gamma}{\partial \xi} = \frac{\lambda}{\beta}\,\frac{M - \frac{1}{\beta}\frac{\partial^2 v}{\partial \tau^2}}{1 - \frac{\lambda}{\beta}} + \frac{1}{\beta}\frac{\partial^2 v}{\partial \tau^2} \tag{13g}$$

The boundary conditions are those given by (9) or (14)

$$u(0, \tau) = 0 \tag{14a}$$

$$v(0, \tau) = 0 \tag{14b}$$

$$\vartheta(0, \tau) = 0 \tag{14c}$$

$$H(1, \tau) = -\lambda \cos \vartheta(1, \tau) \tag{14d}$$

$$\gamma(1, \tau) = 0 \tag{14e}$$

$$V(1, \tau) = -\lambda \sin \vartheta(1, \tau) \tag{14f}$$

$$M(1, \tau) = 0. \tag{14g}$$

From (13) H, u and γ could be solved after the solutions for V, M, v and ϑ are found. Thus we consider first the system consisting of (13b), (13c), (13e) and (13g). This system is further reduced to a single fourth order equation for v

$$A\frac{\partial^4 v}{\partial \xi^4} + \left(1 - \frac{\lambda}{\mu^2}\right)\frac{\partial^2 v}{\partial \tau^2} + \lambda\frac{\partial^2 v}{\partial \xi^2} - B\frac{\partial^4 v}{\partial \xi^2 \partial \tau^2} = 0 \tag{15}$$

where

$$A = \frac{1 - \frac{\lambda}{\beta}}{1 - \frac{\lambda}{\mu^2}}; \qquad B = \frac{\gamma_1^2}{1 - \frac{\lambda}{\mu^2}} \tag{16}$$

Under the following boundary conditions at end (B)

$$v(0, \tau) = 0 \tag{17a}$$

$$\frac{\partial v(0, \tau)}{\partial \xi} = 0 \tag{17b}$$

and at end (C)

$$B \frac{\partial^3 v(1, \tau)}{\partial \xi \, \partial \tau^2} - A \frac{\partial^3 v(1, \tau)}{\partial \xi^3} = 0 \tag{17c}$$

$$\frac{1}{\beta} \frac{\partial^2 v(1, \tau)}{\partial \tau^2} - A \frac{\partial^2 v(1, \tau)}{\partial \xi^2} = 0 \tag{17d}$$

The system (15) and (17) represents the generalization of the Beck's problem for the generalized (Engesser type) elastica with rotary inertia.

The solution to (15) is taken

$$v(\xi, \tau) = X(\xi)\, e^{i\omega\tau} \tag{18}$$

Substituting Eq. (18) into (15) is obtained

$$A\, X^{IV} + (\lambda + B\,\omega^2) X'' - \omega^2 (1 - \frac{\lambda}{\mu^2}) X = 0 \tag{19}$$

where the prime superscript is simple ($d/d\xi$). With Eq. (19) and the boundary conditions the following eigenvalue equation is obtained

$$\begin{gathered} s_1 s_2 [A\,\omega^2 (\frac{1}{\beta} + B)\,(s_2^2 - s_1^2) - A^2 (s_1^4 + s_2^4) - \frac{2B}{\beta}\,\omega^4] - \\ s_1 s_2 \cos(s_2)\, ch(s_1) [A\,\omega^2 (\frac{1}{\beta} + B)\,(s_2^2 - s_1^2) + A^2 s_1^2 s_2^2 - \frac{2B}{\beta}\,\omega^4] + \\ \sin(s_2)\, sh(s_1)\, [\omega^4 \frac{B}{\beta} (s_2^2 - s_1^2) - A \frac{\omega^2}{\beta} (s_1^4 + s_2^4) + 2ABs_1^2 s_2^2 \omega^2 - A^2 s_1^2 s_2^2 (s_2^2 - s_1^2)] = 0 \end{gathered} \tag{20}$$

where

$$s_1 = \left\{ -\frac{1}{2A} (\lambda + B\,\omega^2) + \frac{1}{2A} \,[(\lambda + B\,\omega^2)^2 + 4A\;\omega^2 (1 - \frac{\lambda}{\mu^2})]^{1/2} \right\}^{1/2} \tag{21a}$$

$$s_2 = \left\{ \frac{1}{2A} (\lambda + B\,\omega^2) + \frac{1}{2A} \,[(\lambda + B\,\omega^2)^2 + 4A\;\omega^2 (1 - \frac{\lambda}{\mu^2})]^{1/2} \right\}^{1/2} \tag{21b}$$

3. Results

The flutter type of instability occurs when ω in Eq. (19) has a nonnegative imaginary part. This is equivalent to the condition that the first two roots of Eq. (20) approach a common real value. Fig. 2 shows a typical plot of Eq. (20) in the $\lambda - \omega^2$ plane.

Curve **1** corresponds to (β=10^6, μ^2=10^2, γ_1^2=10^{-6}), curve **2** to (β=∞, μ^2=∞, γ_1^2=0) and curve **3** to (β=10^2, μ^2=10^6, γ_1^2=0.01). The parameters β, μ and γ_1 represent the influence of

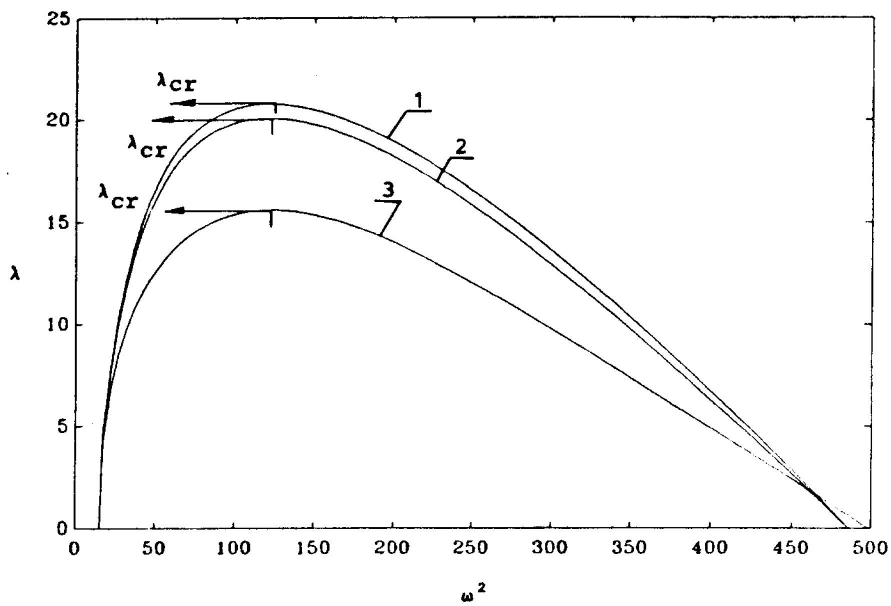

Fig. 2 Follower force as a function of frequency

Table

	λ_{cr}		
	γ_1^2=0	γ_1^2=0.05	γ_1^2=0.1
$\beta=\infty$ $\mu^2=\infty$	20.05095	15.43159	12.30468
β=1000 $\mu^2=\infty$	19.52230	15.00423	11.95776
$\beta=\infty$ μ^2=10000	20.11711	15.47743	12.34050
β=1000 μ^2=10000	19.58338	15.04576	11.99020

shear rigidity, extensional rigidity and rotary inertia, respectively. The critical values of $\lambda=\lambda_{cr}$, for several values of parameters are shown in the Table below.

4. Conclusions

In this paper we treated the generalized Beck's problem. The main results may be summarized as:

1. The nonlinear partial differential equations describing in-plane motion of the rod for which the constitutive equations are taken in the form proposed by Schmidt and DaDeppo, 1971 are derived.

2. The nonlinear system of equations of motion is linearized around the trivial solution defined by Eqs. (10a)-(10g). The resulting linear system is given by Eqs. (13a)-(13g), which

is further reduced to a fourth order partial differential Eq. (15).

3. The stability condition is obtained from the requirement that the solution of Eq. (15) is bounded. The analysis of stability conditions leads to the following conclusions:

The critical load parameter in the case $\beta=\infty$ and $\mu^2=\infty$ (classical case) agrees well with that recently obtained, highly accurate result of Jankovic (1993).

From the results shown in the Table above it is concluded that by increasing the rotary inertia coefficient γ_1^2, the value of the critical force decreases. The same is true for the parameter β, i.e., by reducing the shear rigidity (k_G) the critical force becomes less. The opposite is true for the slenderness ratio μ. That is, by reducing the extensional rigidity (k_E), the critical force is increased.

4. Note that the present generalization could be modified by using different angular acceleration term and different boundary condition at the fixed end. Results of such investigation will be reported elsewhere.

References

Atanackovic, T.M. and Spasic, D.T. (1991), "Effect of shear and compressibility on stability of a heavy column with an end load", *Q. Jl. Mech. Appl Math.*, **44**, 601-617.

Beck, M. (1952), "Die Knicklast des einseitig eingespannten, tangential gedrückten Stabes", *ZAMP* **3**, 225-228.

Becker, M., Hauger, W. and Winzen, W. (1977), "Influence of internal and external damping on the stability of Beck's column on an elastic foundation", *Journal of Sound and Vibration,* **54**, 468-472.

Carr, J. and Malhardeen, M.Z. (1979), "Beck's problem", *SIAM Journal of Applied Mathemaqtics,* **37**, 261-262.

Gjelsvik, A. (1991), "Stability of built-up columns", *ASCE Journal of Engineering Mechanics,* **117**, 1331-1345.

Hauger, W. (1975), "Stability of a compressible rod subjected to nonconservative forces", *ASME Journal of Applied Mechanics,* **42**, 887-888.

Jankovic, M.S. (1993). "Comments on eigenvalue sensitivity in the stability analysis of Beck's column with a concentrated mass at the free end", *Journal of Sound and Vibration,* **167**, 557-559.

Libai, A. (1992), "Equations for the nonlinear planar deformation of beams", *ASME Journal of Applied Mechanics,* **59**, 1028-1030.

Matsuda, H., Sakiyama, T. and Morita, Ch. (1993), "Variable cross sectional Beck's column subjected to nonconservative load", *ZAMM* **73**, 383-385.

Panovko, Y.G. and Gubanova, I.I. (1987), *Stability and Vibration of Elastic Systems,* 4th ed. Moscow: Nauka (in Russian).

Schmidt, R. and DaDeppo, D.A. (1971), "Nonlinear theory of arches with transverse shear deformation and rotary inertia", *The Journal of the Industrial Mathematics Society,* **21**, 33-49.

Smith, T.E. and Herrmann, G. (1972), "Stability of a beam on an elastic foundation subjected to a follower force", *ASME Journal of Applied Mechanics,* **39**, 628-629.

Notations

H : contact force in $\bar{x}$ direction

i : $\sqrt{-1}$=imaginary unit

J : rotary inertia of rod

k_E : parameter describing the column's extensional rigidity

k_G : parameter describing the column's shear rigidity
k_M : parameter describing the column's bending rigidity
L : column's length
M : bending moment
m : bending moment per unit length on the undeformed column axis
N : component of the contact force in the direction tangent to the column axis
P : follower force
Q : component of the contact force in the direction of the sheared planes
q_x : intensity of distributed forces in $\bar{x}$ direction per unit length of the undeformed column axis
q_y : intensity of distributed forces in $\bar{y}$ direction per unit length of the undeformed column axis
S : length of the undeformed column
s : length of the deformed column
t : time
V : contact force in $\bar{y}$ direction
x, y : coordinates of an arbitrary point
β : parameter measuring relative influence of shear rigidity with respect to bending rigidity
ε : strain of the column axis
γ : shear angle
γ_1 : non-dimensional rotary inertia
λ : non-dimensional follower force
μ : slenderness ratio
ω : the frequency of the system
ρ : mass per unit length in the deformed state
ϑ : angle between the tangent to the column and $\bar{x}$ axis