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Analysis of composite plates using various plate theories Part 1: Formulation and analytical solutions

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Abstract. A unified third-order laminate plate theory that contains classical, first-order and third-order theories as special cases is presented. Analytical solutions using the Navier and Lévy solution procedures are presented. The Navier solutions are limited to simply supported rectangular plates while the Lévy solutions are restricted to rectangular plates with two parallel edges simply supported and other two edges having arbitrary combination of simply supported, clamped, and free boundary conditions. Numerical results of bending and vibration for a number of problems are discussed in the second part of the paper.

Key words: higher-order theory; Lévy solutions; finite element solutions; Navier solutions; shear deformation; bending natural vibration.

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1. Introduction

1.1. Background

Most of the structural theories used till now to characterize the behavior of composite laminates fall into the category of "equivalent single layer (ESL) theories". In these theories, the material properties of the constituent layers are "smeared" to form a hypothetical single layer whose properties are equivalent to through-the-thickness integrated sum of its constituents. This category of theories have been found to be adequate in predicting global response characteristics of laminates, like maximum deflections, maximum stresses, fundamental frequencies, or critical buckling loads. The present study deals with a unification and critical evaluation of various ESL theories in predicting global response characteristics of a composite laminate. The following literature review forms a background for the present study.

1.2. Review of literature

The subject of plate and shell theories continue to attract the attention of researchers judging by the number of publications in this area. Beams, plates, and shells being among the most common structural components in use today, their analysis is of considerable interest to

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engineers and scientists. The theories used for modeling laminated composite plates and shells fall into the following three categories (Reddy 1997, 1990, 1989).

1. Equivalent Single Layer (ESL) 2-D theories
2. Layerwise (LW) 2-D theories
3. Continuum based 3-D theories

Since we will be primarily concerned with the ESL theories, the literature related to that class of theories will be discussed in detail. The literature for layerwise and continuum based theories will also be discussed briefly.

1.2.1. ESL theories

In the ESL theories, the displacements or stresses are expanded as a linear combination of the thickness coordinate and undetermined functions of position in the reference surface

$$\phi_i(x, y, z) = \sum_{j=0}^{N_i} \phi_i^j(x, y) z^j, \quad (i = 1, 2, 3) \tag{1}$$

where N_i are the number of terms in the expansion. ϕ_i^j can be either displacements or stresses. This reduces the 3-D elasticity equations to 2-D equations in terms of thickness-averaged forces and moments.

In all the assumed displacement ELS theories, the displacements and their derivatives (and hence the strains) are continuous through the thickness of the plate. However, since the constitutive properties of each layer are different, the stresses are discontinuous at the layer interfaces. The principle of virtual displacements or the method of moments are used to derive the equations of motion.

The simplest ESL theory is the Classical Laminated Plate Theory (CLPT) (Timoshenko and Woinowsky-Kreiger 1961), in which the transverse shear and normal deformations are neglected. The classical plate theory was originally developed for homogeneous isotropic plates and was later extended to laminated composite plates. Laminated plate theories based on CLPT can be found in the works of Lekhnitskii (1981), Stavsky (1961), Reissner and Stavsky (1961), Dong *et al.* (1962), Bert and Mayberry (1969), and Whitney and Leissa (1969). The CLPT underpredicts deflections and overpredicts natural frequencies and buckling loads.

The simplest theory to take into account transverse shear deformation is the First Order Shear Deformation Theory (FSDT). This theory accounts for linear variation of inplane displacements through the thickness:

$$\begin{aligned} u_1(x, y, z) &= u(x, y) + z\,\phi_1(x, y) \\ u_2(x, y, z) &= v(x, y) + z\,\phi_2(x, y) \\ u_3(x, y, z) &= w(x, y) \end{aligned} \tag{2}$$

where ϕ_1 and ϕ_2 are the rotations of a straight line in the xz and yz planes. In FSDT, the normals to the midplane remain straight, but not necessarily normal, to the deformed surface. Although the particular nomenclature, "first order shear deformation theory", was introduced by Reddy (1985), the original idea can be found in the works of Basset (1890), Hildebrand *et al.* (1949), Reissner (1945), Hencky (1947), and Mindlin (1951). Another early paper which included shear deformation effects in homogeneous plates was due to Uflyand (1948). Attempts were first made to include shear deformation in non-homogeneous plates by Stavsky (1960), Ambartsumyan (1969), Yang *et al.* (1966), Whitney (1969), and Whitney and Pagano (1970). FSDT gives a state of constant shear strain through the thickness of the plate.

However, according to 3-D elasticity theory, the shear strains vary at least quadratically through the thickness. The so-called shear correction factors were introduced to correct for the discrepancy in the shear forces of FSDT and 3-D elasticity theory. The value of the shear correction factors depend on the constituent ply properties, lamination scheme, geometry, and boundary conditions, among others (Bert 1973, Whitney 1973, Chatterjee and Kulkarni 1979).

Higher order shear deformation plate theories are those in which the displacements were expanded upto the quadratic or higher powers of the thickness coordinate (see Nelson and Lorch 1974, Sun and co-workers 1971, 1972, 1973, Librescu 1975, Jemeilita 1975, Schmidt 1977, Krishna Murty 1977, 1986, 1987, Lo *et al.* 1977, 1978, Levinson and co-workers 1980, 1983, 1983, Murthy 1981, Reddy 1984, 1987, Bhimaraddi and Stevens 1984, Di Sciuva 1984, Stein 1986, and Tessler 1991, 1991). The third order theory of Reddy (1984, 1987), referred to as the Special Third Order Theory of Reddy (STTR), is based on the displacement fields used by Vlasov (1958), Jemeilita (1975), Schmidt (1977), and Krishna Murty (1977) for isotropic plates. Reddy (1987, 1997) extended the theories in Reddy (1983, 1984) to include geometrically nonlinear effects.

Higher order theories with similar displacement fields were developed by Levinson (1980), Murthy (1981), and Bhimaraddi and Stevens (1984). Schmidt (1977) also accounted for nonlinear strains. However, Schmidt and Levinson used the equilibrium equations of the first order theory, while the stresses are evaluated using strains associated with the higher order displacement field. Murthy also used the same approach to obtain the equations of motion. Such equations are not consistent with the principle of virtual displacements. Krishna Murty (1987) suggested that the transverse deflection be split into two parts, one due to bending, and the other due to shear. A similar representation was given in the 1960s by Huffington (1963) for beams. Of all the higher order theories, Reddy (1984) was the first to obtain the equilibrium equations in a consistent manner using the principle of virtual displacements. This gives rise to self-adjoint equations, and the type and form of the boundary conditions fall out uniquely. Lewinski (1986) investigated the mathematical aspects of the theories, while Bielski and Telega (1988) presented the existence of solutions to Reddy's nonlinear theory (1987).

Lo *et al.* (1977, 1978) proposed a high order theory in which the inplane displacements are expanded upto the cubic power in z, and the transverse displacement is expanded upto the quadratic power in z. The theory will be referred to as the General Third Order Theory (GTOT). Reddy (1990) developed a modification of Lo's theory in which he accounted for the zero transverse shear stress conditions at the top and bottom of the plate. This theory will be referred to as the general third order theory of Reddy (GTTR). It also takes into account the stretching of the transverse normals. Analytical and finite element solutions of this theory (in a slightly modified form) will be carried out in the present study. GTTR contains as special cases STTR, FSDT, and CLPT. The displacement field for GTTR (in its unmodified form) is

$$u_1 = u + z\phi_1 + z^2\left(-\frac{1}{2}\frac{\partial\psi_3}{\partial x}\right) + z^3\left[-\frac{1}{3}\left\{\frac{\partial\theta_3}{\partial x} + \frac{4}{h^2}\left(\phi_1 + \frac{\partial w}{\partial x}\right)\right\}\right]$$

$$u_2 = v + z\phi_2 + z^2\left(-\frac{1}{2}\frac{\partial\psi_3}{\partial y}\right) + z^3\left[-\frac{1}{3}\left\{\frac{\partial\theta_3}{\partial y} + \frac{4}{h^2}\left(\phi_2 + \frac{\partial w}{\partial y}\right)\right\}\right]$$

$$u_3 = w + z\,\psi_3 + z^2\theta_3 \tag{3}$$

Analytical solutions of the CLPT were developed by Whitney and Leissa (1969), and of FSDT by Whitney (1969, 1987), Bert and Chen (1978), Reddy (1984), Reddy and Chao (1981), Reddy, Khdeir, and Librescu (1987), and Chaudhuri *et al.* (1989, 1992). Analytical solutions of the higher order theories are due to Levinson and Cooke (1983), and Khdeir, Reddy, and Librescu (1984, 1985, 1987, 1988, 1989).

Numerous finite element models of the theories have been put forward. Of these, only a few representative contributions will be mentioned. For the first order theory, they include the work of Hughes and co-workers (1977, 1978), Hinton and co-workers (1979, 1984) and Reddy and co-workers (1979, 1980, 1983). Hughes *et al.* (1977) suggested reduced integration for shear terms to prevent "locking". A C^0 penalty finite element formulation was developed by Reddy (1979) that explained the use of selective integration in evaluating the stiffnesses associated with transverse shear deformation. For the third order theories, Phan and Reddy (1985), Ren and Hinton (1986) developed displacement finite element models of Reddy's special third order theory (1984), which we refer to as STTR. They used C^1 interpolation functions for the transverse displacement, and C^0 interpolation functions for the other displacements. Averill and Reddy (1992) analyzed plate finite elements based on STTR and FSDT for distortion sensitivity, accuracy, reliability and efficiency. Putcha and Reddy (1986) developed a mixed finite element formulation of STTR. Doong (1987) used the general third order theory (GTOT) of Lo *et al.* (1977) to study the stability and vibration of initially stressed thick plates. Other finite element models are due to Ren (1987), Di Sciuva (1986), Engblom and Ochoa (1986), and Kant and co-workers (1982, 1988).

In the stress-based theories, the stresses are expanded as a linear combination of z and unknown functions of the inplane coordinates. A stress-based formulation was first presented by Reissner (1944, 1945, 1947). He assumed linear variation of the inplane stresses through the thickness. The transverse stresses were obtained from the equations of equilibrium, which were in turn, derived using the principle of virtual forces. Other stress formulations are due to Panc (1964, 1975), and Kromm (1953, 1955). Gol'denveizer (1958, 1962) generalized Reissner's theory by replacing the linear distribution of stresses through the thickness by a distribution represented by an arbitrary function. Salerno and Goldberg (1960), and Voyiadjis and Baluch (1981, 1988) also presented theories which were modifications or extensions of Reissner's theory to composite plates. Pryor *et al.* (1970) conducted a finite element analysis of Reissner's theory.

Although rarely used, a mixed formulation where both assumed displacement and assumed stress expansions are used was presented by Reissner (1961, 1976). The governing equations are obtained using the mixed variational principle. Other works in this area are by Pagano (1978), Reissner (1984, 1986), and Putcha and Reddy (1986).

1.2.2. Layerwise theories

In the ESL theories, due to the single displacement expansions through the thickness of the laminate, the transverse strains are continuous through the thickness. When the laminate is made of layers of dissimilar materials, the ESL theories do not give very good results. The same is true when it comes to accurately modeling local phenomena like delamination. In such cases, piecewise stress/displacement approximations in the thickness direction have been used in some theories. They are called layerwise theories, or are sometimes referred to in literature as discrete layer theories; see Reddy (1997). The layerwise plate theory expanded as a linear combination of known layerwise continuous functions of the thickness coordinate $\phi_j^i(z)$, and undetermined functions $U_i^j(x, y)$:

$$u_i(x, y, z) = u_i^0(x, y) + \sum_{j=1}^{N_i} U_i^j(x, y)\, \phi_j^i(z) \qquad (i = 1, 2, 3) \tag{4}$$

where N_i is the number of subdivisions (hypothetical layers) through the thickness. The approximation in Eq. (4) can be interpreted as the global semi-discrete finite element approximation of u_is through the thickness. In that case, ϕ_j^i denote the global interpolation functions (they are piecewise continuous functions, defined only on two adjacent layers), and U_i^j are the nodal values. On appropriate selection of the value of N_i, a lamina can have more than one element, or several laminae can be included in one element.

A number of other layerwise theories have been proposed. Yu (1959) and Durocher and Solecki (1975) considered the case of a three-layer plate. Mau (1973) used the first order theory with layerwise generalized displacements. Rehfield and his colleagues (1982, 1983) advanced a layer-wise assumed stress theory, where the layer-wise stresses are expressed in terms of the stress and moment resultants using CLPT. Murakami (1986), and Toledano and Murakami (1987) based their work on Reissner's mixed variational principle. Legendre polynomials were used for the stress and displacement expansions. Seide (1980) also derived a theory for layer-wise linear displacements. Librescu (1975, 1976) presented a multilayer shell theory which included geometric nonlinearity. Pryor and Baker (1971) presented a finite element model based on linear functions on each layer. Hinrichsen and Palazotto (1986) used an idea similar to Reddy's GLPT (1984), but used cubic spline functions to account for the thickness variations. Other publications which fall into the category of discrete layer theories are due to Epstein and co-workers (1977, 1983) and Owen and Li (1987).

1.2.3. Continuum based theories

A 3-D elasticity solution for simply-supported homogeneous isotropic plates was presented by Vlasov (1957). Later, with the development of composite materials, there was increased interest in the accurate prediction of the response characteristics of composite plates. Analytical solutions to the bending, free vibration, and stability of laminated plates were presented by Srinivas *et al.* (1966, 1970, 1970, 1973), Pagano (1969, 1970), Lee (1967), Jones (1970), Pagano and Hatfield (1972), and Seide (1975). Lee and Reismann (1969) studied the dynamic response of rectangular plates. Noor and Burton (1989) and Savoia and Reddy (1992) presented analytical solutions for bending and free vibration problems of rectangular multilayered anisotropic plates. Finite element three-dimensional models were developed by Putcha and Reddy (1982), and Liou and Sun (1987). The finite element modeling of 3-D theories is cumbersome, especially if one is dealing with transient or nonlinear problems. However, the analytical 3-D elasticity solutions are useful for the purpose of comparison with results obtained from other analyses performed using less rigorous theories.

1.3. The present study

In the present study, we shall develop a unified third order theory and evaluate the performance of some displacement based ESL theories in predicting deflections, stresses, and natural frequencies in laminated composite plates. The theories considered here are the classical laminated plate theory (CLPT), first order shear deformation theory (FSDT), the special third order theory of Reddy (STTR), the general third order theory of Reddy based on (1990, 1995) which includes transverse normal stresses (GTTR), and the general third order theory proposed by Lo *et al.* (GTOT); GTTR can be obtained from GTOT by imposing the condition that the transverse shear stresses by zero at the top and bottom planes of the laminated plate. In this study, all commonly used assumed displacement, single layer theories are unified as special cases of the general third order theory of Reddy by the introduction of so-called "tracers". By assigning different values to these tracers (0 or 1), one can obtain the displacement field for either GTTR, STTR, FSDT, or CLPT. While this kind of unified kinematical relations have been presented before (1990, 1995), what is novel about the present study is that the equations of motion, analytical solutions, and finite element models have been derived for the comprehensive unified theory.

2. Governing equations

2.1. Introduction

In this section, the kinematical relations, constitutive equations and the governing equations of motion are presented. As stated in the Section 1, an effort is made to develop a comprehensive theory, which encompasses all the lower order theories. The lower order theories can be obtained from the unified theory by changing the values of the *tracers*.

2.2. Lamina constitutive equations

The linear constitutive equations for the k-th orthotropic lamina are given by

$$\begin{Bmatrix} \sigma_1' \\ \sigma_2' \\ \sigma_3' \\ \sigma_4' \\ \sigma_5' \\ \sigma_6' \end{Bmatrix} = \begin{bmatrix} C_{11}' & C_{12}' & C_{13}' & 0 & 0 & 0 \\ C_{11}' & C_{12}' & C_{13}' & 0 & 0 & 0 \\ C_{13}' & C_{23}' & C_{13}' & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44}' & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{55}' & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{66}' \end{bmatrix} \begin{Bmatrix} \varepsilon_1' \\ \varepsilon_2' \\ \varepsilon_3' \\ \varepsilon_4' \\ \varepsilon_5' \\ \varepsilon_6' \end{Bmatrix} \tag{5}$$

In the above equation, the coordinate system is the principal material coordinate system. However, in general, the coordinate system of a problem will not coincide with the material coordinate system. Hence it is necessary to determine the elastic properties with respect to the (x, y) (problem) coordinate system when the elastic stiffnesses are known relative to the (x', y') (material) coordinate system.

In the laminate coordinates the stress-strain relations can be written in matrix form as

$$\begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ C_{11} & C_{12} & C_{13} & 0 & 0 & C_{26} \\ C_{13} & C_{23} & C_{13} & 0 & 0 & C_{36} \\ 0 & 0 & 0 & C_{44} & C_{45} & 0 \\ 0 & 0 & 0 & C_{45} & C_{55} & 0 \\ C_{16} & C_{26} & C_{36} & 0 & 0 & C_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_1 \\ \varepsilon_2 \\ \varepsilon_3 \\ \varepsilon_4 \\ \varepsilon_5 \\ \varepsilon_6 \end{Bmatrix} \tag{6}$$

where the components C_{ij} of matrix $[C]$ are given by ($c=\cos\theta$, $s=\sin\theta$)

$$
\begin{aligned}
&C_{11}=C'_{11}c^4+2c^2s^2(C'_{12}+2C'_{66})+C'_{22}s^4\\
&C_{12}=c^2s^2(C'_{11}+C'_{22}-4C'_{66})+C'_{12}(c^4+s^4)\\
&C_{13}=C'_{13}c^2+C'_{23}s^2\\
&C_{16}=cs\,[C'_{11}c^2-C'_{22}s^2-(C'_{12}+2C'_{66})(c^2-s^2)]\\
&C_{22}=C'_{11}s^4+2c^2s^2(C'_{12}+2C'_{66})+C'_{22}c^4\\
&C_{23}=C'_{13}s^2+C'_{23}c^2\\
&C_{26}=cs\,[C'_{11}s^2-C'_{22}c^2+(C'_{12}+2C'_{66})(c^2-s^2)]\\
&C_{33}=C'_{33},\quad C_{36}=cs\,(C'_{13}-C'_{23}),\quad C_{44}=C'_{44}c^2+C'_{55}\,s^2\\
&C_{45}=cs\,(C'_{44}-C'_{55}),\quad C_{55}=C'_{44}\,s^2+C'_{55}\,c^2\\
&C_{66}=c^2s^2(C'_{11}+C'_{22}-2C'_{2})+(C)'_{66}(c^2-s^2)^2
\end{aligned}
$$

A plane stress state is one in which the out-of-plane stresses are neglected. For a lamina in the (x, y) plane, this is defined by setting

$$\sigma_z=0 \quad \tau_{yz}=0 \quad \tau_{xz}=0 \tag{8}$$

For a lamina of constant thickness and made of an orthotropic material, the plane stress constitutive equations in the laminate coordinates are

$$\begin{Bmatrix}\sigma_1\\ \sigma_2\\ \sigma_6\end{Bmatrix}=\begin{bmatrix}Q_{11} & Q_{12} & Q_{16}\\ Q_{12} & Q_{22} & Q_{26}\\ Q_{16} & Q_{26} & Q_{66}\end{bmatrix}\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_6\end{Bmatrix} \tag{9}$$

and

$$\begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix}=\begin{bmatrix}Q_{44} & Q_{45}\\ Q_{45} & Q_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_4\\ \varepsilon_5\end{Bmatrix} \tag{10}$$

where Q_{ij} are the transformed reduced stiffnesses. The relations between the Q_{ij}s and the Q'_{ij}s are exactly the same as the relations between their counterpart C_{ij}s and C'_{ij}s in the 3-D stress state.

2.3. Kinematical relations

2.3.1. General third-order theory

We consider a general third-order laminated plate theory in which the inplane displacements (u_1, u_2) are expanded upto the cubic term in the thickness term z, and the transverse displacement u_3 is expanded upto the quadratic term in z. This is done to take into account the parabolic variation of the transverse shear stresses through the thickness of the plate.

$$
\begin{aligned}
u_1(x,y,z,t)&=u(x,y,t)+z\,\phi_1(x,y,t)+z^2\psi_1(x,y,t)+z^3\theta_1(x,y,t)\\
u_2(x,y,z,t)&=v(x,y,t)+z\,\phi_2(x,y,t)+z^2\psi_2(x,y,t)+z^3\theta_2(x,y,t)
\end{aligned}
$$

$$u_3(x,y,z,t)=w(x,y,t)+z\,\psi_3(x,y,t)+z^2\theta_3(x,y,t) \tag{11}$$

where u, v, w, ϕ_1, ϕ_2, ψ_1, ψ_2, ψ_3, θ_1, θ_2, θ_3 are unknown. Here (u, v, w) denote the displacements of a point on the mid-surface of the plate (z=0). ϕ_x and ϕ_y are the rotations of the transverse normal in the xz and yz planes. The term ψ_3 can be interpreted as the stretching of the transverse normal. For the remaining higher-order terms, a physical interpretation is not immediately apparent, but can be taken to the higher-order rotations or extensions. Lo *et al.* carried out an analytical investigation on the behavior of elastic plates based on the general third-order theory given above.

A number of special cases can be derived from the above theory:

- If we assume that the transverse normals are inextensible (ψ_3=0, θ_3=0), the number of unknowns reduce to nine. In this case, the plane stress assumption may be used.
- A third-order theory in which the second-order terms are neglected (ψ_1=0, ψ_2=0, ψ_3=0). The number of unknown functions are eight.
- A second-order theory in which θ_1 and θ_2 are set to zero. In this case, there are nine unknowns. An extension of this is when, additionally, θ_3 is set to zero. If we incorporate the transverse normal inextensibility condition, then ψ_3=0, and the number of unknowns are seven. For this case also, the plane stress reduced stiffnesses can be used.
- A theory in which the inplane displacements (u_1, u_2) are expanded upto the linear power of z and the transverse displacement (u_3) is expanded upto the quadratic power of z.
- A class of theories in which the transverse shear stresses (σ_{yz}, σ_{xz}) are made to vanish on the top and bottom of the plate ($z=\pm h/2$). This class of theories will be discussed in a susbsequent section.

The linear strain field for the general third order theory is obtained from the strain-displacement relations as

$$\begin{Bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{Bmatrix} = \begin{Bmatrix}\varepsilon_1^{(0)}\\ \varepsilon_2^{(0)}\\ \varepsilon_3^{(0)}\\ \varepsilon_4^{(0)}\\ \varepsilon_5^{(0)}\\ \varepsilon_6^{(0)}\end{Bmatrix} + z\begin{Bmatrix}\varepsilon_1^{(1)}\\ \varepsilon_2^{(1)}\\ \varepsilon_3^{(1)}\\ \varepsilon_4^{(1)}\\ \varepsilon_5^{(1)}\\ \varepsilon_6^{(1)}\end{Bmatrix} + z^2\begin{Bmatrix}\varepsilon_1^{(2)}\\ \varepsilon_2^{(2)}\\ \varepsilon_3^{(2)}\\ \varepsilon_4^{(2)}\\ \varepsilon_5^{(2)}\\ \varepsilon_6^{(2)}\end{Bmatrix} + z^3\begin{Bmatrix}\varepsilon_1^{(3)}\\ \varepsilon_2^{(3)}\\ \varepsilon_3^{(3)}\\ \varepsilon_4^{(3)}\\ \varepsilon_5^{(3)}\\ \varepsilon_6^{(3)}\end{Bmatrix} \tag{12}$$

where

$$\varepsilon_1^{(0)}=\frac{\partial u}{\partial x},\quad \varepsilon_1^{(1)}=\frac{\partial \phi_1}{\partial x},\quad \varepsilon_1^{(2)}=\frac{\partial \psi_1}{\partial x},\quad \varepsilon_1^{(3)}=\frac{\partial \theta_1}{\partial x}$$

$$\varepsilon_2^{(0)}=\frac{\partial v}{\partial y},\quad \varepsilon_2^{(1)}=\frac{\partial \phi_2}{\partial y},\quad \varepsilon_2^{(2)}=\frac{\partial \psi_2}{\partial y},\quad \varepsilon_2^{(3)}=\frac{\partial \theta_2}{\partial y}$$

$$\varepsilon_3^{(0)}=\psi_3,\quad \varepsilon_3^{(1)}=2\theta_3,\quad \varepsilon_3^{(2)}=0,\quad \varepsilon_3^{(3)}=0$$

$$\varepsilon_4^{(0)}=\phi_2+\frac{\partial w}{\partial y},\quad \varepsilon_4^{(1)}=2\psi_2+\frac{\partial \psi_3}{\partial y},\quad \varepsilon_4^{(2)}=3\theta_2+\frac{\partial \theta_3}{\partial y},\quad \varepsilon_4^{(3)}=0$$

$$\varepsilon_5^{(0)}=\phi_1+\frac{\partial w}{\partial x},\quad \varepsilon_5^{(1)}=2\psi_1+\frac{\partial \psi_3}{\partial x},\quad \varepsilon_5^{(2)}=3\theta_1+\frac{\partial \theta_3}{\partial x},\quad \varepsilon_5^{(3)}=0$$

$$\varepsilon_6^{(0)}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x},\quad \varepsilon_6^{(1)}=\frac{\partial \phi_1}{\partial y}+\frac{\partial \phi_2}{\partial x},\quad \varepsilon_6^{(2)}=\frac{\partial \psi_1}{\partial y}+\frac{\partial \psi_2}{\partial x},\quad \varepsilon_6^{(3)}=\frac{\partial \theta_1}{\partial y}+\frac{\partial \theta_2}{\partial x} \tag{13}$$

2.3.2. General third-order theory of Reddy

The displacement field for the general third order theory of Reddy (GTTR) will be derived in this section. In a particular lamina, the transverse shear stresses are given by (see Reddy 1990, 1995).

$$\begin{Bmatrix}\sigma_4\\ \sigma_5\end{Bmatrix}=\begin{bmatrix}Q_{44} & Q_{45}\\ Q_{45} & Q_{55}\end{bmatrix}\begin{Bmatrix}\varepsilon_4\\ \varepsilon_5\end{Bmatrix} \tag{14}$$

If the transverse shear stresses σ_4 and σ_5 are to vanish at the bounding planes of the plate (at $z=\pm h/2$), the transverse shear strains ε_4 and ε_5 should also vanish there. That is

$$\varepsilon_4\left(x,y,\pm\frac{h}{2}\right)=\varepsilon_5\left(x,y,\pm\frac{h}{2}\right)=0 \tag{15}$$

which imply the following conditions

$$\begin{aligned}\psi_1&=-\frac{1}{2}\frac{\partial \psi_3}{\partial x},\quad \theta_1=-\frac{1}{3}\left[\frac{\partial \theta_3}{\partial x}+\frac{4}{h^2}\left(\phi_1+\frac{\partial w}{\partial x}\right)\right]\\ \psi_2&=-\frac{1}{2}\frac{\partial \psi_3}{\partial y},\quad \theta_2=-\frac{1}{3}\left[\frac{\partial \theta_3}{\partial y}+\frac{4}{h^2}\left(\phi_2+\frac{\partial w}{\partial y}\right)\right]\end{aligned} \tag{16}$$

Putting the above conditions in Eq. (14) leads to the following displacement field

$$\begin{aligned}u_1&=u+z\phi_1+z^2\left(-\frac{1}{2}\frac{\partial \psi_3}{\partial x}\right)+z^3\left[-\frac{1}{3}\left\{\frac{\partial \theta_3}{\partial x}+\frac{4}{h^2}\left(\phi_1+\frac{\partial w}{\partial x}\right)\right\}\right]\\ u_2&=v+z\phi_2+z^2\left(-\frac{1}{2}\frac{\partial \psi_3}{\partial y}\right)+z^3\left[-\frac{1}{3}\left\{\frac{\partial \theta_3}{\partial y}+\frac{4}{h^2}\left(\phi_2+\frac{\partial w}{\partial y}\right)\right\}\right]\\ u_3&=w+z\psi_3+z^2\theta_3\end{aligned} \tag{17}$$

In order to simplify the analysis, the following substitution is made:

$$\zeta=\theta_3+\frac{4}{h^2}w \tag{18}$$

The introduction of ζ prevents the occurence of redundant boundary conditions in the equations of motion. It also simplifies the finite element modeling using this theory.

The number of variables requiring C^1 continuity is reduced from three to two. Also, at this point, we introduce some parameters called tracers into the displacement field of Eq. (17). As these tracers assume different values, the displacement field reduces to different special cases.

$$u_1 = u + z\left[1 - \chi\frac{4}{3}\left(\frac{z}{h}\right)^2\right]\phi_1 + \lambda z^2\left(-\frac{1}{2}\frac{\partial\psi_3}{\partial x}\right) + \gamma z^3\left(-\frac{1}{3}\frac{\partial\zeta}{\partial x}\right)$$
$$u_2 = v + z\left[1 - \chi\frac{4}{3}\left(\frac{z}{h}\right)^2\right]\phi_2 + \lambda z^2\left(-\frac{1}{2}\frac{\partial\psi_3}{\partial y}\right) + \gamma z^3\left(-\frac{1}{3}\frac{\partial\zeta}{\partial y}\right)$$
$$u_3 = w + z\,\psi_3 + z^2(\zeta - c_1 w) \tag{19}$$

where $c_1=4/h^2$, and χ, λ and γ are tracers. The different theories that can be obtained by assigning different values to the tracers are

- Classical Theory (CLPT): $\chi=\lambda=\gamma=0$, $\phi_1=-\partial w/\partial x$, $\phi_2=-\partial w/\partial y$
- First Order Shear Deformation Theory (FSDT): $\chi=\lambda=\gamma=0$
- Higher Order Shear Deformation Theory (STTR): $\chi=1$, $\lambda=0$, $\gamma=1$, $\zeta=c_1 w$
- Third Order Theory which has provision for transverse normal stresses in its kinematics (GTTR): $\chi=\lambda=\gamma=1$

The strain field associated with the displacement field in Eq. (17) is of the same form as in Eq. (16) with the strain components

$$\varepsilon_1^{(0)} = \frac{\partial u}{\partial x},\quad \varepsilon_1^{(1)} = \frac{\partial\phi_1}{\partial x},\quad \varepsilon_1^{(2)} = -\frac{\lambda}{2}\frac{\partial^2\psi_3}{\partial x^2},\quad \varepsilon_1^{(3)} = -\frac{1}{3}\left(\chi c_1\frac{\partial\phi_1}{\partial x} + \gamma\frac{\partial^2\zeta}{\partial x^2}\right)$$
$$\varepsilon_2^{(0)} = \frac{\partial v}{\partial y},\quad \varepsilon_2^{(1)} = \frac{\partial\phi_2}{\partial y},\quad \varepsilon_2^{(2)} = -\frac{\lambda}{2}\frac{\partial^2\psi_3}{\partial y^2},\quad \varepsilon_2^{(3)} = -\frac{1}{3}\left(\chi c_1\frac{\partial\phi_2}{\partial y} + \gamma\frac{\partial^2\zeta}{\partial y^2}\right)$$
$$\varepsilon_3^{(0)} = \lambda\psi_3,\quad \varepsilon_3^{(1)} = 2\gamma(\zeta - c_1 w),\quad \varepsilon_3^{(2)} = 0,\quad \varepsilon_3^{(3)} = 0$$
$$\varepsilon_4^{(0)} = \phi_2 + \frac{\partial w}{\partial y},\quad \varepsilon_4^{(1)} = 0,\quad \varepsilon_4^{(2)} = -c_1\left(\chi\phi_2 + \gamma\frac{\partial w}{\partial y}\right),\quad \varepsilon_4^{(3)} = 0$$
$$\varepsilon_5^{(0)} = \phi_1 + \frac{\partial w}{\partial x},\quad \varepsilon_5^{(1)} = 0,\quad \varepsilon_5^{(2)} = -c_1\left(\chi\phi_1 + \gamma\frac{\partial w}{\partial x}\right),\quad \varepsilon_5^{(3)} = 0$$
$$\varepsilon_6^{(0)} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x},\quad \varepsilon_6^{(1)} = \frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x},\quad \varepsilon_6^{(2)} = -\lambda\frac{\partial^2\psi_3}{\partial x\,\partial y},$$
$$\varepsilon_6^{(3)} = -\frac{1}{3}\left[\chi c_1\left(\frac{\partial\phi_1}{\partial y} + \frac{\partial\phi_2}{\partial x}\right) + 2\gamma\frac{\partial^2\zeta}{\partial x\,\partial y}\right] \tag{20}$$

2.4. Laminate constitutive equations

First we define the overall laminate stiffnesses A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij} and H_{ij}.

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}) + \int_{-h/2}^{h/2} C_{ij}^{(k)}\,(1, z, z^2, z^3, z^4, z^5, z^6)\,dz$$
$$(i, j = 1, 2, 3, 4, 5, 6) \tag{21}$$

If we write the A_{ij}, B_{ij}, etc., in terms of the ply stiffnesses $C_{ij}^{(k)}$ and the ply coordinates z_k and z_{k+1}, we have

$$(A_{ij}, B_{ij}, D_{ij}, E_{ij}, F_{ij}, G_{ij}, H_{ij}) = \frac{1}{n} \sum_{k=1}^{N} C_{ij}^{(k)}(z_{k+1}^{n} - z_{k}^{n}) \quad (n = 1, 2, 3, 4, 5, 6, 7) \tag{22}$$

For the plane stress case, we substitute the $C_{ij}^{(k)}$s in the above equations with the plane stress reduced stiffnesses $Q_{ij}^{(k)}$. Next, the stress resultants N_i, M_i, P_i, S_i are defined.

$$(N_i, M_i, P_i, S_i) = \int_{-h/2}^{h/2} \sigma_i (1, z, z^2, z^3)\, dz = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \sigma_i^{(k)} (1, z, z^2, z^3)\, dz \tag{23}$$

If we write Eq. (12) as

$$\{\varepsilon\} = \{\varepsilon^{(0)}\} + z\{\varepsilon^{(1)}\} + z^2\{\varepsilon^{(2)}\} + z^3\{\varepsilon^{(3)}\} \tag{24}$$

then from the constitutive relations and the above equation, the stress resultants can be written in compact form as

$$\begin{Bmatrix} \{N\} \\ \{M\} \\ \{P\} \\ \{S\} \end{Bmatrix} = \begin{bmatrix} [A] & [B] & [D] & [E] \\ [B] & [D] & [E] & [F] \\ [D] & [E] & [F] & [G] \\ [E] & [F] & [G] & [H] \end{bmatrix} \begin{Bmatrix} \{\varepsilon^{(0)}\} \\ \{\varepsilon^{(1)}\} \\ \{\varepsilon^{(2)}\} \\ \{\varepsilon^{(3)}\} \end{Bmatrix} \tag{25}$$

For cross-ply laminates, the plate stiffnesses can be simplified. The 16, 26, 36, and 45 terms are zero. Also, for symmetric laminates, $B_{ij}=E_{ij}=G_{ij}=0$, and for antisymmetric cross-ply laminates, $B_{12}=B_{66}=E_{12}=E_{66}=G_{12}=G_{66}=0$.

We introduce the following plate inertias:

$$(I_1, I_2, I_3, I_4, I_5, I_6, I_7) = \sum_{k=1}^{N} \int_{z_k}^{z_{k+1}} \rho^{(k)}(1, z, z^2, z^3, z^4, z^5, z^6)\, dz \tag{26}$$

If the material for all the layers are identical, that is if the density $\rho^{(k)}$ is the same for all k, then we have

$$I_2 = I_4 = I_6 = 0$$

$$I_1 = \rho h, \quad I_3 = \frac{1}{12} \rho h^3, \quad I_5 = \frac{1}{80} \rho h^5, \quad I_3 = \frac{1}{448} \rho h^7 \tag{27}$$

2.5. Equations of motion

The Hamilton's principle for an elastic body is

$$\int_{t_1}^{t_2} (\delta U + \delta V - \delta K)\, dt = 0 \tag{28}$$

where δU is the virtual strain energy, δV is the virtual work done by external forces, and δK is the virtual kinetic energy:

$$\delta U = \int_{\Omega} \int_{-h/2}^{h/2} (\sigma_i\, \delta\varepsilon_i)\, dz\, dx\, dy$$

$$=\int_{\Omega}(N_i\,\delta\varepsilon_i^{(0)}+M_i\,\delta\varepsilon_i^{(1)}+P_i\,\delta\varepsilon_i^{(2)}+S_i\,\delta\varepsilon_i^{(3)})\,dz\,dx\,dy \tag{29}$$

$$\delta V=-\int_{\Omega}[q(x,y)\,\delta u_3]\,dx\,dy \tag{30}$$

$$\delta K=\int_{\Omega}\int_{-h/2}^{h/2}\rho(\dot{u}_j\,\delta\dot{u}_j)\,dz\,dx\,dy \qquad i=1,2,...,6.\;\; j=1,2,3. \tag{31}$$

For the general third order theory, we obtain the following equations of motion from (28):

$$\begin{aligned}
\delta u&:\ \frac{\partial N_1}{\partial x}+\frac{\partial N_6}{\partial y}=I_1\ddot{u}+I_2\ddot{\phi}_1+I_3\ddot{\psi}_1+I_4\ddot{\theta}_1\\
\delta v&:\ \frac{\partial N_6}{\partial x}+\frac{\partial N_2}{\partial y}=I_1\ddot{v}+I_2\ddot{\phi}_2+I_3\ddot{\psi}_2+I_4\ddot{\theta}_2\\
\delta w&:\ \frac{\partial N_5}{\partial x}+\frac{\partial N_4}{\partial y}+q=I_1\ddot{w}+I_2\ddot{\psi}_3+I_3\ddot{\theta}_3\\
\delta\phi_1&:\ \frac{\partial M_1}{\partial x}+\frac{\partial M_6}{\partial y}-N_5=I_2\ddot{u}+I_3\ddot{\phi}_1+I_4\ddot{\psi}_1+I_5\ddot{\theta}_1\\
\delta\phi_2&:\ \frac{\partial M_6}{\partial x}+\frac{\partial M_2}{\partial y}-N_4=I_2\ddot{v}+I_3\ddot{\phi}_2+I_4\ddot{\psi}_2+I_5\ddot{\theta}_2\\
\delta\psi_1&:\ \frac{\partial P_1}{\partial x}+\frac{\partial P_6}{\partial y}-2M_5=I_3\ddot{u}+I_4\ddot{\phi}_1+I_5\ddot{\psi}_1+I_6\ddot{\theta}_1\\
\delta\psi_2&:\ \frac{\partial P_6}{\partial x}+\frac{\partial P_2}{\partial y}-2M_4=I_3\ddot{u}+I_4\ddot{\phi}_2+I_5\ddot{\psi}_2+I_6\ddot{\theta}_2\\
\delta\psi_3&:\ \frac{\partial M_5}{\partial x}+\frac{\partial M_4}{\partial y}-N_3+q_1=I_2\ddot{w}+I_3\ddot{\psi}_3+I_4\ddot{\theta}_3\\
\delta\theta_1&:\ \frac{\partial S_1}{\partial x}+\frac{\partial S_6}{\partial y}-3P_5=I_4\ddot{u}+I_5\ddot{\phi}_1+I_6\ddot{\psi}_1+I_7\ddot{\theta}_1\\
\delta\theta_2&:\ \frac{\partial S_6}{\partial x}+\frac{\partial S_2}{\partial y}-3P_4=I_4\ddot{v}+I_5\ddot{\phi}_2+I_6\ddot{\psi}_2+I_7\ddot{\theta}_2\\
\delta\theta_3&:\ \frac{\partial P_5}{\partial x}+\frac{\partial P_4}{\partial y}-2M_3+q_2=I_3\ddot{w}+I_4\ddot{\psi}_3+I_5\ddot{\theta}_3
\end{aligned} \tag{32}$$

where q is the transverse load on the top surface and $q_1=qh/2$, $q_2=qh^2/4$.
The primary variables (i.e., generalized displacements) and secondary variables (i.e., generalized forces) of the theory are

$$\begin{aligned}
&\text{primary variables: } u_n,\ u_s,\ w,\ \phi_n,\ \phi_s,\ \psi_n,\ \psi_s,\ \psi_z,\ \theta_n,\ \theta_s,\ \theta_z\\
&\text{secondary variables: } N_n,\ N_{ns},\ N_z,\ M_n,\ M_{ns},\ P_n,\ P_{ns},\ M_z,\ S_n,\ S_{ns},\ P_z
\end{aligned} \tag{33}$$

Proceeding in a similar manner, we use Eq. (19) and Eq. (20) in Hamilton's principle to get the equations of motion for the special third order theory.

$$\begin{aligned}
\delta u&:\ \frac{\partial N_1}{\partial x}+\frac{\partial N_6}{\partial y}=I_1\ddot{u}+\bar{I}_2\ddot{\phi}_1-\frac{\lambda}{2}I_3\frac{\partial\ddot{\psi}_3}{\partial x}-\frac{\gamma}{3}I_4\frac{\partial\ddot{\zeta}}{\partial x}\\
\delta v&:\ \frac{\partial N_6}{\partial x}+\frac{\partial N_2}{\partial y}=I_1\ddot{v}+\bar{I}_2\ddot{\phi}_2-\frac{\lambda}{2}I_3\frac{\partial\ddot{\psi}_3}{\partial y}-\frac{\gamma}{3}I_4\frac{\partial\ddot{\zeta}}{\partial y}
\end{aligned}$$

$$\delta w:\ \left(\frac{\partial N_5}{\partial x}+\frac{\partial N_4}{\partial y}\right)+2\gamma c_1M_3-\gamma c_1\left(\frac{\partial P_5}{\partial x}+\frac{\partial P_4}{\partial y}\right)+q=\bar{I}_1\ddot{w}+\lambda\tilde{I}_2\ddot{\psi}_3+\gamma\tilde{I}_3\ddot{\zeta}$$

$$\delta\phi_1:\ \left(\frac{\partial M_1}{\partial x}+\frac{\partial M_6}{\partial y}\right)-\chi c_2\left(\frac{\partial S_1}{\partial x}+\frac{\partial S_6}{\partial y}\right)-(N_5-\chi c_1P_5)=\bar{I}_2\ddot{u}+\bar{I}_3\ddot{\phi}_1-\frac{\lambda}{2}\bar{I}_4\frac{\partial\ddot{\psi}_3}{\partial x}-\frac{\gamma}{3}\bar{I}_5\frac{\partial\ddot{\zeta}}{\partial x}$$

$$\delta\phi_2:\ \left(\frac{\partial M_6}{\partial x}+\frac{\partial M_2}{\partial y}\right)-\chi c_2\left(\frac{\partial S_6}{\partial x}+\frac{\partial S_2}{\partial y}\right)-(N_4-\chi c_1P_4)=\bar{I}_2\ddot{v}+\bar{I}_3\ddot{\phi}_2-\frac{\lambda}{2}\bar{I}_4\frac{\partial\ddot{\psi}_3}{\partial y}-\frac{\gamma}{3}\bar{I}_5\frac{\partial\ddot{\zeta}}{\partial y}$$

$$\delta\psi_3:\ \frac{\lambda}{2}\left[\left(\frac{\partial^2P_1}{\partial x^2}+2\frac{\partial^2P_6}{\partial x\partial y}+\frac{\partial^2P_2}{\partial y^2}\right)-2N_3\right]+q_1=\frac{\lambda}{2}I_3\left(\frac{\partial\ddot{u}}{\partial x}+\frac{\partial\ddot{v}}{\partial y}\right)+\frac{\lambda}{2}\bar{I}_4\left(\frac{\partial\ddot{\phi}_1}{\partial x}+\frac{\partial\ddot{\phi}_2}{\partial y}\right)$$
$$+\lambda\tilde{I}_2\ddot{w}+\lambda^2I_3\ddot{\psi}_3-\frac{\lambda^2}{4}I_5\left(\frac{\partial^2\ddot{\psi}_3}{\partial x^2}+\frac{\partial^2\ddot{\psi}_3}{\partial y^2}\right)+\lambda\gamma I_4\ddot{\zeta}-\frac{\lambda\gamma}{6}I_6\left(\frac{\partial^2\ddot{\zeta}}{\partial x^2}+\frac{\partial^2\ddot{\zeta}}{\partial y^2}\right)$$

$$\delta\zeta:\ \frac{\gamma}{3}\left[\left(\frac{\partial^2S_1}{\partial x^2}+2\frac{\partial^2S_6}{\partial x\partial y}+\frac{\partial^2S_2}{\partial y^2}\right)-6M_3\right]+q_2=\frac{\gamma}{3}I_4\left(\frac{\partial\ddot{u}}{\partial x}+\frac{\partial\ddot{v}}{\partial y}\right)+\frac{\gamma}{3}\bar{I}_5\left(\frac{\partial\ddot{\phi}_1}{\partial x}+\frac{\partial\ddot{\phi}_2}{\partial y}\right)$$
$$+\gamma\tilde{I}_3\ddot{w}+\lambda\gamma I_4\ddot{\psi}_3-\frac{\lambda\gamma}{6}I_6\left(\frac{\partial^2\ddot{\psi}_3}{\partial x^2}+\frac{\partial^2\ddot{\psi}_3}{\partial y^2}\right)+\gamma^2I_5\ddot{\zeta}-\frac{\gamma^2}{9}I_7\left(\frac{\partial^2\ddot{\zeta}}{\partial x^2}+\frac{\partial^2\ddot{\zeta}}{\partial y^2}\right)\tag{34}$$

where

$$\begin{gathered}\bar{I}_1=I_1-2\gamma c_1I_3+\gamma^2c_2I_5,\quad \bar{I}_2=I_2-\chi c_2I_4\\ \tilde{I}_2=I_2-\gamma c_1I_4,\quad \bar{I}_3=I_3-2\chi c_2I_5+\chi^2d_2I_7\\ \tilde{I}_3=I_3-\gamma c_1I_5,\quad \bar{I}_4=I_4-\chi c_2I_6,\quad \bar{I}_5=I_5-\chi c_2I_7\\ c_1=4/h^2,\quad c_2=4/3h^2,\quad d_1=c_1^2,\quad d_2=c_2^2\end{gathered}\tag{35}$$

The primary variables and secondary variables of the theory are

$$\text{primary variables:}\quad u_n,\ u_s,\ w,\ \phi_n,\ \phi_s,\ \psi_3,\ \frac{\partial\psi_3}{\partial n},\ \zeta,\ \frac{\partial\zeta}{\partial n}$$
$$\text{secondary variables:}\quad N_n,\ N_{ns},\ N_z,\ M_n,\ M_{ns},\ Q_z,\ P_n,\ R_z,\ S_n\tag{36}$$

3. Analytical solutions

3.1. Introduction

In this section, we develop the exact solutions of various theories for rectangular plates. An exact solution satisfies the governing equations of the problem at every point of the domain and the boundary conditions. Both the Navier and Lévy methods assume solutions in the form of an infinite trigonometric series. In both cases, however, the series can be truncated after a few terms to get a sufficiently accurate solution.

Since we shall find out analytical solutions for the special third order theory and its particular derivative cases, we write the equations of motion (2.37)-(2.43) in terms of the displacements. We shall consider the case of symmetric and anti-symmetric cross ply plates in which all the laminae are of the same material. The equations of motion in terms of generalized displacements are:

$\underline{\delta u}$:

$$A_{11}\frac{\partial^2 u}{\partial x^2}+A_{12}\frac{\partial^2 v}{\partial x\partial y}+A_{13}\lambda\frac{\partial \psi_3}{\partial x}+B_{11}\frac{\partial^2\phi_1}{\partial x^2}+2B_{13}\gamma\left(\frac{\partial\zeta}{\partial x}-c_1\frac{\partial w}{\partial x}\right)-\frac{1}{2}\lambda D_{11}\frac{\partial^3\psi_3}{\partial x^3}$$
$$-\frac{1}{2}\lambda D_{12}\frac{\partial^3\psi_3}{\partial x\partial y^2}-\frac{1}{3}E_{11}\left(\chi c_1\frac{\partial^2\phi_1}{\partial x^2}+\gamma\frac{\partial^3\zeta}{\partial x^3}\right)+A_{66}\left(\frac{\partial^2 u}{\partial y^2}+\frac{\partial^2 v}{\partial x\partial y}\right)-\lambda D_{66}\frac{\partial^3\psi_3}{\partial x\partial y^2}$$
$$=I_1\ddot{u}-\frac{1}{2}\lambda I_3\frac{\partial\ddot{\psi}_3}{\partial x}\qquad(37)$$

$\underline{\delta v}$:

$$A_{66}\left(\frac{\partial^2 u}{\partial x\partial y}+\frac{\partial^2 v}{\partial x^2}\right)-\lambda D_{66}\frac{\partial^3\psi_3}{\partial x\partial y^2}+A_{12}\frac{\partial^2 u}{\partial x\partial y}+A_{22}\frac{\partial^2 v}{\partial y^2}+\lambda A_{23}\frac{\partial\psi_3}{\partial y}+B_{22}\frac{\partial^2\phi_2}{\partial y^2}$$
$$+2B_{23}\gamma\left(\frac{\partial\zeta}{\partial y}-c_1\frac{\partial w}{\partial y}\right)\frac{1}{2}\lambda D_{12}\frac{\partial^3\psi_3}{\partial x^2\partial y}-\frac{1}{2}\lambda D_{22}\frac{\partial^3\psi_3}{\partial y^3}-\frac{1}{3}E_{22}\left(\chi c_1\frac{\partial^2\phi_2}{\partial y^2}+\gamma\frac{\partial^3\zeta}{\partial y^3}\right)$$
$$=I_1\ddot{v}-\frac{1}{2}\lambda I_3\frac{\partial\ddot{\psi}_3}{\partial y}\qquad(38)$$

$\underline{\delta w}$:

$$A_{55}\left(\frac{\partial\phi_1}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)+A_{44}\left(\frac{\partial\phi_2}{\partial y}+\frac{\partial^2 w}{\partial y^2}\right)-c_1D_{55}\left(\chi\frac{\partial\phi_1}{\partial x}+\gamma\frac{\partial^2 w}{\partial x^2}\right)-c_1D_{44}\left(\chi\frac{\partial\phi_2}{\partial y}+\gamma\frac{\partial^2 w}{\partial y^2}\right)$$
$$-\gamma c_1D_{55}\left(\frac{\partial\phi_1}{\partial x}+\frac{\partial^2 w}{\partial x^2}\right)-\gamma c_1D_{44}\left(\frac{\partial\phi_2}{\partial y}+\gamma\frac{\partial^2 w}{\partial y^2}\right)+\gamma c_2F_{55}\left(\chi\frac{\partial\phi_1}{\partial x}+\gamma\frac{\partial^2 w}{\partial x^2}\right)$$
$$+\gamma c_2F_{44}\left(\chi\frac{\partial\phi_2}{\partial y}+\gamma\frac{\partial^2 w}{\partial y^2}\right)+2\gamma c_1\left[B_{13}\frac{\partial u}{\partial x}+B_{23}\frac{\partial v}{\partial y}+\lambda B_{33}\psi_3+D_{13}\frac{\partial\phi_1}{\partial x}+D_{23}\frac{\partial\phi_2}{\partial y}\right.$$
$$+2\gamma D_{33}(\zeta-c_1w)-\frac{1}{2}\lambda E_{13}\frac{\partial^2\phi_3}{\partial x^2}-\frac{1}{2}\lambda E_{23}\frac{\partial^2\psi_3}{\partial y^2}-\frac{1}{3}F_{13}\left(\chi c_1\frac{\partial\phi_1}{\partial x}+\gamma\frac{\partial^2\zeta}{\partial x^2}\right)$$
$$\left.-\frac{1}{3}F_{23}\left(\chi c_1\frac{\partial\phi_2}{\partial x}+\gamma\frac{\partial^2\zeta}{\partial y^2}\right)\right]+q=\bar{I}_1\ddot{w}+\gamma\tilde{I}_3\ddot{\zeta}\qquad(39)$$

$\underline{\delta\phi_1}$:

$$B_{11}\frac{\partial^2 u}{\partial x^2}+\lambda B_{13}\frac{\partial\psi_3}{\partial x}+D_{11}\frac{\partial^2\phi_1}{\partial x^2}+D_{12}\frac{\partial^2\phi_2}{\partial x\partial y}+2\gamma D_{13}\left(\frac{\partial\zeta}{\partial x}-c_1\frac{\partial w}{\partial x}\right)-\frac{1}{2}\lambda E_{11}\frac{\partial^3\psi_3}{\partial x^3}$$

$$-\frac{1}{3}F_{11}\left(\chi c_1\frac{\partial^2\phi_1}{\partial x^2}+\gamma\frac{\partial^3\zeta}{\partial x^3}\right)-\frac{1}{3}F_{12}\left(\chi c_1\frac{\partial^2\phi_2}{\partial x\,\partial y}+\gamma\frac{\partial^3\zeta}{\partial x\,\partial y^2}\right)+D_{66}\left(\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_2}{\partial x\,\partial y}\right)$$
$$-\chi d_1F_{66}\left(\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_2}{\partial x\,\partial y}\right)-\frac{2}{3}\gamma F_{66}\frac{\partial^3\zeta}{\partial x\,\partial y^2}-\chi d_1\left[E_{11}\frac{\partial^2u}{\partial x^2}+\lambda E_{13}\frac{\partial^2\phi_1}{\partial x^2}+F_{11}\frac{\partial^2\phi_1}{\partial x^2}\right.$$
$$+F_{12}\frac{\partial^2\phi_2}{\partial x\,\partial y}+2\gamma F_{13}\left(\frac{\partial\zeta}{\partial x}-c_1\frac{\partial w}{\partial x}\right)-\frac{1}{2}\lambda G_{11}\frac{\partial^3\psi_3}{\partial x^3}-\frac{1}{3}H_{11}\left(\chi c_1\frac{\partial^2\phi_1}{\partial x^2}+\gamma\frac{\partial^3\zeta}{\partial x^3}\right)$$
$$-\frac{1}{3}H_{12}\left(\chi c_1\frac{\partial^2\phi_2}{\partial x\,\partial y}+\gamma\frac{\partial^3\zeta}{\partial x\,\partial y^2}\right)+F_{66}\left(\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_2}{\partial x\,\partial y}\right)-\chi d_1H_{66}\left(\frac{\partial^2\phi_1}{\partial x^2}+\frac{\partial^2\phi_2}{\partial x\,\partial y}\right)$$
$$\left.-\frac{2}{3}\gamma H_{66}\frac{\partial^3\zeta}{\partial x\,\partial y^2}\right]-A_{55}\left(\phi_1+\frac{\partial w}{\partial x}\right)+c_1D_{55}\left(\chi\phi_1+\gamma\frac{\partial w}{\partial x}\right)+\chi c_1D_{55}\left(\phi_1+\frac{\partial w}{\partial x}\right)$$
$$-\chi c_2F_{55}\left(\chi\phi_1+\gamma\frac{\partial w}{\partial x}\right)=\bar{I}_3\ddot{\phi}_1-\frac{1}{3}\gamma I_5\frac{\partial\ddot{\zeta}}{\partial x}\tag{40}$$

$\underline{\delta\phi_2}$:

$$+D_{66}\left(\frac{\partial^2\phi_1}{\partial x\,\partial y}+\frac{\partial^2\phi_2}{\partial x^2}\right)-\chi d_1F_{66}\left(\frac{\partial^2\phi_1}{\partial x\,\partial y}+\frac{\partial^2\phi_2}{\partial x^2}\right)-\frac{2}{3}\gamma F_{66}\frac{\partial^3\zeta}{\partial x^2\partial y}$$
$$B_{22}\frac{\partial^2v}{\partial y^2}+\lambda B_{23}\frac{\partial\psi_3}{\partial y}+D_{12}\frac{\partial^2\phi_1}{\partial x\,\partial y}+D_{22}\frac{\partial^2\phi_2}{\partial y^2}+2\gamma D_{23}\left(\frac{\partial\zeta}{\partial y}-c_1\frac{\partial w}{\partial y}\right)-\frac{1}{2}\lambda E_{22}\frac{\partial^3\psi_3}{\partial y^3}$$
$$-\frac{1}{3}F_{12}\left(\chi c_1\frac{\partial^2\phi_1}{\partial x\,\partial y}+\gamma\frac{\partial^3\zeta}{\partial x^2\partial y}\right)-\frac{1}{3}F_{22}\left(\chi c_1\frac{\partial^2\phi_2}{\partial y^2}+\gamma\frac{\partial^3\zeta}{\partial y^3}\right)-\chi d_1\left[F_{66}\left(\frac{\partial^2\phi_1}{\partial x\,\partial y}+\frac{\partial^2\phi_2}{\partial x^2}\right)\right.$$
$$-\chi d_1H_{66}\left(\frac{\partial^2\phi_1}{\partial x\,\partial y}+\frac{\partial^2\phi_2}{\partial x^2}\right)-\frac{2}{3}\gamma H_{66}\frac{\partial^3\zeta}{\partial x^2\partial y}+E_{22}\frac{\partial^2v}{\partial y^2}+\lambda E_{23}\frac{\partial\psi_3}{\partial y}+F_{12}\frac{\partial^2\phi_1}{\partial x\,\partial y}$$
$$+F_{22}\frac{\partial^2\phi_2}{\partial y^2}+2\gamma F_{23}\left(\frac{\partial\zeta}{\partial y}-c_1\frac{\partial w}{\partial y}\right)-\frac{1}{2}\lambda G_{22}\frac{\partial^3\psi_3}{\partial y^3}-\frac{1}{3}H_{12}\left(\chi c_1\frac{\partial^2\phi_1}{\partial x\,\partial y}+\gamma\frac{\partial^3\zeta}{\partial x^2\partial y}\right)$$
$$\left.-\frac{1}{3}H_{22}\left(\chi c_1\frac{\partial^2\phi_2}{\partial y^2}+\gamma\frac{\partial^3\zeta}{\partial y^3}\right)\right]-A_{44}\left(\phi_2+\frac{\partial w}{\partial y}\right)+c_1D_{44}\left(\chi\phi_2+\gamma\frac{\partial w}{\partial y}\right)$$
$$+\chi c_1D_{44}\left(\phi_2+\frac{\partial w}{\partial y}\right)-\chi c_2F_{44}\left(\chi\phi_2+\gamma\frac{\partial w}{\partial y}\right)=\bar{I}_3\ddot{\phi}_2-\frac{1}{3}\gamma I_5\frac{\partial\ddot{\zeta}}{\partial y}\tag{41}$$

$\underline{\delta\psi_3}$:

$$\frac{1}{2}\lambda\left[D_{11}\frac{\partial^3u}{\partial x^3}+D_{12}\frac{\partial^3v}{\partial x^2\partial y}+\lambda D_{13}\frac{\partial^2\psi_3}{\partial x^2}+E_{11}\frac{\partial^3\phi_1}{\partial x^3}+2\gamma E_{13}\left(\frac{\partial^2\zeta}{\partial x^2}-c_1\frac{\partial^2w}{\partial x^2}\right)\right.$$
$$-\frac{1}{2}\lambda F_{11}\frac{\partial^4\psi_3}{\partial x^4}-\frac{1}{2}\lambda F_{12}\frac{\partial^4\psi_3}{\partial x^2\partial y^2}-\frac{1}{3}G_{11}\left(\chi c_1\frac{\partial^3\phi_1}{\partial x^3}+\gamma\frac{\partial^4\zeta}{\partial x^4}\right)$$

$$+2D_{66}\left(\frac{\partial^3 u}{\partial x\,\partial y^2}+\frac{\partial^3 v}{\partial x^2\partial y}\right)-2\lambda F_{66}\frac{\partial^4\psi_3}{\partial x^2\partial y^2}+D_{12}\frac{\partial^3 u}{\partial x\,\partial y^2}+D_{22}\frac{\partial^3 v}{\partial y^3}+\lambda D_{23}\frac{\partial^2\psi_3}{\partial y^2}$$

$$+E_{22}\frac{\partial^3\phi_2}{\partial y^3}+2\gamma E_{23}\left(\frac{\partial^2\zeta}{\partial y^2}-c_1\frac{\partial^2 w}{\partial y^2}\right)-\frac{1}{2}\lambda F_{12}\frac{\partial^4\psi_3}{\partial x^2\partial y^2}-\frac{1}{2}\lambda F_{22}\frac{\partial^4\psi_3}{\partial y^4}$$

$$-\frac{1}{3}G_{22}\left(\chi c_1\frac{\partial^3\phi_2}{\partial y^3}+\gamma\frac{\partial^4\zeta}{\partial y^4}\right)\Bigg]-\lambda\Bigg[A_{13}\frac{\partial u}{\partial x}+A_{23}\frac{\partial v}{\partial y}+\lambda A_{33}\psi_3+B_{13}\frac{\partial\phi_1}{\partial x}+B_{23}\frac{\partial\phi_2}{\partial y}$$

$$+2\gamma B_{33}(\zeta-c_1 w)-\frac{1}{2}\lambda D_{13}\frac{\partial^2\psi_3}{\partial x^2}-\frac{1}{2}\lambda\Delta_{23}\frac{\partial^2\psi_3}{\partial y^2}-\frac{1}{3}E_{13}\left(\chi c_1\frac{\partial\phi_1}{\partial x}+\gamma\frac{\partial^2\zeta}{\partial x^2}\right)$$

$$-\frac{1}{3}E_{23}\left(\chi c_1\frac{\partial\phi_2}{\partial y}+\gamma\frac{\partial^2\zeta}{\partial y^2}\right)\Bigg]=\frac{1}{2}\lambda I_3\left(\frac{\partial\ddot{u}}{\partial x}+\frac{\partial\ddot{v}}{\partial y}\right)$$

$$+\lambda^2 I_3\ddot{\psi}_3-\frac{1}{4}\lambda^2 I_5\left(\frac{\partial^2\ddot{\psi}_3}{\partial x^2}+\frac{\partial^2\ddot{\psi}_3}{\partial y^2}\right)\tag{42}$$

<u>$\delta\zeta$</u>:

$$\frac{1}{3}\gamma\Bigg[E_{11}\frac{\partial^3 u}{\partial x^3}+\lambda E_{13}\frac{\partial^2\psi_3}{\partial x^2}+F_{11}\frac{\partial^3\phi_1}{\partial x^3}+F_{12}\frac{\partial^3\phi_2}{\partial x^2\partial y}+2\gamma F_{13}\left(\frac{\partial^2\zeta}{\partial x^2}-c_1\frac{\partial^2 w}{\partial x^2}\right)$$

$$-\frac{1}{2}\lambda G_{11}\frac{\partial^4\psi_3}{\partial x^4}-\frac{1}{3}H_{11}\left(\chi c_1\frac{\partial^3\phi_1}{\partial x^3}+\gamma\frac{\partial^4\zeta}{\partial x^4}\right)-\frac{1}{3}H_{12}\left(\chi c_1\frac{\partial^3\phi_2}{\partial x^2\partial y}+\gamma\frac{\partial^4\zeta}{\partial x^2\partial y^2}\right)$$

$$+2F_{66}\left(\frac{\partial^3\phi_1}{\partial x\,\partial y^2}+\frac{\partial^3\phi_2}{\partial x^2\partial y}\right)-2\chi d_1 H_{66}\left(\frac{\partial^3\phi_1}{\partial x\,\partial y^2}+\frac{\partial^3\phi_2}{\partial x^2\partial y}\right)-\frac{4}{3}\gamma H_{66}\frac{\partial^4\zeta}{\partial x^2\partial y^2}$$

$$E_{22}\frac{\partial^3 v}{\partial y^3}+\lambda E_{23}\frac{\partial^2\psi_3}{\partial y^2}+F_{12}\frac{\partial^3\phi_1}{\partial x\,\partial y^2}+F_{22}\frac{\partial^3\phi_2}{\partial y^3}+2\gamma F_{23}\left(\frac{\partial^2\zeta}{\partial y^2}-c_1\frac{\partial^2 w}{\partial y^2}\right)$$

$$-\frac{1}{2}\lambda G_{22}\frac{\partial^4\psi_3}{\partial y^4}-\frac{1}{3}H_{12}\left(\chi c_1\frac{\partial^3\phi_1}{\partial x\,\partial y^2}+\gamma\frac{\partial^4\zeta}{\partial x^2\partial y^2}\right)-\frac{1}{3}H_{22}\left(\chi c_1\frac{\partial^3\phi_2}{\partial y^3}+\gamma\frac{\partial^4\zeta}{\partial y^4}\right)\Bigg]$$

$$-2\gamma\Bigg[B_{13}\frac{\partial u}{\partial x}+B_{23}\frac{\partial v}{\partial y}+\lambda B_{33}\psi_3+D_{13}\frac{\partial\phi_1}{\partial x}+D_{23}\frac{\partial\phi_2}{\partial y}+2\gamma D_{33}(\zeta-c_1 w)$$

$$-\frac{1}{2}\lambda E_{13}\frac{\partial^2\psi_3}{\partial x^2}-\frac{1}{2}\lambda E_{23}\frac{\partial^2\psi_3}{\partial y^2}-\frac{1}{3}F_{13}\left(\chi c_1\frac{\partial\phi_1}{\partial x}+\gamma\frac{\partial^2\zeta}{\partial x^2}\right)$$

$$-\frac{1}{3}F_{23}\left(\chi c_1\frac{\partial\phi_2}{\partial y}+\gamma\frac{\partial^2\zeta}{\partial y^2}+\gamma\frac{\partial^2\zeta}{\partial y^2}\right)\Bigg]=\gamma\tilde{I}_3\ddot{w}+\frac{1}{3}\gamma\bar{I}_5\left(\frac{\partial\ddot{\phi}_1}{\partial x}+\frac{\partial\ddot{\phi}_1}{\partial y}\right)$$

$$+\gamma^2 I_5\ddot{\zeta}-\frac{1}{9}\gamma^2 I_7\left(\frac{\partial^2\ddot{\zeta}_3}{\partial x^2}+\frac{\partial^2\ddot{\zeta}_3}{\partial y^2}\right)\tag{43}$$

3.2. Navier solutions

We consider the cases of symmetric and anti-symmetric cross-ply rectangular plates which are simply-supported on all sides. The dependent unknowns (u, v, w, ϕ_1, ϕ_2, ψ_3, ζ) and the load term q (in the case of bending analysis) are expanded in a double trigonometric series in terms of unknown coefficients. The trigonometric functions of the series may be either sine or cosine functions depending upon the particular boundary conditions of the problem. The series expansions of the generalized displacements are then substituted into the governing equations of motion. This results in a set of simultaneous algebraic equations in the undetermined coefficients, which can be easily solved in the case of bending analysis. For free vibration analysis, the substitution leads to a generalized eigenvalue problem, which can be solved for the eigenvalues and eigenvectors. See Reddy (1997) for additional details.

3.2.1. Boundary conditions

The Navier solutions are developed for the following set of simply-supported boundary conditions (for a rectangular plate of dimension a by b).

$$
\begin{array}{rcl}
u(x,0)=u(x,b)=0, & , & N_2(x,0)=N_2(x,b)=0,\\
v(0,y)=v(a,y)=0, & , & N_1(0,y)=N_1(a,y)=0,\\
w(x,0)=w(x,b)=0 & = & w(0,y)=w(a,y)=0,\\
\phi_1(x,0)=\phi_1(x,b)=0, & , & M_2(x,0)=M_2(x,b)=0,\\
\phi_2(0,y)=\phi_2(a,y)=0, & , & M_1(0,y)=M_1(a,y)=0,\\
\psi_3(x,0)=\psi_3(x,b) & = & \psi_3(0,y)=\psi_3(a,y)=0,\\
\zeta(x,0)=\zeta(x,b) & = & \zeta(0,y)=\zeta(a,y)=0.
\end{array}
\tag{44}
$$

3.2.2. Assumed displacements

The following displacement field satisfies the above boundary conditions.

$$
\begin{aligned}
u(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} U_{mn}\cos\alpha x\sin\beta y\\
v(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} V_{mn}\sin\alpha x\cos\beta y\\
w(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} W_{mn}\sin\alpha x\sin\beta y\\
\phi_1(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} X_{mn}\cos\alpha x\sin\beta y\\
\phi_2(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Y_{mn}\sin\alpha x\cos\beta y\\
\psi_3(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}^{(1)}\sin\alpha x\sin\beta y\\
\zeta(x,y)&=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Z_{mn}^{(2)}\sin\alpha x\sin\beta y
\end{aligned}
\tag{45}
$$

where $\alpha = m\pi/a$ and $\beta = n\pi/b$.

3.2.3. Bending analysis

The distributed transverse load q is expanded in a double trigonometric series

$$q(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} Q_{mn} \sin \alpha x \sin \beta y \tag{46}$$

Where Q_{mn} can be evaluated for different types of loading conditions. For example

$$Q_{mn} = \begin{cases} (16q_0)/(mn\,\pi^2) & \text{for uniformly distributed load (UL)} \\ (4P/ab)\sin(m\,\pi/2)\sin(n\,\pi/2) & \text{for point load at the center (PL)} \end{cases} \tag{47}$$

where m=1, 3, 5, ⋯ and n=1, 3, 5, ⋯.

Substituting Eq. (45) and Eq. (47) into the static equilibrium equations of Section 2, for each (m, n), we get a set of equations of the form

$$[S]\{\Delta\}=\{F\} \tag{48}$$

where

$$\{\Delta\}=\{U_{mn}\ \ V_{mn}\ \ W_{mn}\ \ X_{mn}\ \ Y_{mn}\ \ Z_{mn}^{(1)}\ \ Z_{mn}^{(2)}\}^T \tag{49}$$

$$\{F\}=\{0\ \ 0\ \ 0\ \ 0\ \ 0\ \ Q_{mn}^{(1)}\ \ Q_{mn}^{(2)}\}^T \tag{50}$$

$$Q_{mn}^{(1)}=Q_{mn}(h/2), \qquad Q_{mn}^{(2)}=Q_{mn}(h^2/4) \tag{51}$$

and the coefficients S_{ij}=S_{ji} are given in Bose (1995).

Eq. (48) is solved for the undetermined coefficients (U_{mn}, V_{mn}, ⋯ etc.) for each (m, n). The actual displacements (u, v, ⋯ etc.) are then obtained from Eq. (45). Note that the equations have been developed for the special comprehensive third order theory (GTTR). All the other theories (CLPT, FSDT, STTR) can be obtained from GTTR by assigning different values to the tracers.

3.2.4. Free vibration analysis

For free vibration analysis, we assume that the displacements are of the form

$$\begin{Bmatrix} u(x,y,t) \\ v(x,y,t) \\ w(x,y,t) \\ \phi_1(x,y,t) \\ \phi_2(x,y,t) \\ \psi_3(x,y,t) \\ \zeta(x,y,t) \end{Bmatrix} = \begin{Bmatrix} u(x,y) \\ v(x,y) \\ w(x,y) \\ \phi_1(x,y) \\ \phi_2(x,y) \\ \psi_3(x,y) \\ \zeta(x,y) \end{Bmatrix} e^{-i\omega t} \tag{52}$$

where ω is the frequency of natural vibration.

On substituting the above relations in the dynamic equilibrium equations of Section 2, we get

$$([S]-\omega_{mn}^2\,[M])\{\Delta\}=\{0\} \tag{53}$$

The coefficients $M_{ij}=M_{ji}$ are given in Bose (1995). The above equations can be solved for eigenvalues and eigenvectors for each (m, n).

3.3. Lévy solution

In this case, we consider a rectangular plate which is simply-supported at $y=0$ and $y=b$. The other two sides ($x=-a/2$ and $x=a/2$) can have a combination of free, clamped, and simply supported boundary conditions. The generalized displacements are expressed as a product of undetermined functions of x and known trigonometric functions of y. The trigonometric functions (sine or cosine) are chosen such that the simply supported boundary conditions at $y=0$ and $y=b$ are identically satisfied.

3.3.1. Assumed displacements

We assume solution in the form

$$
\begin{aligned}
u(x,y) &= \sum_{m=1}^{\infty} U_m \sin\beta y \\
v(x,y) &= \sum_{m=1}^{\infty} V_m \cos\beta y \\
w(x,y) &= \sum_{m=1}^{\infty} W_m \sin\beta y \\
\phi_1(x,y) &= \sum_{m=1}^{\infty} X_m \sin\beta y \\
\phi_2(x,y) &= \sum_{m=1}^{\infty} Y_m \cos\beta y \\
\psi_3(x,y) &= \sum_{m=1}^{\infty} Z_m^{(1)} \sin\beta y \\
\zeta(x,y) &= \sum_{m=1}^{\infty} Z_m^{(2)} \sin\beta y
\end{aligned}
\tag{54}
$$

where $\beta=m\pi/b$.

3.3.2. Solution technique

Unlike the Navier method, for the Lévy method, the solution technique is similar for static and free vibration problems. The procedure will be described in the subsequent paragraphs. First, the distributed transverse load q is expanded in a manner similar to the displacements.

$$
q(x,y) = \sum_{m=1}^{\infty} Q_m \sin\beta y \tag{55}
$$

As before, Q_m can be evaluated for different types of loading conditions.

$$
Q_m = \begin{cases} (4q_0)/(m\pi) & \text{for uniformly distributed load (UL)} \\ (2P/b)\sin(m\pi/2) & \text{for line load along the centerline } y=b/2 \text{ (LL)} \end{cases} \tag{56}
$$

where $m=1, 3, 5, \cdots$.

Second, the time-dependent displacements are expressed in exactly the same manner as in the Navier method (see Eq. (52)). Of course, the generalized displacements u, v, $\cdots$ etc. are defined differently in this case. Substitution of this displacement field into the dynamic equilibrium equations gives rise to a system of ordinary differential equations in x of the form

$$
\begin{aligned}
&e_1 U_m + e_2 U_m'' + e_3 V_m' + e_4 W_m'' + e_5 X_m'' + e_6 Z_m^{(1)'} + e_7 Z_m^{(1)'''} + e_8 Z_m^{(2)'} + e_9 Z_m^{(2)'''} = 0 \\
&e_{10} U_m' + e_{11} V_m + e_{12} V_m'' + e_{13} W_m + e_{14} Y_m + e_{15} Z_m^{(1)} + e_{16} Z_m^{(1)''} + e_{17} Z_m^{(2)} = 0 \\
&e_{18} U_m' + e_{19} V_m + e_{20} W_m + e_{21} W_m'' + e_{22} X_m' + e_{23} Y_m + e_{24} Z_m^{(1)} + e_{25} Z_m^{(1)''} + e_{26} Z_m^{(2)} + e_{27} Z_m^{(2)''} = 0 \\
&e_{28} U_m'' + e_{29} W_m' + e_{30} X_m + e_{31} X_m'' + e_{32} Y_m' + e_{33} Z_m^{(1)'} + e_{34} Z_m^{(1)'''} + e_{35} Z_m^{(2)'} + e_{36} Z_m^{(2)'''} = 0 \\
&e_{37} V_m + e_{38} W_m + e_{39} X_m' + e_{40} Y_m + e_{41} Y_m'' + e_{42} Z_m^{(1)} + e_{43} Z_m^{(2)} + e_{44} Z_m^{(2)''} = 0 \\
&e_{45} U_m' + e_{46} U_m''' + e_{47} V_m + e_{48} V_m'' + e_{49} W_m + e_{50} W_m'' + e_{51} X_m' + e_{52} X_m''' + e_{53} Y_m + e_{54} Z_m^{(1)} \\
&\quad + e_{55} Z_m^{(1)''} + e_{56} Z_m^{(1)''''} + e_{57} Z_m^{(2)} + e_{58} Z_m^{(2)''} + e_{59} Z_m^{(2)''''} + Q_m^{(1)} = 0 \\
&e_{60} U_m' + e_{61} U_m''' + e_{62} V_m + e_{63} V_m'' + e_{64} W_m + e_{65} W_m'' + e_{66} X_m' + e_{67} X_m''' + e_{68} Y_m + e_{69} Z_m^{(1)} \\
&\quad + e_{70} Z_m^{(1)''} + e_{71} Z_m^{(1)''''} + e_{72} Z_m^{(2)} + e_{73} Z_m^{(2)''} + e_{74} Z_m^{(2)''''} + Q_m^{(2)} = 0
\end{aligned} \tag{57}
$$

where

$$
Q_m^{(1)} = Q_m (h/2) \qquad Q_m^{(2)} = Q_m (h^2/4) \tag{58}
$$

and the coefficients e_i are given in (Bose 1995). The primes over the variables indicate differentiation with respect to x.

The next step is to modify Eq. (57) to make it suitable for the state-space approach. We try to bring the highest powers of each variable to the left hand side and express it in terms of its lower powers as well as the lower powers of the other variables. This will convert the set of equations to a tractable form which can be solved by the state-space procedure. We introduce a new set of variables

$$
\begin{Bmatrix} \Theta_1 \\ \Theta_2 \\ \Theta_3 \\ \Theta_4 \\ \Theta_5 \\ \Theta_6 \\ \Theta_7 \\ \Theta_8 \\ \Theta_9 \\ \Theta_{10} \\ \Theta_{11} \\ \Theta_{12} \\ \Theta_{13} \\ \Theta_{14} \\ \Theta_{15} \\ \Theta_{16} \\ \Theta_{17} \\ \Theta_{18} \end{Bmatrix} = \begin{Bmatrix} U_m \\ U_m' \\ V_m \\ V_m' \\ W_m \\ W_m' \\ X_m \\ X_m' \\ Y_m \\ Y_m' \\ Z_m^{(1)} \\ Z_m^{(1)'} \\ Z_m^{(1)''} \\ Z_m^{(1)'''} \\ Z_m^{(2)} \\ Z_m^{(2)'} \\ Z_m^{(2)''} \\ Z_m^{(2)'''} \end{Bmatrix} \tag{59}
$$

Eq. (57) is written in the form as described in the previous paragraph.

$$\{\Theta\}' = [A]\{\Theta\} + \{\Gamma\} \tag{60}$$

were $\{\Gamma\}$ is the load vector after manipulation. Obviously, for a free vibration problem, $\{\Gamma\}$ will not be present. The general structure of the matrix $[A]$ and the vector $\{\Gamma\}$ are given in Bose (1995). Their components are not written out explicitly since they are very long. However, the actual effort involved in manipulating Eq. (57) to bring it to the form of Eq. (60) is not too much. Note that the equations above have been derived for the comprehensive case of the special third order theory (GTTR). The other theories, viz. CLPT, FSDT, and STTR can be derived from GTTR quite easily. Also, the definition of Θ_is change with the different theories as do the form of $[A]$ and $\{\Gamma\}$, $\{\Theta\}$ is given below for the different cases and $[A]$ and $\{\Gamma\}$ are given explicitly for CLPT, FSDT, and STTR in Bose (1995).

CLPT:

$$\begin{aligned} &\Theta_1 = U_m, \quad \Theta_2 = U_m', \quad \Theta_3 = V_m, \quad \Theta_4 = W_m' \\ &\Theta_5 = W_m, \quad \Theta_6 = W_m', \quad \Theta_7 = W_m'', \quad \Theta_8 = W_m''' \end{aligned} \tag{61}$$

FSDT

$$\begin{aligned} &\Theta_1 = U_m, \quad \Theta_2 = U_m', \quad \Theta_3 = V_m, \quad \Theta_4 = V_m', \quad \Theta_5 = W_m, \\ &\Theta_6 = W_m', \quad \Theta_7 = X_m, \quad \Theta_8 = X_m', \quad \Theta_9 = Y_m, \quad \Theta_{10} = Y_m' \end{aligned} \tag{62}$$

STTR

$$\begin{aligned} &\Theta_1 = U_m, \quad \Theta_2 = U_m', \quad \Theta_3 = V_m, \quad \Theta_4 = V_m', \quad \Theta_5 = W_m, \quad \Theta_6 = W_m' \\ &\Theta_7 = W_m'', \quad \Theta_8 = W_m''', \quad \Theta_9 = X_m, \quad \Theta_{10} = X_m', \quad \Theta_{11} = Y_m, \quad \Theta_{12} = Y_m' \end{aligned} \tag{63}$$

3.3.3. State space solution and imposition of boundary conditions

The solution to the state space Eq. (60) is given by Franklin (1968) and Brogan (1985).

$$\{\Theta(x)\} = [e^{Ax}]\{K\} + [e^{Ax}]\int [e^{-A\xi}]\{\Gamma\}\, d\xi \tag{64}$$

where $\{K\}$ is a constant vector which is to be determined from the boundary conditions of the problem and $[e^{Ax}]$ is given by

$$[e^{Ax}] = [T]\begin{bmatrix} e^{\lambda_1 x} & 0 \\ \vdots & \\ 0 & e^{\lambda_n x} \end{bmatrix}[T]^{-1} \tag{65}$$

and λ_i denotes the distinct eigenvalues of $[A]$, and $[T]$ is the matrix of distinct eigenvectors of $[A]$. The value of n is 8 for CLPT, 10 for FSDT, 12 for STTR, and 18 for GTTR. The second term on the right hand side of Eq. (64) is present only for bending problems.

Eq. (64) is then substituted into the boundary conditions associated with the two adges $x=\pm a/2$. In the case of bending analysis, this results in a system of non-homogeneous equations which can be solved for $\{K\}$

$$[P]\{K\} + \{R\} = \{0\} \tag{66}$$

The K_is can then be put back into Eq. (64) to obtain U_m, V_m, W_m, $\cdots$, etc.

In the case of free vibration analysis, we get a system of homogeneous equations

$$[P]\{K\}=\{0\} \tag{67}$$

For a non-trivial solution, the determinant of $[P]$ should be zero.

$$|P|=0 \tag{68}$$

Eq. (68) gives the natural frequencies for each value of m.

3.3.4. Boundary conditions

The boundary conditions for simply supported (S), clamped (C), and free (F) at the edges $x=\pm a/2$ for the different theories are

CLPT

$$\begin{aligned}
&S:\ v=w=N_1=M_1=0\\
&C:\ u=v=w=\frac{\partial w}{\partial x}=0\\
&F:\ N_1=N_6=M_1=\left(\frac{\partial M_1}{\partial x}+2\frac{\partial M_6}{\partial y}\right)=0
\end{aligned} \tag{69}$$

FSDT

$$\begin{aligned}
&S:\ v=w=\phi_2=N_1=M_1=0\\
&C:\ u=v=w=\phi_1=\phi_2=0\\
&F:\ N_1=N_6=N_5=M_1=M_6=0
\end{aligned} \tag{70}$$

STTR

$$\begin{aligned}
&S:\ v=w=\phi_2=N_1=M_1=S_1=0\\
&C:\ u=v=w=\frac{\partial w}{\partial x}=\phi_1=\phi_2=0\\
&F:\ N_1=N_6=\left[(N_5-c_1P_5)+d_1\left(\frac{\partial S_1}{\partial x}+\frac{\partial S_6}{\partial y}\right)\right]=0\\
&\qquad M_1=S_1=M_6-d_1S_6=0
\end{aligned} \tag{71}$$

GTTR

$$\begin{aligned}
&S:\ v=w=\phi_2=\psi_3=\zeta=N_1=M_1=P_1=S_1=0\\
&C:\ u=v=w=\phi_1=\phi_2=\psi_3=\frac{\partial \psi_3}{\partial x}=\zeta=\frac{\partial \zeta}{\partial x}=0\\
&F1:\ N_1=N_6=(N_5-c_1P_5)=M_1=S_1=(M_6-d_1S_6)=0\\
&\qquad P_1=\left(\frac{\partial P_2}{\partial x}+\frac{\partial P_6}{\partial y}\right)=\left(\frac{\partial S_1}{\partial x}+\frac{\partial S_6}{\partial y}\right)=0\\
&F2:\ N_1=N_6=(N_5-c_1P_5)=0\\
&\qquad M_1=M_6=P_1=P_6=S_1=S_6=0
\end{aligned} \tag{72}$$

Note that two sets of boundary conditions have been given for a free edge in GTTR.

3.3.5. Computational aspects

Some computational problems encountered while implementing the Lévy method and their suggested solutions are discussed here.

There are a multitude of methods avaliable for calculating the matrix exponential $[e^{Ax}]$. The method which immediately attracts attention is one in which it is evaluated in the form of an infinite series Chen (1984):

$$[e^{Ax}]=[I]+[A]+\frac{x^2}{2!}[A]^2+\frac{x^3}{3!}[A]^3+\cdots \tag{73}$$

Although this procedure is easy to program and does not involve complex numbers, the series will diverge if even one of the eigenvalues of $[A]$ fall on the right half of the complex z-plane.

The closed form expression for the matrix exponential follows from the Cayley-Hamilton Theorem (1985). Due to the sparse nature of the $[A]$ matrix, the matrix $[P]$ in Eq. (66) obtained after imposing the boundary conditions is often ill-conditioned and computer overflow or underflow is a common occurence. This problem is overcome in the following manner Nosier and Reddy (1992). Eq. (64) is rewritten in the form

$$\{\Theta(x)\}=[T]\begin{bmatrix} e^{\lambda_1 x} & & 0 \\ & \vdots & \\ 0 & & e^{\lambda_n x}\end{bmatrix}[T]^{-1}\{K\}+[e^{Ax}]\int[e^{-A\xi}]\{\Gamma\}\,d\xi$$

or

$$\{\Theta(x)\}=[T]\begin{bmatrix} e^{\lambda_1 x} & & 0 \\ & \vdots & \\ 0 & & e^{\lambda_n x}\end{bmatrix}\{\hat{K}\}+[e^{Ax}]\int[e^{-A\xi}]\{\Gamma\}\,d\xi \tag{74}$$

where $\{\hat{K}\}=\{T\}^{-1}\{K\}$. In effect, we will be solving for $\{\hat{K}\}$ instead of $\{K\}$. After imposing the boundary conditions, we arrive at an equation similar to Eq. (66).

$$[Q]\{\hat{K}\}+\{R\}=\{0\} \tag{75}$$

But, in this case, the matrix $[Q]$ is not ill-conditioned and can be easily handled. In the bending case, after solving for $\{\hat{K}\}$, $\{K\}$ can be obtained by

$$\{K\}=[T]\{\hat{K}\} \tag{76}$$

For free vibration problems, $|P|$=0, Eq. (68), implies

$$|[Q]\{T\}^{-1}|=0$$

$$\text{or}\quad |Q|/|T|=0 \tag{77}$$

However, it should be kept in mind that while, in the previous case, $[P]$ and $[K]$ were both real matrices, after the transformation, $[Q]$ and $[\hat{K}]$ will, in general, both be complex.

In another scenario, difficulties might arise during the computation of the eigenvalues and eigenvectors of $[A]$ since the diagonal elements of $[A]$ are always all zero. This can be circumvented by adding a constant non-zero number to the diagonal elements of $[A]$. The eigenvalues of $[A]$ can be obtained by subtracting the same number from the eigenvalues of the new matrix. The eigenvectors are the same in both cases.

4. Conclusions

In this paper a unified third order theory of laminated plates is developed. All existing plate theories can be obtained as special cases of the theories developed herein. Analytical solutions using the Navier and Lévy methods are developed for cross-ply rectangular laminates.

Althogh the Lévy method is more general than the Navier method in its range of applications, it has two limitations.

1. Two opposite edges of the plate have to be simply supported. In our case, the simple supports are assumed to be at the edges $y=0$ and $y=b$.
2. The shape of the loading function should be the same for all sections parallel to the other two edges. In our case, the loading function should have the same shape parallel to the y-axis.

But, the convergence of the Lévy solution is much faster compared to the Navier solution. Thus, very accurate results are obtained with only few terms of the Lévy solution.

In the second part of this paper, we develop the finite element model of the plate theories discussed here, and present numerical results for bending and natural vibration.

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