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Cylindrical bending of laminated cylindrical shells using a modified zig-zag theory

Ugo Icardi†

Politecnico di Torino, Dipartimento di Ingegneria Aeronautica e spaziale
Corso Duca degli Abruzzi 24-10129 Torino, Italy

Abstract. A relatively simple two-dimensional multilayered shell model is presented for predicting both global quantities and stress distributions across the thickness of multilayered thick shells, that is based on a third-order zig-zag approach. As for any zig-zag model, the layerwise kinematics is accounted for, with the stress continuity conditions at interfaces met a priori. Moreover, the shell model satisfies the zero transverse shear stress conditions at the upper and lower free surfaces of the shell, irrespective of the lay-up. By changing the parameters in the displacement model, some higher order shell models are obtained as particular cases. Although it potentially has a wide range of validity, application is limited to cylindrical shell panels in cylindrical bending, a lot of solutions of two-dimensional models based on rather different simplyfying assumptions and the exact three-dimensional elasticity solution being available for comparisons for this benchmark problem. The numerical investigation performed by the present shell model and by the shell models derived from it illustrates the effects of transverse shear modeling and the range of applicability of the simplyfying assumptions introduced. The implications of retaining only selected terms depending on the radius-to-thickness ratio are focused by comparing the present solutions to the exact one and to other two-dimensional solutions in literature based on rather different simplyfying assumptions.

Key words: laminated shells modeling; layer-wise theories; stress analysis.

1. Introduction

The usage of composite multilayered shells has grown much lately in the aerospace, automotive, shipbuilding and other industries where the demand is for high strength-to-weight ratio shell structures.

Many different approaches for modeling the response of laminated shells are already available. Basically they can be classified as:

(i) *smeared laminate models*, whose goal is to accurately predict the global response of both thin and moderately thick shell structures;

(ii) *layer-wise models*, i.e. recent models capable of predicting both the response and the thickness-wise stress distribution even in multilayered thick shells.

A characteristic feature in the analysis of laminated structures is the extreme importance of transverse shear deformation, both for global and local scale effects, this is well documented by Reddy (1984). Multilayered shell models differ in the way transverse shear deformability is accounted for.

† Substitute Professor of Spacecraft structures

First-order shear deformation theory (FSDST) based on Reissner-Mindlin hypotheses have been extensively used. Being not able to meet the zero transverse shear conditions at free surfaces,[1] this thery requires the use of shear correction factors. Higher-order smeared laminate shell models (HSDST) have been developed to overcome this setback. One example is the shell model of Reddy and Liu (1985), which assumed cubic variation of in-plane disolacements across the thickness and the zero stress conditions are met, at least for shells with symmetric lay-ups. Since FSDST and HSDST smeared laminate shell models consider the laminated shell as an equivalent sigle-layer anisotropic shell eg. representing an extension of isotropic shell models to multilayered shells, they are confined to global response predictions.

Inclusion of the effect of the transverse normal stress have been analysed by Voyiadjis and Shi (1991). The result of this research activity on smeared laminate shell models is improve the accuracy in predicting global quantities by an improved description of the distorsion of the normal.

Multilayered shell models capable of predicting both the response and the thickness-wise stress distribution of thick shells have been developed, and layer-wise kinematics is usually included.

Dennis and Palazotto (1991), who considered cylindrical shells in cylindrical bending, compared their results with the exact solution of Ren (1987). They proved layer-wise approaches necessary for thick shells, the smeared laminate shell models providing inaccurate responses, both for deflections and stress distributions.

Two different approaches have been quoted (see, Noor and Burton 1989, 1990) for multilayered shells accounting for layerwise kinematics:

(i) discrete-layer models that consider each layer as a shell, thus imposing the continuity of transverse stresses as constrain conditions at interfaces;

(ii) zig-zag shell models that assume a displacement field with discontinuous first derivatives, to satisfy them a priori.

Extension of discrete-layer approaches to laminated shell has been given by Barbero and Reddy (1990), who developed a generalized discrete-layer shell theory where, with a suitable selection of variable and functions involved, any desired degree of approximation can be achieved in describing the zig-zag variation of in-plane displacements across the thickness.

Due to the way continuity of transverse stresses at interfaces is accounted, discrete-layer plate and shell models exploit a high accuracy, but at the expense of an increasing number of unknowns with increasing the number of layers and the degree of approximation.

In contrast, zig-zag shell models always contain the same number of unknowns, coinciding with the five generalized displacements of FSDST and HSDST shell models, irrespective of the number of constituent layers.

There is no doubt that a number of unknowns independent of the number of layer is a great advantage, if accuracy in predicting global and local quantities can be preserved. Since the pioneering work of Di Sciuva (1986), a considerable amount of research have been produced to improve the zig-zag modeling of plates. In this respect, we cite, among others, the works of Bhaskar and Varadan (1989), Lee *et al.* (1990), (1993), Lee and Liu (1991), (1993). Today piecewise cubic variations of in-plane displacements are included into zig-zag plate models, to satisfy the continuity conditions of transverse shear stresses at interfaces and, contemporaneously, the zero transverse shear stress conditions at free surfaces irrespective the

[1] Here we refer to stresses computed by constitutive equations.

lay-up might be. Examples are the works of Lee *et al.* (1990), Di Sciuva (1990), Cho and Parmerter (1993), Di Sciuva and Icardi (1995).

Di Sciuva (1987) and Di Sciuva and Icardi (1993) developed first and third-order zig-zag shell models, respectively. Xavier *et al.* (1993) developed an improved third-order zig-zag shell model for laminated shells of arbitrary lay-up. With the exception of previously cited work by Dennis and Palazotto (1991), where finite element results were presented, and by Xavier *et al.* (1993), where analytical results were presented, comparisons of results of two-dimensional shell formulations with exact three-dimensional solutions (see, Ren 1987, Varadan & Bhaskar 1991) are not found in the open literature.

Thus, the versatility and the limitations of most of the proposed shell models for both global and local predictions has not been fully explored. As a consequence, further research work is still required to settle what are the effects of kinematic assumptions across the thickness pertaining displacements and of approximations made in retaining in a consistent way, or not, the thickness coordinate-to-radius ratio terms ζ/R_α, h/R_β, on assuming R_α, R_β as the raddi in the principal directions α, β, h being the thickness. Another question still open to discussion is whether a non-constant transverse displacement should be included.

In fact, when the results of various shell theories are compared, discrepancies can be due to both the displacement model used and to retaining in some different ways the thickness coordinate-to-radius ratio terms. For a better understanding, the influence of these two coupled effects should be investigated separately.

In view of the above, the present work develops a third-order zig-zag shell model for the analysis of shells with general lay-up, that can be reduced to particular cases of the shell models of Di Sciuva (1995) and Di Sciuva and Icardi (1993). The present shell model represent the extension of the zig-zag plate model developed by Di Sciuva and Icardi (1995) for the analysis of plates of general lay-up, derived on the basis of the generalized zig-zag model of Di Sciuva (1994), to laminated shells.

Then the results of the present shell model, that are obtained under the same assumptions for the ζ/R_α, ζ/R_β terms retained, as in Di Scicuva (1995), are compared to the elasticity solution of Ren (1987), to assess the accuracy of kinematic assumptions made.

The comparison with the two-dimensional solutions of Dennis and Palazotto (1991), and Xavier, Lee & Chew (1993), for corresponding shell displacement models, allows us to assess the effect of a different selection of the thickness coordinate-to-radius ratio terms retained. In addition, predictions of different two-dimensional shell models can be compared together.

The comparison of present numerical results with available published results show that: (i) zig-zag models are superior to other two-dimensional model in predicting both deflections and stress distributions; (ii) the present simplyfying assumptions in defining strain-displacement relations seems to provide accurate enough results even for thick shells; (iii) the major discrepancies with the exact solution seem to be ascribed either to the neglected transverse normal stress, due to the constant transverse displacement assumed across the thickness, or to neglected higher-order ζ/R_α, ζ/R_β terms, since results of Xavier, Lee & Chew (1993), where all these terms are retained in a consistent way do not considerably improve predictions.

2. Shell kinematics

Consider a multilayered cylindrical shell of total thickness h made by an arbitrary number N of orthotropic layers perfetly bonded together and refer it to the curvilinear tri-orthogonal system of coordinates (α, β, ζ) of the lines of principal curvature. Let the reference surface Ω be the middle surface of the cylindrical shell. Let R_β be the radius of the shell.

We begin by assuming the following displacement field across the thickness of the shell:

$$u_\alpha(\alpha,\beta)=u_\alpha^{(0)}(\alpha,\beta)+\zeta\Gamma_\alpha(\alpha,\beta)+\zeta^2\Phi_\alpha(\alpha,\beta)+ \zeta^3\Psi_\alpha(\alpha,\beta)+\sum_{k=1}^{s-1}\Delta_\alpha^{(k)}(\alpha,\beta)(Z-Z^{(k)})H_k \quad (1)$$

$$u_\beta(\alpha,\beta)=(1-\frac{\zeta}{R_\beta})u_\beta^{(0)}(\alpha,\beta)+\zeta\Gamma_\beta(\alpha,\beta)+\zeta^2\Phi_\beta(\alpha,\beta)+ \zeta^3\Psi_\beta(\alpha,\beta)+\sum_{k=1}^{s-1}\Delta_\beta^{(k)}(\alpha,\beta)(Z-Z^{(k)})H_k \quad (2)$$

$$u_\zeta(\alpha,\beta)=u_\zeta^{(0)}(\alpha,\beta) \quad (3)$$

Hereafter we will specialize it for the sample problem of cylindrical shells in cylindrical bending we are interested to. In previous equations, u_α, u_β, u_ζ are the three components of the elastic displacement of the points of the shell; u_α^0, u_β^0, u_ζ^0 are the corresponding quantities of the points of Ω;

$$\Gamma_\alpha=\gamma_\alpha-\frac{u_{\zeta,\alpha}}{A};\ \Gamma_\beta=\gamma_\beta-\frac{u_{\zeta,\beta}}{B}$$

are the rotations of the normal in the (α, ζ) and (β, ζ) planes respectively, γ_α, γ_β being the transverse shear rotations of the points of Ω; $\zeta^{(k)}$) are the coordinates of interfaces; Z, $Z^{(k)}$ are the thickness coordinate and the coordinate of interfaces measured from the upper bounding surface of the plate; H_k is the Heaviside unit function: it has the value of 0 before the interface k and 1 from this interface; $\Delta_\alpha^{(k)}(x_\alpha, x_\beta)$, $\Delta_\beta^{(k)}(x_\alpha, x_\beta)$ are functions to be determined by imposing the continuity of transverse shear stresses to be met at interfaces.

The reason for the introduction of Z, $Z^{(k)}$ is that continuity functions $\Delta_\beta^{(k)}(x_\alpha, x_\beta)$ can be conveniently determined by starting from the top bounding surface, on the account that H_k is defined only for positive values of $\zeta_3^{(k)}$, and that the choice of the surface to start with is immaterial for the physical meaning of these quantities[2].

Φ_α, Φ_β, Ψ_α, Ψ_β are unknown functions to be determined by imposing the zero transverse shear conditions to be met at the upper and lower free surfaces of the shell.

Consequently to a constant transverse displacement u_ζ across the thickness, the transverse normal strain is identically zero and hence the transverse normal stress will be also assumed to be identically zero.

We will now reduce this displacement model for the special case of cylindrical shells in cylindrical bending that we want to solve. Dealing with our sample problem, let the α-coordinate be chosen along the axis of the cylindrical shell and extend from $-\infty$ to $+\infty$; let the β and ζ-coordinates be chosen along the circumferential and normal directions, respectively. Under these assumptions, the displacement fields are:

$$u_\beta(\beta)=(1-\frac{\zeta}{R_\beta})u_\beta^{(0)}(\beta)+\zeta\Gamma_\beta(\beta)+\zeta^2\Phi\beta_\beta(\beta)+\zeta^3\Psi_\beta(\beta)+$$

[2] Otherwise, by starting to compute the continuity functions from the plate reference plane, two sums for positive and negative values of ζ are required.

$$\sum_{k=1}^{s-1} \Delta_{\beta}^{(k)}(\beta)(Z - Z^{(k)}) H_k \tag{4}$$

$$u_{\zeta}(\beta) = u_{\zeta}^{(0)}(\beta) \tag{5}$$

The first step is to determine the appropriate expressions of Φ_{β}, Ψ_{β} in order the present zig-zag cylindrical shell model satisfy the zero transverse shear stress conditions at the upper and lower free surfaces, irrespective of what the lay-up might be (see, Section 3).

Note that (i) by this suitable choice of Φ_{β}, Ψ_{β}, the setback of the first-order zig-zag shell model Di Sciuva (1987) to never meet the zero transverse shear conditions, and that of the third-order zig-zag shell model Di Sciuva (1993) to meet these conditions only for symmetric lay-ups are overcame; (ii) the choice of a cubic displacement field is dictated by the desire to represent transverse shear strains by piecewise quadratic functions across the thickness, in order to have piecewise quadratic transverse shear stresses that are continuous at interfaces, already when predicted by constitutive equations.

Note that the presence of second order power of ζ giving an antisymmetric contribution to transverse shear stresses is essential for obtaining a zig-zag shell model suitable for unsymmetric lay-ups, i.e., with transverse shear stresses predicted from constitutive equations that met the zero stress conditions at free surfaces.

2.1. Strain-displacement relations

Inconsistent with the infinitesimal deformations theory, the strain-displacement relations to be used in the analysis of a generic shell are[3]:

$$H_{\alpha}\varepsilon_{\alpha\alpha} = u_{\alpha,\alpha} + u_{\beta}\frac{A_{,\beta}}{B} - u_{\zeta}\frac{A}{R_{\alpha}};\quad H_{\beta}\varepsilon_{\beta\beta} = u_{\beta,\beta} + u\frac{B_{,\alpha}}{A} - u_{\zeta}\frac{B}{R_{\beta}}$$

$$H_{\alpha}H_{\beta}\varepsilon_{\alpha\beta} = H_{\alpha}^{2}\left(\frac{u_{\alpha}}{H_{\alpha}}\right)_{,\beta} + H_{\beta}^{2}\left(\frac{u_{\beta}}{H_{\beta}}\right)_{,\alpha}$$

$$H_{\alpha}\varepsilon_{\alpha\zeta} = H_{\alpha}^{2}\left(\frac{u_{\alpha}}{H_{\alpha}}\right)_{,\zeta} + u_{\zeta,\alpha};\quad H_{\beta}\varepsilon_{\beta\zeta} = H_{\beta}^{2}\left(\frac{u_{\beta}}{H_{\beta}}\right)_{,\zeta} + u_{\zeta,\beta}$$

R_{α} and R_{β} being the radii of curvature in the principal directions α, β, A and B being the surface metrics of Ω and $H_{\alpha} = A(1-\zeta/R_{\alpha})$, $H_{\beta} = A(1-\zeta/R_{\beta})$ being the Lam'e coefficients.

For the case of a cylindrical shell in cylindrical bending, these strain-displacement relations simplify to:

$$H_{\beta}\varepsilon_{\beta\beta} = u_{\beta,\beta} - u_{\zeta}\frac{1}{R_{\beta}} \tag{6}$$

$$H_{\beta}\varepsilon_{\beta\zeta} = H_{\beta}^{2}\left(\frac{u_{\beta}}{H_{\beta}}\right)_{,\zeta} + u_{\zeta,\beta} \tag{7}$$

In the following developments, the simplifying assumption to neglect all ζ/R_{β} terms appearing in the expression of the transverse shear strain $\varepsilon_{\beta\zeta}$ will be made, conforming to Di Sciuva (1987) and Di Sciuva and Icardi (1993). Once substituted into previous relation the expressions of displacements by the assumed displacement model, the expression of $\varepsilon_{\beta\zeta}$

[3] Into the sequel, the notation $(.)_{,\alpha}$ will be used for indicating differentiation, i.e. $\partial(.)/\partial_{\alpha}$

simplify to:

$$H_{\beta}\varepsilon_{\beta\zeta}=\gamma_{\alpha}+2\zeta\Phi_{\beta}+3\zeta^{2}\Psi_{\beta}+\sum_{k=1}^{s-1}\Delta_{\beta}^{(k)}(1-\frac{Z^{(k)}}{R_{\beta}})H_{k} \tag{8}$$

Our choice to retain only selected terms in the expressions of $\varepsilon_{\beta\zeta}$ conforming to Di Sciuva (1987) and (1993) allows us to provide a numerical assessment of the zig-zag shell theories there developed, aware from numerical solutions. This give us also the opportunity to make a comparison of present results to the results of shell theories where all terms of the order of ζ/R_{β} are retained consistently, such as in Xavier, Lee and Chew (1993), and with other theories where simplyfying assumptions are made, such as in Dennis and Palazotto (1991), in order to assess the effect of this aspect of the modeling for thick shells, i.e., with a low radius-to-thickness ratio.

Once defined the transverse shear strain, the transverse shear stress expression that follows directly from it can be used to determine the expressions of Φ_{β}, Ψ_{β} by imposing the zero transverse shear conditions to be met at the shell free surfaces $\zeta=\pm h/2$.

3. Derivation of the expressions of Φ_{β}, Ψ_{β}

In view of the above, it is noted that the imposition of the zero stress conditions at free surfaces is equivalent to the requirement that corresponding transverse shear strains vanish on these surfaces. By imposing the vanishing of the transverse shear strain at the shell bounding surfaces, we obtain the following expressions for Φ_{β}, Ψ_{β}:

$$\Phi_{\beta}(\alpha,\beta)=-\frac{1}{2h}\sum_{k=1}^{N-1}\Delta_{\beta}^{(k)}(1-\frac{Z^{(k)}}{R_{\beta}})H_{k} \tag{9}$$

$$\Psi_{\beta}(\alpha,\beta)=-\frac{4}{3h^{2}}[\gamma_{\beta}+\frac{1}{2}\sum_{k=1}^{N-1}\Delta_{\beta}^{(k)}(1-\frac{Z^{(k)}}{R_{\beta}})H_{k}] \tag{10}$$

the sum being now extended to all interfaces. Use of previous expressions allows us to derive the expressions for the continuity functions $\Delta_{\beta}^{(k)}$, with k ranging from 1 to N-1, by imposing transverse shear stresses to be continuous at the N-1 interfaces. Note that here summations are extended to all N-1 interfaces.

4. Derivation of the expression of the continuity functions $\Delta_{\beta}^{(k)}$

As for any zig-zag model, the continuity functions $\Delta_{\alpha}^{(k)}$ can be determined by imposing the contact conditions to be satisfyed at interfaces. Let us indicate with $\sigma_{\beta\zeta}^{-}$ the value of the transverse shear stress before the generic interface k have been encountered and with $\zeta=\zeta_{(k)}$-0 the corresponding thickness coordinate. Conversely, $\sigma_{\beta\zeta}^{+}$, $\zeta=\zeta^{(k)}+0$ let be the corresponding quantities after the interface k has been encountered. Thus, the contact conditions at interfaces write:

$$\sigma_{\beta\zeta}^{+}=\sigma_{\beta\zeta}^{-}\quad \text{i.e.,}\quad Q_{44}(k+1)[\varepsilon_{\beta\zeta}^{-}-\Delta_{\beta}^{(k)}]=Q_{44}(k+1)\,\varepsilon_{\beta\zeta}^{-}$$

It is worthwhile to note that $\sigma_{\beta\zeta}$ being function only of the transverse shear rotation γ_{β}, the continuity function $\Delta_\beta^{(k)}$ can be written purely in term of γ_β by:

$$\Delta_\beta^{(k)} = a_k\, \gamma_\beta$$

the constants a_k at interfaces being unknown at this stage. Once substituted the previous expression of $\Delta_\beta^{(k)}$ into the contact conditions and the expressions of $\varepsilon_{\beta\zeta}$ given by Eq. (8), the following algebraic system, that allows to determine the N-1 values of the continuity functions a_k at interfaces, is written:

$$\begin{aligned}
&[Q_{44}(k+1)-Q_{44}(k)]\,[-\frac{Z^{(k)}}{h}\sum_{s=1}^{N-1} a_s\,\gamma_\beta(1-\frac{Z^{(k)}}{R_\beta})H_k-\frac{2(Z^{(k)})^2}{h^2}\sum_{s=1}^{N-1} a_s\,\gamma_\beta(1-\frac{Z^{(k)}}{R_\beta})H_k]+\\
&[Q_{44}(k+1)-Q_{44}(k)]\sum_{s=1}^{k-1} a_s\,\gamma_\beta(1-\frac{Z^{(k)}}{R_\beta})H_k+Q_{44}(k+1)\,a_k\,\gamma_\beta(1-\frac{Z^{(k)}}{R_\beta})=\\
&\qquad -[Q_{44}(k+1)-Q_{44}(k)]\,\gamma_\beta(1-\frac{4(Z^{(k)})^2}{h^2})
\end{aligned}\tag{11}$$

After this system of N-1 algebraic equations in the N-1 unknown a_k is solved, all the quantities appearing in the displacement model are completely determined. The final expression of the displacement model is thus:

$$\begin{aligned}
u_\beta(\beta)=&(1-\frac{\zeta}{R_\beta})u_\beta^{(0)}+\zeta(\delta_F\,\gamma_\beta-u_{\zeta},\ \beta)+\delta_A\,\zeta^2[-\frac{1}{2h}\sum_{k=1}^{N-1} a_k\,\gamma_\beta(1-\frac{Z^{(k)}}{R_\beta})H_k]+\\
&\delta_A\,\zeta^3\,\frac{3h^2}{4}[-\delta_T\,\gamma_\beta-\frac{1}{2}\sum_{k=1}^{N-1} a_k\,\gamma_\beta(1-\frac{Z^{(k)}}{R_\beta})H_k]+\sum_{k=1}^{s-1} a_k\,\gamma_\beta(Z-Z^{(k)})H_k
\end{aligned}\tag{12}$$

$$u_\zeta(\beta)=u_\zeta^{(0)}(\beta)\tag{13}$$

Deleted Quantitiy	Model Obtained
none	PRESENT
δ_A	RHSDST Di Sciuva, Icardi (1993)
δ_A, δ_T	RFSDST Di Sciuva (1987)
δ_A, $\Delta_\beta^{(k)}$	HSDST Reddy, Liu (1985), Reddy (1984)
δ_A, δ_T, $\Delta_\beta^{(k)}$	FSDST
δ_A, δ_F, δ_T, $\Delta_\beta^{(k)}$	CPT

where the tracer operators δ_A, δ_F and δ_T have been introduced. Assuming the values 0 or 1, they identify contributions brought in the present shell model, as specified into the following, to obtain, as particular cases, some shell models proposed in literature, to be used for sake of comparison. The shell models we can particularize are:

5. Equilibrium equations in terms of averaged resultants

Substituting the present shell model of Eqs. (12) and (13) into strain-displacement relations of Eqs. (6), (7), and expressions for in-plane and transverse shear strains so obtained into the virtual work principle, equilibrium equations and consistent boundary conditions are then derived.

Table 1 Non-dimensional maximum deflection $\bar{u}_\zeta = \dfrac{10E_T\, u_\zeta^0(0, \psi/2)}{q^0 h S^4}$ in (90^0) cylindrical orthotropic shell, as predicted by various shell models

R/h	Dennis *et al.* (1991) (f.e.m.)	Xavier *et al.* (1993)	HSDST (present)	RFSDST (present)	RHSDST (present)	IRHSDST (present)	Exact Ren (1987)
2	0.803 (0.8304)	0.9406	1.0006	0.9409 (0.9736)	1.0006	1.0006	0.9986
4	0.278 (0.2824)	0.3125	0.3071	0.2653 (0.2838)	0.3071	0.3071	0.312
10	0.108 (0.1092)	0.1140	0.1109	0.1053 (0.1086)	0.1109	0.1109	0.115
50	0.0762 (0.0764)	0.0772	0.0773	0.0772 (0.0769)	0.0773	0.0773	0.077
100	0.0751 (0.0754)	0.0758	0.0763	0.0763 (0.0757)	0.0763	0.0763	0.0755
500	0.0746 (0.0750)	0.0751	0.0760	0.0760 (0.0751)	0.0760	0.0760	0.0749

Results in brackets are obtained from Reddy and Liu (1985).

5.1. Force and moment stress resultants

The governing equations in terms of averaged resultants are obtained by introducing the following force and moment stress resultants for unit length[4]

$$(N_{\beta\beta}, Q_\beta, M_{\beta\beta}, T_\beta, P_{\beta\beta}) = <(\sigma_{\beta\beta}, \sigma_{\beta\zeta}, \zeta\sigma_{\beta\beta}, \zeta^2\sigma_{\beta\zeta}, \zeta^3\sigma_{\beta\beta})>$$

$$M^a_{\beta\beta} = <\sigma_{\beta\beta}\sum_{k=1}^{s-1} a_k(\zeta-\zeta^{(k)})>\,;\;\; Q^a_\beta = <\sigma_{\beta\zeta}\sum_{k=1}^{s-1} a_k(1-\frac{\zeta^{(k)}}{R_\beta})>$$

$$M2^a_{\beta\beta} = <\sigma_{\beta\beta}\zeta^2>\sum_{k=1}^{N-1} a_k(1-\frac{\zeta^{(k)}}{R_\beta});\;\; M3^a_{\beta\beta} = <\sigma_{\beta\beta}\zeta^3>\sum_{k=1}^{N-1} a_k(1-\frac{\zeta^{(k)}}{R_\beta})$$

$$Q1^a_{\beta\beta} = <\sigma_{\beta\zeta}\zeta>\sum_{k=1}^{N-1} a_k(1-\frac{\zeta^{(k)}}{R_\beta});\;\; Q2^a_{\beta\beta} = <\sigma_{\beta\zeta}\zeta^2>\sum_{k=1}^{N-1} a_k(1-\frac{\zeta^{(k)}}{R_\beta})$$

where

$$<\ldots> = \sum_{s=1}^{N} \int_{\zeta_{s-1}}^{\zeta_s} (\ldots)\, d\zeta$$

5.2. Force and moment resultants of external loadings

Let us assume the cylindrical shell be acted upon by a distributed tranverse load $\bar{q}_\zeta$ at the upper surface $\zeta_q = -h/2$. The resultant of this external loading write

$$\bar{Q}_\zeta = -(1-\frac{\zeta_q}{2R_\beta})\,\bar{q}_\zeta$$

[4] By now $\zeta^{(k)}$ are the coordinates of interfaces measured from the shell reference surface Ω.

5.3. Field equations

Use of previous definitions gives the following equilibrium equations in terms of stress and moment resultants:

$$N_{\beta\beta,\beta}-\frac{M_{\beta\beta,\beta}}{R_\beta}=0 \tag{14}$$

$$M_{\beta\beta,\beta\beta}+\frac{N_{\beta\beta}}{R_\beta}+\bar{Q}_\zeta=0 \tag{15}$$

$$\delta_F(M_{\beta\beta,\beta}-Q_\beta)+(M^a_{\beta\beta,\beta}-Q^a_\beta)-\delta_T\frac{4}{3h^2}[P_{\beta\beta,\beta}+3T_\beta]-$$
$$\delta_A[\frac{1}{2h}M2^a_{\beta\beta}+\frac{2}{3h^2}M3^a_{\beta\beta}]-\frac{1}{h}Q1^a_{\beta\beta}-\frac{2}{h^2}Q2^a_{\beta\beta}=0 \tag{16}$$

5.4. Boundary conditions

The variationally consistent boundary conditions to be associated to previous field equations write:

Natural	Prescribed	
$N_{\beta\beta}-R_\beta^{-1}M_{\beta\beta}=\bar{N}_{\beta\beta}-R_\beta^{-1}\bar{M}_{\beta\beta}$	$u^0_\beta=\bar{u}^0_\beta$	(17)
$M_{\beta\beta,\beta}=0$	$u^0_\zeta=\bar{u}^0_\zeta$	(18)
$M_{\beta\beta}=0$	$u^0_{\zeta,\beta}=\bar{u}^0_{\zeta,\beta}$	(19)

$$\delta_F M_{\beta\beta}+M^a_{\beta\beta}-\delta_T\frac{4}{3h^2}P_{\beta\beta}-\delta_A[\frac{1}{2h}M2^a_{\beta\beta}+\frac{2}{3h^2}M3^a_{\beta\beta}]$$
$$=\delta_F\bar{M}_{\beta\beta}+\bar{M}^a_{\beta\beta}-\delta_T\frac{4}{3h^2}\bar{P}_{\beta\beta}-\delta_A[\frac{1}{2h}\bar{M}2^a_{\beta\beta}+\frac{2}{3h^2}\bar{M}3^a_{\beta\beta}] \qquad \gamma_\beta=\bar{\gamma}_\beta \tag{20}$$

Here overlined quantities are the prescrived values of quantities.

Stresses appearing in the previous resultants can be expressed in terms of strains by using usual constitutive equations of orthotropic layers with reduced plane stress elastic constants. This will give equilibrium equations written purely in terms of generalied coordinates.

5.5. Laminated shell stiffnesses

Equilibrium equations can be expressed in terms of generalized coordinates, introducing the following shell stiffnesses[5]

$$[A_{22},B_{22},D_{22},E_{22},F_{22},H_{22}]=\langle Q_{22}[1,\zeta,\zeta^2,\delta_T(\zeta^3,\zeta^4,\zeta^6)]\rangle$$
$$[A_{55},D_{55},F_{55}]=\langle Q_{55}[1,\zeta^2,\delta_T(\zeta^4)]\rangle$$
$$[B^a_{22},D^a_{22},F^a_{22}]=\langle Q_{22}[1,\zeta,\delta_T\zeta^3]\sum_{k=1}^{s-1}a_k(\zeta-\zeta^{(k)})\rangle$$

[5] Stiffness and elasticity terms $(.)_{22}$ relate to the β-direction.

$$[A^a_{55}, D^a_{55}] = < Q_{55}[1, \zeta^2] \sum_{k=1}^{s-1} a_k >$$

$$D^{aa}_{22} = < Q_{22} \sum_{k=1}^{s-1} a_k(\zeta - \zeta^{(k)}) \sum_{r=1}^{s-1} r(\zeta - \zeta^{(r)}) >$$

$$A^{aa}_{55} = < Q_{55} \sum_{k=1}^{s-1} \alpha_k \sum_{r=1}^{s-1} \beta_r >$$

$$[D^a_{22}2, \varepsilon^a_{22}2, F^a_{22}2] = < \delta_A Q_{22}[\zeta^2, \zeta^3, \zeta^5(-\frac{4}{3h^2})] > \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta})$$

$$[B^a_{55}2, \varepsilon^a_{22}2] = < \delta_A Q_{22}[\zeta, \zeta^2(-\frac{4}{h^2})] > \sum_{k=1}^{N-1} a_k(1 - \zeta\frac{^{(k)}}{R_\beta})$$

$$[D^{aa}_{22}2, \varepsilon^{aa}_{22}2] = < \delta_A Q_{22}[\zeta^4(-\frac{1}{2h}), \zeta^5(-\frac{2}{3h^2})] > \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta}) \sum_{r=1}^{N-1} a_r(1 - \frac{\zeta^{(r)}}{R_\beta})$$

$$[B^{aa}_{55}2, D^{aa}_{55}2] = < \delta_A Q_{55}[\zeta^2(-\frac{1}{h}), \zeta^3(-\frac{2}{h^2})] > \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta}) \sum_{r=1}^{N-1} a_r(1 - \frac{\zeta^{(r)}}{R_\beta})$$

$$F^{aa}_{22}2 = < \delta_A Q_{22} x_3^2 \sum_{r=1}^{k-1} a_r(\zeta - \zeta^{(r)}) > \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta})$$

$$F^{aa}_{55}2 = < \delta_A Q_{55}\zeta \sum_{r=1}^{k-1} a_r > (1 - \frac{\zeta^{(r)}}{R_\beta}) \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta})$$

$$[D^a_{22}3, \varepsilon^a_{22}3, F^a_{22}3] = < \delta_A Q_{22}[\zeta^3, \zeta^4, \zeta^6(-\frac{4}{3h^2})] > \sum_{k=1}^{N-1} \alpha_k(1 - \frac{\zeta^{(k)}}{R_\beta})$$

$$[B^a_{55}3, \varepsilon^a_{55}3] = < \delta_A Q_{55}[\zeta^2, \zeta^3(-\frac{4}{h^2})] > \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta})$$

$$[D^{aa}_{22}3, \varepsilon^{aa}_{22}3] = < \delta_A Q_{22}[\zeta^5(-\frac{1}{2h}), \zeta^6(-\frac{2}{3h^2})] > \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta}) \sum_{r=1}^{N-1} a_r(1 - \frac{\zeta^{(r)}}{R_\beta})$$

$$[B^{aa}_{55}3, \varepsilon^{aa}_{55}3] = < \delta_A Q_{55}[\zeta^3(-\frac{1}{h}), \zeta^4(-\frac{2}{h^2})] \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(r)}}{R_\beta}) > \sum_{r=1}^{N-1} a_r(1 - \frac{\zeta^{(k)}}{R_\beta})$$

$$F^{aa}_{22}3 = < \delta_A Q_{22}\zeta^3 \sum_{r=1}^{k-1} \beta_r(\zeta - \zeta^{(r)}) > \sum_{k=1}^{N-1} \alpha_k(1 - \frac{\zeta^{(k)}}{R_\beta})$$

$$F^{aa}_{55}3 = < \delta_A Q_{55}\zeta^2 \sum_{r=1}^{k-1} a_r(1 - \zeta\frac{^{(r)}}{R_\beta}) > \sum_{k=1}^{N-1} a_k(1 - \frac{\zeta^{(k)}}{R_\beta})$$

5.6. Force and moment stress resultants in terms of generalized displacements

Using the previous definitions for shell stiffness gives the following expressions of force and moment stress resultants in terms of generalized displacements:

$$N_{\beta\beta} = A_{22}(1 - \frac{\zeta}{R_\beta}) u_\beta^{(o)} - A_{22}\frac{u_\zeta}{R_\beta} - B_{22} u^o_{\zeta,\beta\beta} + (\delta_F B_{22} + B^a_{22} - \delta_T \frac{4}{3h^2} E_{22}) \gamma_{\beta,\beta}$$

$$M_{\beta\beta} = B_{22}(1 - \frac{\zeta}{R_\beta}) u_\beta^{(o)} - B_{22}\frac{u_\zeta}{R_\beta} - D_{22} u^o_{\zeta,\beta\beta} + (\delta_F D_{22} + D^a_{22} - \delta_T \frac{4}{3h^2} F_{22}) \gamma_{\beta,\beta}$$

$$P_{\beta\beta}=E_{22}(1-\frac{\zeta}{R_\beta})u_\beta^{(o)}-E_{22}\frac{u_\zeta}{R_\beta}-F_{22}w^o_{\zeta,\beta\beta}+(\delta_F F_{22}+F^a_{22}-\delta_T\frac{4}{3h^2}H_{22})\gamma_{\beta,\beta}$$

$$Q_\beta=(\delta_F D_{55}-\delta_T\frac{4}{h^2}F_{55}+D^c_{45}+D^b_{55})\gamma_\beta$$

$$T_\beta=(\delta_F A_{55}-\delta_T\frac{4}{h^2}D_{55}+A^c_{45}+A^a_{55})\gamma_\beta$$

$$M2^a_{\beta\beta}=\delta_A\{D^a_{22}2\,u^o_{\beta,\beta}-\varepsilon^a_{22}2\,u^o_{\zeta,\beta\beta}+[D^{aa}_{22}2+\varepsilon^a_{22}2+\varepsilon^{aa}_{22}2+F^a_{22}2+F^{aa}_{22}2]\,\gamma_{\beta,\beta}\}$$
$$M3^a_{\beta\beta}=\delta_A\{D^a_{22}3\,u^o_{\beta,\beta}-\varepsilon^a_{22}3\,u^o_{\zeta,\beta\beta}+[D^{aa}_{22}3+\varepsilon^a_{22}3+\varepsilon^{aa}_{22}2+F^a_{22}3+F^{aa}_{22}3]\,\gamma_{\beta,\beta}\}$$
$$Q1^a_{\beta\beta}=\delta_A[B^a_{44}2+B^{aa}_{44}2+D^a_{44}2+\varepsilon^a_{44}2+F^{aa}_{44}2]\,\gamma_{\beta,\beta}$$
$$Q2^a_{\beta\beta}=\delta_A[B^a_{44}3+B^{aa}_{44}3+D^a_{44}3+\varepsilon^a_{44}3+F^{aa}_{44}3]\,\gamma_{\beta,\beta}$$

The expressions of ${M_{\beta\beta}}^a$ and ${Q_\beta}^a$ appearing into equilibrium equations that are not written here follows from those of $M_{\beta\beta}$ and Q_β by adding the superscript a to the stiffnesses quantities involved.

Substituting previous expressions for averaged resultants into equilibrium Eqs. (14) and (16) we obtain the corresponding ones in terms of generalized coordinates. Once equilibrium equations are in this form, analytical approaches, such as the Galerkin and Rayleigh-Ritz's methods, can be used.

6. Numerical results

In general, exact solution is not possible, even for the simple two-dimensional case. Nevertheless, in the special case of simply-supported cylindrical shells with a symmetric or antisymmetric cross-ply lamination scheme, exact solution can be found. Provided that the transverse load $\bar{q}_\zeta$ is distributed sinusoidally, exact solution is obtained by assuming appropriate sine and cosine distributions of generalized displacements. As mentioned earlier, for the cylindrical bending problem the exact three-dimensional solution is available, Ren (1987). Hence, the problem of cross-ply, simply-supported cylindrical shells in cylindrical bending can be used as a benchmark problem for testing the effective capability of shell models to predict stress distributions and displacements. This allows us to test contempraneously the validity of the displacement model used and the limits of simplfying assumptions made in deriving governing equations, in particular, of approximations made in retaining selected terms where powers of various order of ζ/R_β appears.

In addition, we can compare present shell formulation with that of Dennis and Palazotto (1991), where selected terms of ζ/R_β are retained, and to that of Xavier *et al.* (1993), where all terms are retained in a consistent way, the generalized displacements assumed being the same of the present shell model.

Comparison with the exact three-dimensional solution also allows us to evaluate if a non-constant transverse displacement u_ζ and the transverse normal stress $\sigma_{\zeta\zeta}$ which are neglected in shell theories, should be or not included.

6.1. Benchmark problems examined

Let us assume a cylindrical shell having an angle substented by the ends ψ equal to $\pi/3$;

the β coordinate then traces a circumferential path of length $R_{\beta}\psi$. The problem investigated is that of deflections and through-the-thickness stress distributions of simply-supported, cylindrical shell panels in cylindrical bending under a sinusoidal distributed loading

$$\bar{q}_{\zeta}=q^{0}\sin(\frac{\pi\beta}{\psi})$$

that acts at the upper surface. The benchmark problems examined are: (i) the cylindrical bending problem of a single-layer (90°) orthotropic shell;

(ii) the cylindrical bending problem of a two-layer (0°/90°) antisymmetric shell, i.e., with fibers parallel to the circumferential direction β and perpendicular to it, respectively in the top and bottom layers;

(iii) the cylindrical bending problem of a three-layer (90°/0°/90°) shell.

The costituent layers are assumed to have equal thickness. Conforming to Ren (1987), their material properties are:

$$\frac{E_L}{E_T}=25;\ \frac{G_{LT}}{E_T}=0.5;\ \frac{G_{TT}}{E_T}=0.2;\ \nu_{LT}=0.25$$

L and T being used to indicate the direction of fibers and that perpendicular to it.

The simply supported boundary conditions at the ends, i.e., β=0, ψ, write:

$$u_{\zeta}=N_{\beta\beta}=M_{\beta\beta}=M^{a}_{\beta\beta}=M2^{a}_{\beta\beta}=M3^{a}_{\beta\beta}=P_{\beta\beta}=P^{a}_{\beta\beta}=0$$

that means:

$$u^{(0)}_{\zeta}=u^{(0)}_{\zeta},_{\beta\beta}=0;\ u^{(0)}_{\beta}=\gamma_{\beta}=free$$

Exact solution to equilibrium equations under previous boundary conditions is obtained by assuming the following expressions for the three generalized displacements:

$$u^{(0)}_{\beta}=A^{u\beta}\cos(\frac{\pi\beta}{\psi})$$

$$u^{(0)}_{\zeta}=A^{u\zeta}\sin(\frac{\pi\beta}{\psi})$$

$$\gamma_{\beta}=A^{\gamma\beta}\cos(\frac{\pi\beta}{\psi})$$

Substitution of the above into the equilibrium equations yields algebraic equations in terms of the unknown constants $A^{u\beta}$, $A^{u\zeta}$, $A^{\gamma\beta}$.

With a suitable choice of tracer operators δ_A, δ_F, δ_T, previous displacement field also gives the solution to the shell models that can be particularized in the present one. Here below numerical results will be given only for higher-order ones of these models, i.e., for the RHSDST, RFSDST, HSDST models.

It is worthwhile to note that the HSDST displacement model coincides with that of Reddy and Liu (1985) and Dennis and Palazotto (1991), whereas corresponding shell models little differ, due to different approximations made for h/R_{β} terms. Comparison results to the HSDST shell model, in addition to those of Dennis *et al.* (1991), are found in Xavier *et al.* (1993), together with results for RFSDST shell model, which have been obtained retaining in a consistent way all the first-order terms h/R_β. This allows us to assess accuracy of some different shell displacement models. Accordingly with Ren. (1987) and Xavier *et al.* (1993), through-the-thickness flexural and transverse shear stresses are presented in the following nondimensional form:

$$\bar{\sigma}_{\beta\beta}=\frac{\sigma_{\beta\beta}(\zeta,\frac{\psi}{2})}{q^0S^2};\ \bar{\sigma}_{\beta\zeta}=\frac{\sigma_{\beta\beta}(\zeta,0)}{q^0S}$$

where $S=R_\beta/h$, these stresses assuming their maximum values at $\beta=\psi/2$ and $\beta=0$, respectively. Deflections are normalized as follows:

$$\bar{u}_\zeta=10\frac{E_T u_\zeta(0,\frac{\psi}{2})}{q^0hS^4}$$

6.2. Numerical illustrations

The influence of kinematic simplifying assumptions made being greater for thick shell, as shown by the comparison of results of Dennis and Palazotto (1991), Xavier, *et al.* (1993) with the exact solution of Ren (1987), the main body of numerical results for stress distributions given here concerns shells with a radius-to-thickness ratio $S=R_\beta/h$ equal to 4. Some results for S=2 and S=10 are also presented for comparison.

For the case of a single layer, stress distributions are not reported here for sake of brevity. This comply with the object of the present research to assess validity of the developed model for the analysis of multilayered shells, all models contained in the present formulation reducing to the FSDT and HSDT models for the (90^0) lay-up, then with the effect of continuities disappearing. Thus the different response of the various shell models for (90^0) shells is here compared only for deflections. Whereas distributions are limited to S=4, deflections are investigated for S ranging from 2 to 500. They are reported in Tables 2, 3, 4.

The following observations are made concerning deflections. In the case of a single-layer orthotropic shell of Table 2, the effect of continuities disappears and the RHSDST and HSDST models coincide; in the same way, the RFSDST and FSDST shell models coincide. In this case, comparing the present results with the exact solution of Ren (1987), the two-dimensional results of the shell model of Xavier *et al.* (1993) and with the results of shell models reported there, allows us to observe that, excepted for extremely thick shells, i.e., with S=2 all models compared provide accurate enough predictions. Practically, the influence of the displacement model used seems to disappears for shells with $S\leq 10$. For $S<10$ rather different predictions are provided by the varios shell models compared, with discrepancies increasing with decreasing S. In particular, comparison of linear and cubic models shows the differences are significant for S=2, 4, whereas they disappears for thin shells.

The comparison of present results for the RFSDST shell model with the corresponding ones of Xavier *et al.* (1993) give us some insights on the effect of h/R_β terms retained. For the case of a single-layer orthotropic shell, a little influence of these terms is shown.

Table 3 refers to $(90^\circ/0^\circ/90^\circ)$ layered shells. It allows us to assess the effect played by continuities of transverse shear stresses at interfaces. Basically, previous observations are substantially confirmed for thin shells, except for the fact that discrepancies which are greater

Table 2 Non-dimensional maximum deflection $\bar{u}_\zeta = \frac{10E_T u_\zeta^0(0,\psi/2)}{q^0 h S^4}$ in $(90^0/0^0/90^0)$ cylindrical laminated shell, as predicted by various shell models

R/h	Dennis *et al.* (1991) (f.e.m.)	Xavier *et al.* (1993)	HSDST (present)	RFSDST (present)	RHSDST (present)	IRHSDST (present)	Exact Ren (1987)
2	1.141 (1.1686)	1.0878	1.1133	1.2382 (1.3763)	1.2401	1.3394	1.436
4	0.382 (0.3847)	0.4410	0.3817	0.4224 (0.4377)	0.4450	0.4478	0.457
10	0.128 (0.1287)	0.1426	0.1270	0.1341 (0.1369)	0.1416	0.1422	0.144
50	0.0796 (0.0798)	0.0810	0.0799	0.0801 (0.0807)	0.0804	0.0804	0.0808
100	0.0781 (0.0783)	0.0788	0.0786	0.0786 (0.0787)	0.0787	0.0787	0.0787
500	0.0774 (0.0778)	0.0779	0.0773	0.0773 (0.0779)	0.0773	0.0773	0.0773

Results in brackets are obtained from Reddy and Liu (1985).

than in the case of single-layer shells continues to appear until S=50. It is observed that the accuracy of piecewise third-order models is greater of that of piecewise linear models, and that the effect of continuities, as shown by comparing the HSDST shell model with the RFSDST, RHSDST and IRHSDST models is consistent. Comparisons of results of the third-order zig-zag shell model of Xavier *et al.* (1993) (with all terms h/R_β retained in a consistent way) with the present ones of the RHSDST and IRHSDST models and with those of the linear zig-zag model of Xavier *et al.* (1993) with the present ones of the RFSDST model focuse a greater importance played by h/R_β terms with respect the single-layer shell.

It is observed that none of the models compared can predict deflections with a great

Table 3 Non-dimensional maximum deflection $\bar{u}_\zeta = \frac{10E_T u_\zeta^0(0,\psi/2)}{q^0 h S^4}$ in $(0^0/90^0)$ cylindrical laminated shell, as predicted by various shell models

R/h	Model of Dennis *et al.* (1991)	Xavier *et al.* (1993)	HSDST (present)	RFSDST (present)	RHSDST (present)	IRHSDST (present)	Exact Ren (1987)
2	– (1.441)	1.534	1.4186	1.3312 (1.433)	1.4280	1.4973	2.079
4	– (0.6993)	0.7196	0.6659	0.5850 (0.6998)	0.6168	0.7013	0.854
10	– (0.4593)	0.4730	0.4339	0.4206 (0.4705)	0.4432	0.4673	0.493
50	– (0.4048)	0.4090	0.3937	0.3931 (0.4089)	0.3945	0.4024	0.409
100	– (0.4018)	0.4039	0.3925	0.3923 (0.4038)	0.3962	0.4015	0.403
500	– (0.3998)	0.4002	0.3921	0.3922 (0.4002)	0.3968	0.3988	0.399

Results in brackets are obtained from Reddy and Liu (1985).

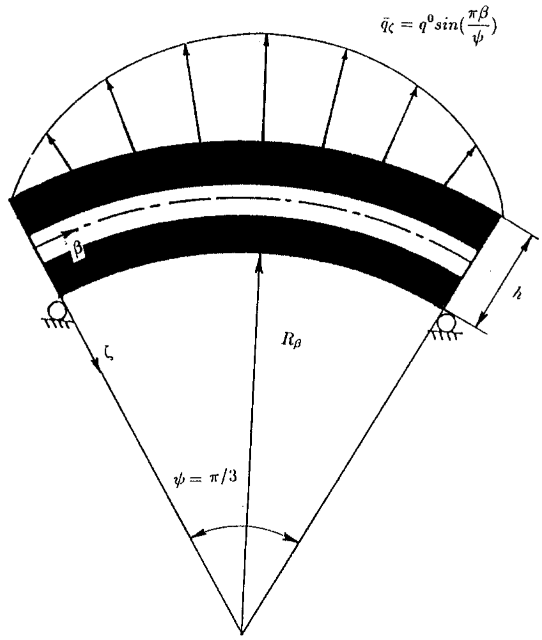

Fig. 1 Laminated shell geometry and loading

accuracy when shells are thick. This could be due either to neglected h/R_β terms, or to use of a constant u_ζ through-the-thickness. It is in the author's opinion that further research work should be done toward this direction, to clarify what terms should be included for thick shells.

Discrepancies among the predictions of the models considered and the exact solution, when shells are thick, are presented in Figs. 2 through 4, relative to shells with S=4. In the remaining figures, results for transverse shear stresses are reported either as given by constitutive equations or by integrating the local differential equilibrium equations. In the last case, it is observed that all models improve consistently their predictions, which otherwise are rather poor[6].

Figs. 2 and 3 gives the through-the-thickness distribution of the transverse shear stress $\sigma_{\beta\zeta}$, as predicted by integrating the first of local equilibrium equations $\sigma_{\beta\beta,\beta}+\sigma_{\beta\zeta,\zeta}=0$ and from constitutive equations, respectively. From Fig. 2 it is observed the importance of satisfying continuity conditions at interfaces, since the IRHSDST, RHSDST, RFSDST shell models give better estimated than the HSDST shell model. Also observed is the importance of including higher-order terms of ζ in the displacement model, as shown by the comparison of IRHSDST and RHSDST shell models with the RFSDST shell model. Inclusion of ζ^2 terms in the IRHSDST model improves consistently predictions, as it results in a more precise prediction of unsymmetric shear stresses, as those of shells are, with respect to the predictions of RHSDST, RFSDST and HSDST models.

When considering estimates from constitutive equations, as given in Fig. 3, only the IRHSDST and RHSDST shell models provide acceptable results. Fig. 4 shows the through-

[6] It is worthwhile to note, in particular, that the HSDT model predicts transverse shear stresses discontinuous at interfaces from constitutive equations, whereas these stresses are continuous when predicted by integrating local equilibrium equations.

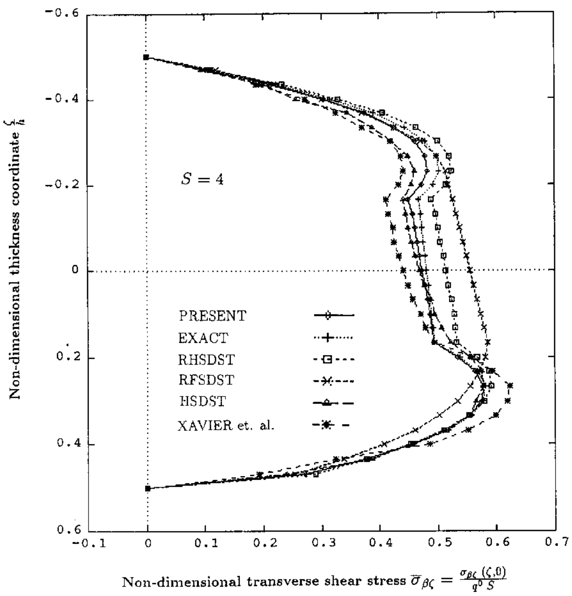

Fig. 2 Through-the-thickness distribution of transverse shear stress in a (90°/0°/90°) cylindrical laminated thick shell in cylindrical bending under sinusoidal loading, as predicted by integrating local differential equilibrium equations

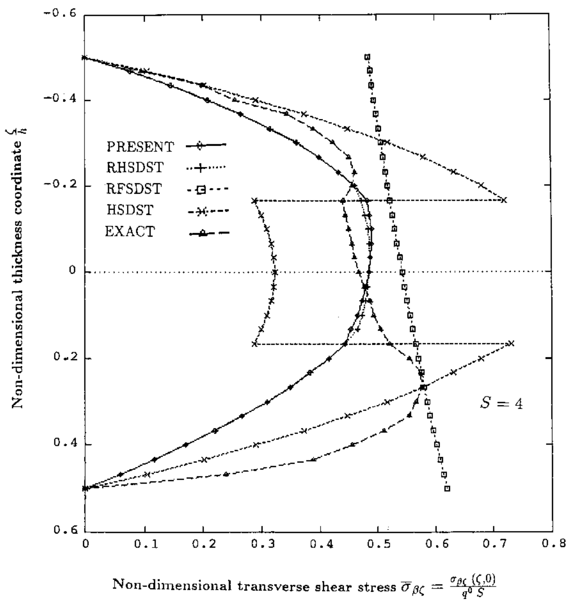

Fig. 3 Through-the-thickness distribution of transverse shear stress in a (90°/0°/90°) cylindrical laminated thick shell in cylindrical bending under sinusoidal loading, as predicted from constitutive equations

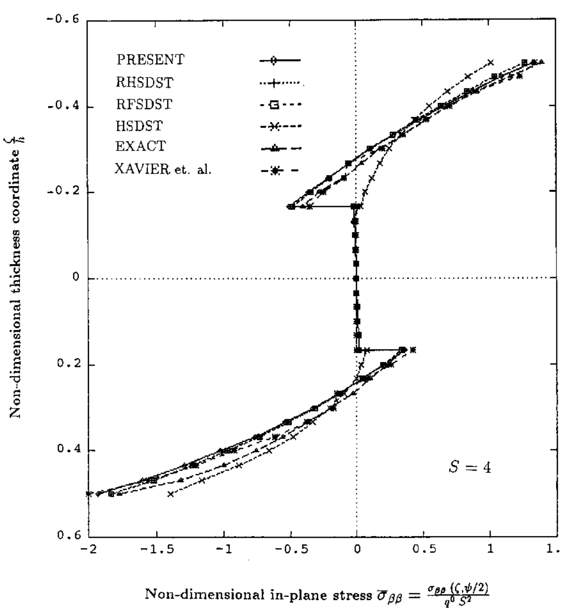

Fig. 4 Through-the-thickness distribution of in-plane stress in a (90°/0°/90°) cylindrical laminated thick shell in cylindrcal bending under sinusoidal loading

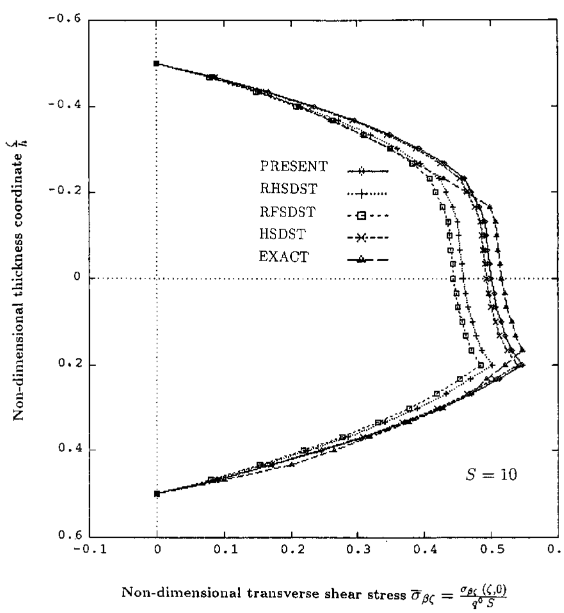

Fig. 5 Through-the-thickness distribution of transverse shear stress in a (90°/0°/90°) cylindrical laminated moderately thick shell in cylindrical bending under sinuoidal loading, as predicted by integrating local differential equilibrium equations

the-thickness distribution of the inplane stress $\sigma_{\beta\beta}$ which confirms observations made in Fig. 2.

The capability of all the shell models considered to provide accurate estimates when shells are moderately thick is shown in Figs. 5 through 7, relative to shells with S=10. Fig. 5 and 6 give the distribution of the transverse shear stress, as predicted by integrating equilibrium equations and by constitutive equations, respectively. Fig. 7 gives the distribution of the in-plane stress. In such a case, discrepancies among shell models reduce considerably.

Examination of Figs. 2 to 7 allows us to conclude that accurate enough predictions are obtained only by shell models accounting for the stress continuities at interfaces. It is observed that these models are the only ones giving acceptable predictions when transverse shear stresses are computed by constitutive equations. It is also observed their accuracy in predicting $\sigma_{\beta\beta}$: this is the reason for accurate predictions of $\sigma_{\beta\zeta}$, the transverse shear stress being obtained by integrating $\sigma_{\beta\beta}$ in ζ.

Confirming results for deflections of Table 2, it is observed that all the shell models here compared seem inadeguate to accurately predict stresses of (90^0/0^0/90^0) thick shells.

Table 4 refers to two-layered (0°/90°) antisymmetric shells. This case allows us to compare accuracy of the IRHSDST shell model, specifically developed to account for unsymmetry in the lay-up (i.e., satisfying the zero stress conditions at free surfaces irrespective the lay-up might be), to other models that are obtained from it that are unable to satisfy these stress-free conditions. A cursory examination of the results of Table 4 allows us to conclude that, in general, the effect of zero stress conditions on deflections is greater in the case of unsymmetric shells. Thus, the shell models suited for shells with general lay-up are those including a series

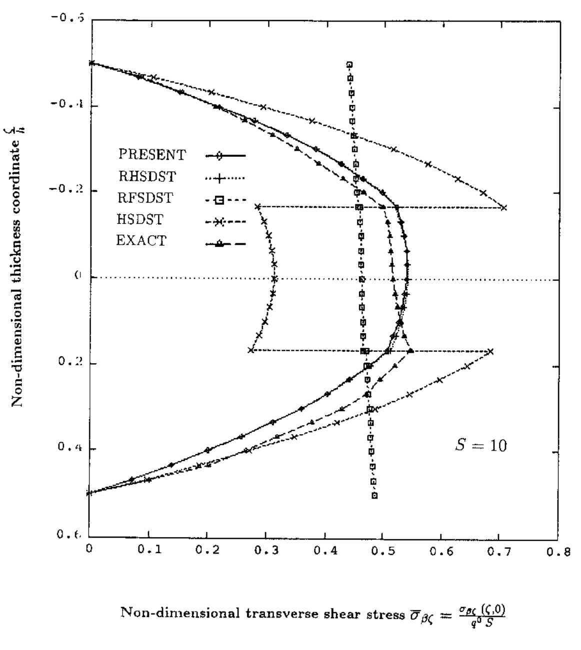

Fig. 6 Through-the-thickness distribution of transverse shear stress in a (90°/0°/90°) cylindrical laminated moderately thick shell in cylindrical bending under sinusoidal loading, as predicted from constitutive equations

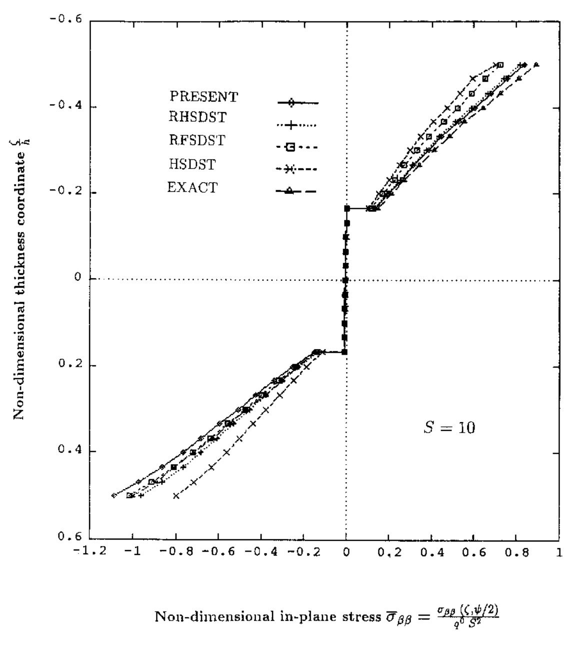

Fig. 7 Through-the-thickness distribution of in-plane stress in a (90°/0°/90°) cylindrical laminated moderately thick shell in cylindrical bending under sinusoidal loading

expansion of in-plane displacements comprising odd and even powers of ζ. This conclusion is substantiated by results of the shell model developed by Xavier *et al.* (1993).

When shells are thick, as previously observed for symmetric laminated shells, but with a great degree, all the models examined cannot provide very accurate predictions, though the IRHSDST shell model gives better estimates than the other models. This is the conclusion that is drawn from Figs. 8 through 10, which shows the transverse shear stress predicted by integrating local equilibrium equations and from constitutive equations, and the in-plane stress for $(0^o/90^o)$ shells with S=4. The same behavior of various shell models discussed previously for $(90^0/0^0/90^0)$ is still observed. In particular IRHSDST and RHSDST shell models appears superior to other models examined; inclusion of ζ^2 terms in the IRHSDST model improves accuracy. Results from constitutive equations show great errors throughout the thickness, irrespective the model. The models which were unable to satisfy the zero stress conditions at the upper and lower bounding surfaces pay a high penalty when the lay-up becomes unsymmetric, as can be seen in Fig. 9.

From the previous results it is concluded that further developments are required for thick shells and shells with unsymmetric lay-up.

7. Concluding remarks

A third-order zig-zag shell model has been developed for the bending of multilayered

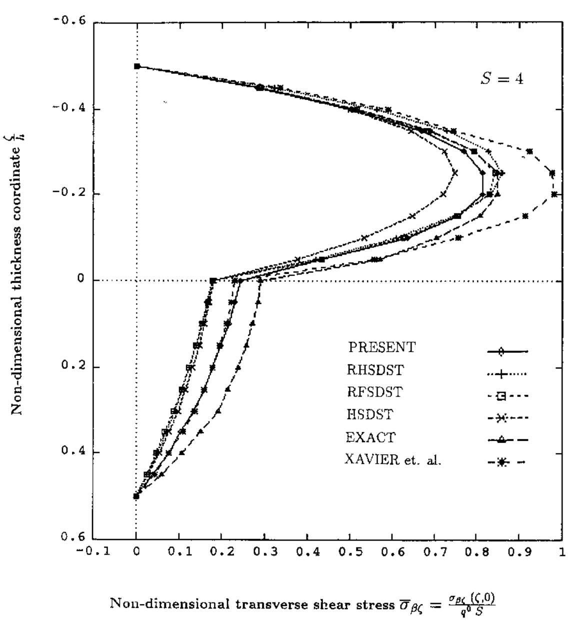

Fig. 8 Through-the-thickness distribution of transverse shear stress in a $(0^o/90^o)$ cylindrical laminated thick shell in cylindrical bending under sinusoidal loading, as predicted by integrating local differential equilibrium equations

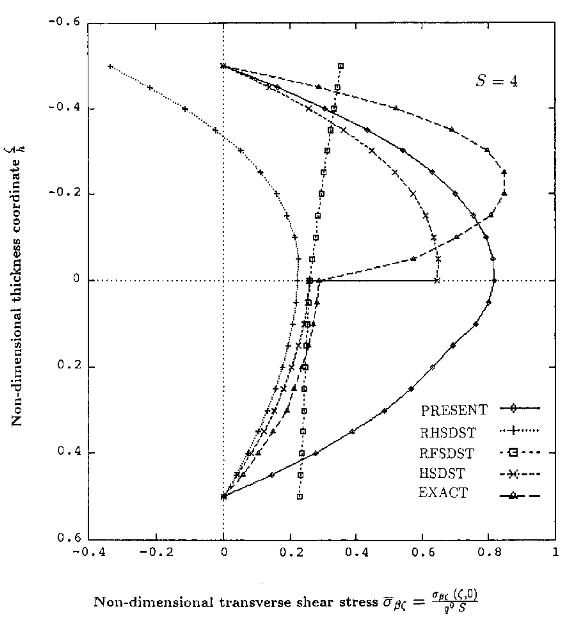

Fig. 9 Through-the-thickness distribution of transverse shear stress in a $(0^o/90^o)$ cylindrical laminated thick shell in cylindrical bending under sinusoidal loading, as predicted from constitutive equations

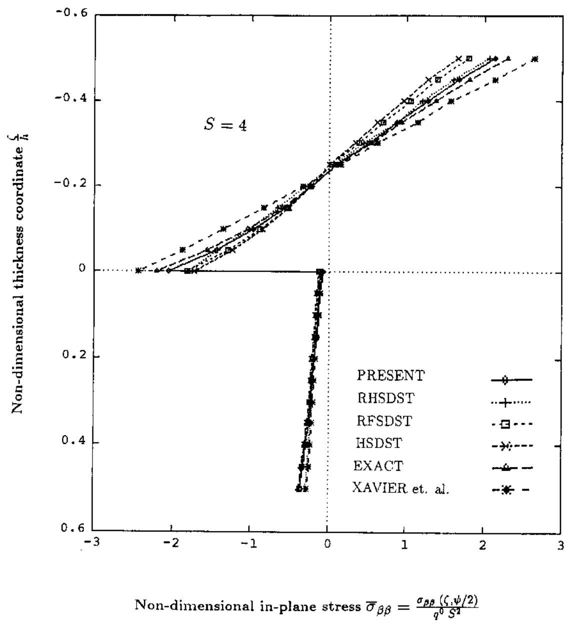

Fig. 10 Through-the-thickness distribution of in-plane stress in a (0°/90°) cylindrical laminated thick shell in cylindrical bending under sinusoidal loading

composite thick shells. As for any zig-zag model, tranverse shear stresses continuity at interfaces is met a priori. Furthermore, the model is able to satisfy the zero transverse shear stress conditions at the upper and lower free surfaces of the shell, irrespective for the lay-up. Some lower-order models can be particularized from it and used for comparisons.

Analytical and numerical application has been limited to cylindrical shells in cylindrical bending, the exact three-dimensional solution being available in this case for certain lay-ups and radius-to-thickness ratios, together with results of some different shell models. Comparison of present results with the exact solution and with approximate solutions of other shell models illustrates the effects of transverse shear modeling and the implications of retaining only selected terms containing the radius-to-thickness ratio.

The following conclusions were found. For homogeneous orthotropic and three-layer symmetric cross-ply shells, present assumptions are proved to be accurate both for deflections and through-the-thickness stress distributions. Inclusion of zig-zag layerwise kinematics, which appears essential for obtaining accurate prediction of stress distributions, also improves prediction of deflections. Inclusion of both even and odd terms into the displacement model allows for a more accurate prediction of the transverse shear stress and, consequently, of deflections. However, none of the shell models compared appears to be acurate when shells are very thick.

For antisymmetric two-layer cross-ply shells, previous conclusions still holds, but agreement with exact solution deviates. In this case, only shell models accounting for the zero stress conditions at free surfaces provide satisfactory predictions of deflections when shells are thick. The present shell model gives better estimates than other models examined, but errors are still large. For all the shell models examined the main discrepancy with the exact solution is shown at the interface, but also high errors are shown everywhere through the thickness. In view of this, future research work should be addressed either to include higher-order h/R_β terms, to improve displacement models by including higher order expansions, or to include the effect of transverse normal deformability.

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