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On boundary discretization and integration in frequency-domain boundary element method

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Abstract. The computation size and accuracy in the boundary element method are mutually coupled and strongly influenced by the formulations in boundary discretization and integration. This aspect is studied numerically for two-dimensional elastodynamic problems in the frequency-domain. The localized nature of error is observed in the computed results. A boundary discretization criterion is examined. The numer of integration points in the boundary integration is studied to find the optimum number for accuracy. Useful information is obtained concerning the optimization in boundary discretization and integration.

Key words: boundary element method; boundary discretization; element integration.

1. Introduction

Although the computation error in the boundary element method (BEM) can not be totally eliminated, it can be reduced to a certain level with elaborate treatments in boundary discretization and integration. This, however, increases the computation size and thus the optimum degree of elaboration should be defined in view of computation size and accuracy.

In the last decade, the self-adaptive technique has been developed to estimate the computation error in the BEM analysis. The nature of multi-moment and multi-frequency in elastodynamics, however, generally makes such posterior error estimation very difficult. Thus, more studies on this technique are needed for practical use at the present time. Meanwhile, it is simple and convenient to establish the boundary discretization criterion in a manner that is often used in finite element analysis. Du and Xiong (1987) conducted the numerical study on impedance of a cylindrical rigid body in an infinite medium, in order to define a discretization criterion for two-dimensional linear problems in this manner. Such a criterion is dependent on element type and error evaluation index but neither were a part of their investigation. In the process of constructing the coefficient matrices in BEM, signular and non-sigular boundary integrations are performed. There are various spacial ways of evaluating singular boundary integration and the choice depends on a fundamental solution, boundary geometry and element type (Hall 1988). For non-singular element integration, Barnejee and Ahmad (1985) investigated the accuracy numerically in three-dimensional elastodynamic problems and concluded that 3×3, 4×4 and 5×5 Gauss-Legendre integration rules could provide the best combination of computation accuracy and size.

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In this study they examined directly the computed displacements rather than the numerical integration rule itself.

The present paper discusses errors which result from boundary discretization and integration, in two-dimensional elastodynamic frequency-domain BEM analyses. A specific element type and numerical integration rule are used in the discussion.

2. Element type and integration rule used

A constant element was used first in BEM. Lachat and Watson (1976) introduced isoparametric transformation in boundary elements. Since then, high-order elements (interpolation functions of higher order) have been widely used, beacuse they can generally describe the actual deformations within an element better and thus provide better accuracy. However, excessively high order elements make the computation effort excessive. It is generally considered that quadratic or cubic boundary elements are the best choices in view of the computation effort and accuracy. This has been demonstrated by Seabra Pereira, *et al.* (1981) in elastostatics problems, Brebbia, *et al.* (1984) in the wave potential problems, and Dominguez and Gellege (1989) in the time domain elastodynamics problems. Therefore, quadratic boundary elements are used herein.

There are two quadratures available for performing non-singular integration over boundary elements. In the Newton-Cotes integration rule, integration points are uniformly distributed over the integrated region. With the aid of this rule, and arbitrary polynomial of order n can be integrated accurately when the number of integration points n is an odd number. When n is an even number, the highest order of the polynomial integrated accurately is $n-1$. In the Gauss-Legendre integration rule with non-uniformly distributed integration points, the highest order of an arbitrary polynomial integrated accurately can reach $2n$-1 (Chan 1991). The Gauss-Legendre rule is generally more efficient in element integration than the other and thus is used herein.

3. Boundary discretization

A circular cavity, in an infinite homogeneous elastic medium, is considered. A harmonic plane wave with a unit amplitude is assumed to propagate in the positive direction along the x axis as shown in Fig. 1. The boundary is uniformly discretized with quadratic elements. The singularity in the boundary integration is obviated by collocating the source point outside of the

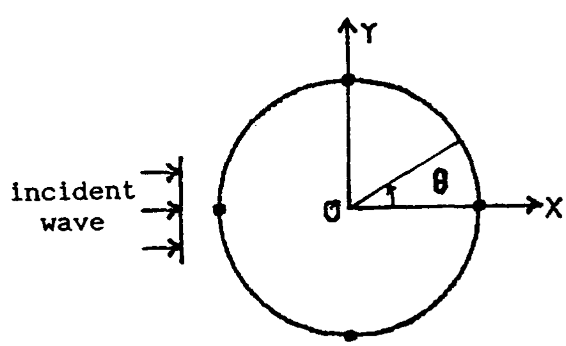

Fig. 1 A circular cavity under incident harmonic plane wave.

Table 1 Errors in computed displacement amplitudes (%)

	SH wave									P wave								
a_0	5			10			15			5			10			15		
$\lambda/\Delta x$	3.2	3.6	4.0	3.2	3.6	4.0	3.2	3.6	4.0	3.2	3.6	4.0	3.2	3.6	4.0	3.2	3.6	4.0
0°	0.8	0.6	0.5	1.0	0.7	0.5	0.6	0.4	0.3	6.2	3.8	0.2	5.2	1.2	0.6	2.0	0.8	0.2
90°	2.7	1.7	2.5	4.4	3.6	2.9	4.8	2.4	3.0	1.4	5.8	3.1	6.5	3.2	2.4	5.4	1.3	1.3
180°	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	0.1	4.7	0.2	2.3	9.6	2.6	0.4	3.7	1.1	0.5
270°	2.7	1.7	2.5	4.4	3.6	2.9	4.8	2.4	3.0	1.4	5.8	3.1	6.5	3.2	2.4	5.4	1.3	1.3

region being studied in two-dimensional elastodynamic problems. Each exterior source point is placed at a distance from its corresponding node, equal to 1.5 times the adjacent node distance (Fu 1995). The Gauss-Legendre rule is applied with 12 integration points. The integration error in this case is, as demonstrated later, very small and thus the error in the computed boundary displacement is reasonably assumed to be due to the discretization.

For these conditions, the boundary displacements in scattering field are calculated by using BEM for SH wave and P wave incidences. Table 1 lists errors in percentage for the calculated boundary displacement amplitudes, in which Δx and λ are respectively the dimension of quadratic element and the wavelength of shear wave in the medium, and a_0 is the dimensionless frequency defined as

$$a_0 = \frac{\omega r}{C_s} = \frac{2\pi r}{\lambda} \tag{1}$$

where r is the radius of the cavity, ω is the circular frequency of the incident wave, and C_s, is the shear wave velocity of the medium. For P wave incidences, errors at θ=0° and 180° are computed for horizontal displacements U_x, whereas errors at θ=90° and 270° are computed for vertical displacements U_y. The table shows two clear trends. First, the computation error generally decreases with an increasing value of $\lambda/\Delta x$. The reason for this decrease is esaily understood. Second, the values of error are not uniform along the boundary despite uniform discretization along the boundary; being smallest at θ=0° and 180° but largest at θ=90° and 270°. This indicates that the errors at two positions on the uniformly discretized boundary are different from each other, if the local incident angles with respect to the boundary at these points are mutually different. Namely, errors in the computed results exhibit the localized nature. A simple and direct explanation for this phenomenon may resort to a quadratic element on which specific boundary value distribution is introduced by the plane incident wave. The displacements distributed over the element are uniform when the local incident angle is normal to the boundary, whereas they are described by a trigonometric function when the local incident angle is tangent to the boundary. The quadratic interpolation functions can reproduce exactly the former distribution but can not reproduce exactly the latter distribution. This results in the above localized nature of computation error.

The general criteria often used in discretization is

$$\frac{\lambda}{\Delta x} \geq 4 \tag{2}$$

which states that the length of the quadratic boundary element is less than or equal to 1/4 of the shear wave length. As observed in Table 1, if this criteria is adopted, errors in the computed boundary displacement are less than 5% even with the localized nature of errors. This magnitude of computational error is acceptable in practical engineering analysis.

4. Element integration

In element integration, a general form of the integrand is $F^*_{ij}\ \phi_m\ |J\ |$ in which F^*_{ij} is the com-

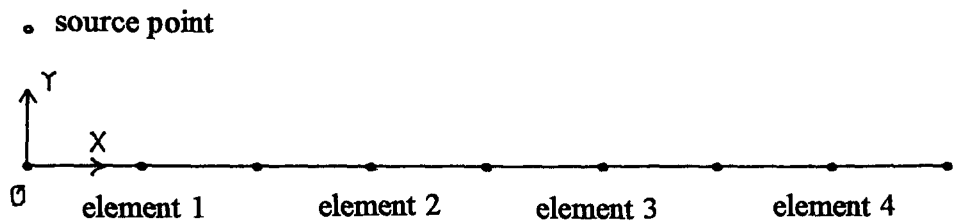

Fig. 2 A straight line boundary discretized by quadratic elements.

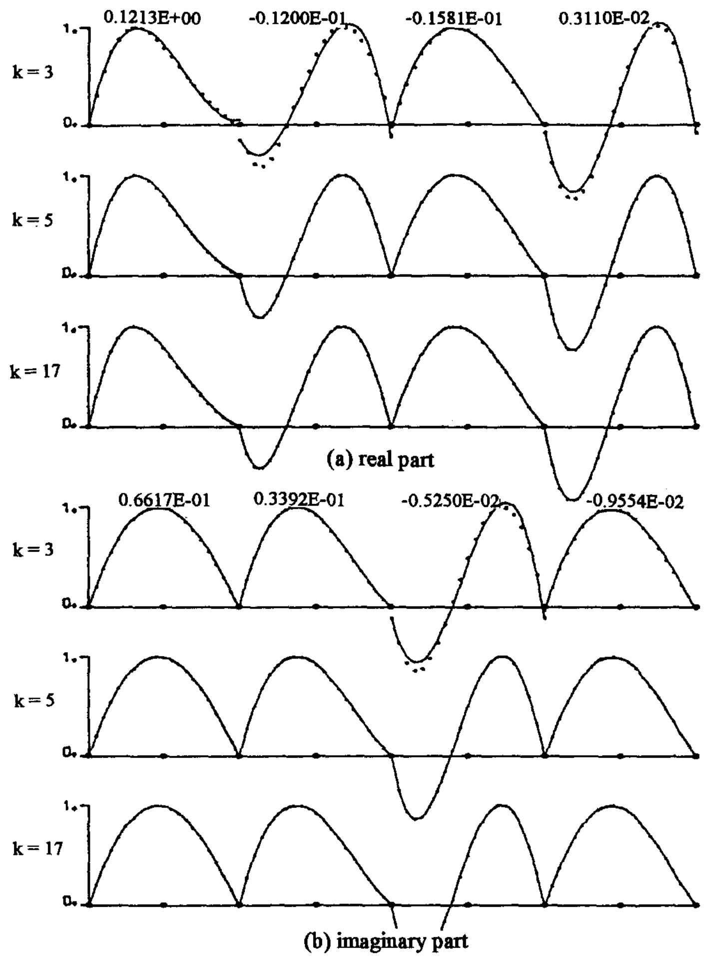

Fig. 3 Variation of $P^*_{33}\ \phi_3 |J\ |$ over element for 2-D anti-plane motion.

ponent of the fundamental solution U_{ij}^* (displacement) or P_{ij}^* (stress), ϕ_m is the standard interpolation function in quadratic element, and $|J|$ is Jacobian. The behavior of an integrand within an element is described by a polynomial of a certain order which is defined by curve fitting. Then, based on the numerical integration rule, the number of integration points in the element is determined. "Integration error" was caused by the inadequate precision in element integration and thus is influenced directly by the number of integration points each element.

A simple straight line boundary as shown in Fig. 2 is considered first. The length of each quadratic element along the boundary is equal to $\lambda/4$ and the source point is located at a distance equal to $\lambda/8$ from the boundary in order to consider the influence of the singularity of the fundamental solutions. For two-dimensional anti-plane motion, the variation of the integrand F_{33}^* $\phi_3|J|$ over elements and the corresponding fitting curves are shown in Fig. 3. The dots in the figures are the values computed directly from the integrand at 21 points in each element, whereas the solid lines show the computed values by the polynomial of order k which is defined by curve fitting. Both values are normalized by the common largest value in each element,

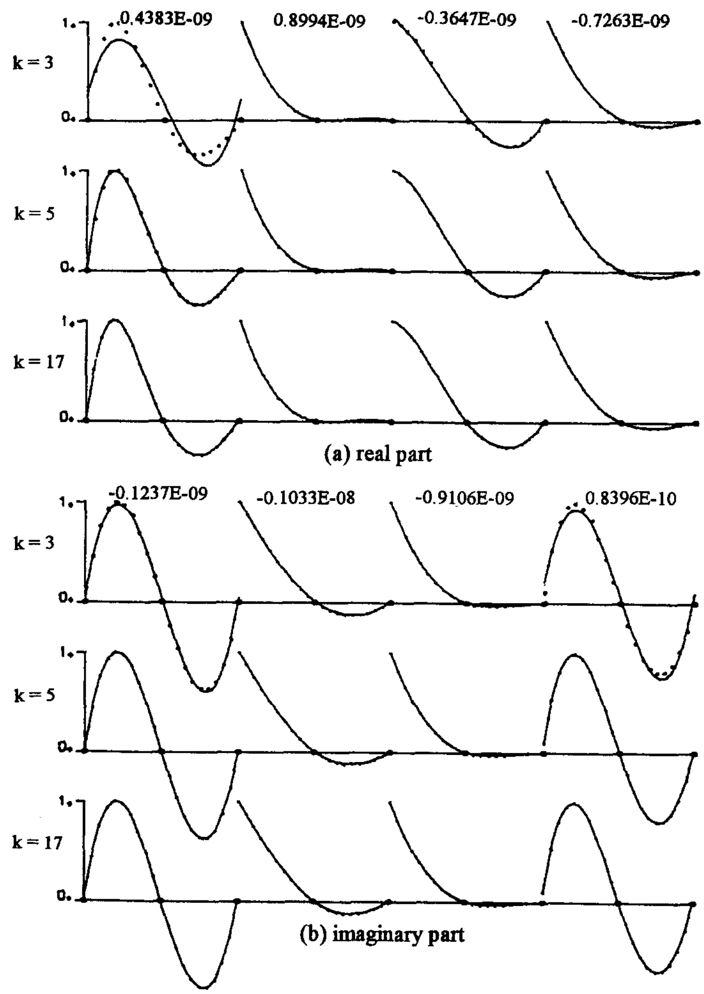

Fig. 4 Variation of U_{12}^* $\phi_1|J|$ over element for 2-D anti-plane motion.

Table 2 Errors in computed displacement amplitudes (%)

		SH wave				P wave			
θ		0°	90°	180°	270°	0°	90°	180°	270°
n	2	9914807	22047436	796212	2205001	2346970	68561639	29015324	68561639
	3	5.2	5.9	5.5	5.9	5.6	5.5	4.7	5.5
	4	0.8	2.8	0.4	2.8	0.1	3.4	4.3	3.3
	5	0.4	3.0	0.1	3.0	0.6	2.3	0.5	2.3
	6	0.5	2.9	0.1	2.9	0.5	2.3	0.4	2.3
	12	0.5	2.9	0.1	2.9	0.6	2.4	0.4	2.4

which is computed directly from the integrand. Fig. 4 shows the integrand $F_{12}^* \phi_1 |J|$ for two-dimensional in-plane motion. Although the curve fitting is improved by increasing k from 2 to 5, it is hardly improved by increasing k from 5 to 17. Thus, the polynomial of order about 5 is adequate to accurately describe the integrand. The number of integration points, n, corresponding to k=5 is equal to 3 according to $n=(k+1)/2$.

The above finding is further studied to see if it is still applicable for a more complex case such as the one shown in Fig. 1. The boundary displacement amplitudes in the scattering field are computed for SH wave and P wave incidences by BEM. In the computation, a_0=10 and $\lambda/\Delta x$=4 are used and exterior source points are collocated at the same position as done earlier. Errors in percentage are listed in Table 2 for various values of n. It is seen in the table that, as n increases, errors decrease sharply to 5-6% from very large numbers at n=2 and then tend to become stationary after approximately n=5. Thus, this and previous results suggest that the optimum number of integration points an element is about 4.

5. Conclusions

Boundary discretization and integration are two important aspects which influence the computation size and accuracy in BEM. This is discussed systematically for two-dimensional elastodynamic problems in the frequency domain in the present paper.

The localized nature of error is observed in the computed displacements. Thus, the discretization error is related not only to the degree of boundary discretization but also to the local direction of the incident wave with respect to the boundary at the location where the displacement is observed. Along the uniformly discretized boundary, the computed error is smallest where the local direction of the incident wave is normal to the boundary, while it is largest where the local direction of incident wave is tangent to the boundary.

The error can be effectively controlled by regulating the boundary discretization mesh and the number of integration points in an element. If the length of each quadratic boundary element is less than 1/4 of shear wavelength in the medium, the error in the computed displacement amplitude is less than 5%. The recommended number of integration points in the boundary integration is 4 for efficient computation.

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