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Minimum stiffness of bracing for multi-column framed structures

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Abstract. A method that determines the minimum stiffness of baracing to achieve non-sway buckling conditions at a given story level of a multi-column elastic frame is proposed. Condensed equations that evaluate the required minimum stiffness of the lateral and torsional bracing are derived using the classical stability functions. The proposed method is applicable to elastic framed structures with rigid, semirigid, and simple connections. It is shown that the minimum stiffness of the bracing required by a multi-column system depends on: 1) the plan layout of the columns; 2) the variation in height and cross sectional properties among the columns; 3) the applied axial load pattern on the columns; 4) the lack of symmetry in the loading pattern, column layout, column sizes and heights that cause torsion-sway and its effects on the flexural bucking capacity; and 5) the flexural and torsional end restrains of the columns. The proposed method is limited to elastic framed structures with columns of doubly symmetrical cross section with their principal axes parallel to the global axes. However, it can be applied to inelastic structures when the non-linear behavior is concentrated at the end connections. The effects of axial deformations in beams and columns are neglected. Three examples are presented in detail to show the effectiveness of the proposed method.

Key words: buckling; bracing; building codes; columns; construction types; computer applications; frames; loads; P-Δ effects; reinforced concrete; seismic loads; steel; stability.

1. Introduction

The question of what constitutes bracing and how to design bracing in real framed structures is of major concern to designers since it has to do with their stability performance under working loading conditions, as well as with their integrity under extreme lateral loadings such as those caused by severe earthquakes and winds. Bracing can be divided into three major categories:

1) component bracing to avoid local or individual member buckling;
2) sub-system bracing to avoid excessive distortion in vertical or horizontal assemblages such as roofs and floor diaphragms; and
3) system bracing to prevent sidesway buckling of the structure as a whole and maintain the lateral stability of the structure, including overturning effects of drift caused by severe lateral loads.

Component and sub-system bracing may consist of cross tension members where the axial stiffness of the bracings is utilized; they may be provided at concentrated locations by other flexural members framing transversely to the member being braced, wherein both axial and flexural

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stiffnesses of the bracing members are utilized; or such bracings may be provided continuously by the wall panels to the columns or by the floor system to the compression flange of the beams. Steel construction codes offer adequate guidance on these two types of bracings (AISC versions ASD 1990, LRFD 1986 and 1994).

On the other hand, system bracing is intimately related to the interstory drift control, generally referred to as "lateral" stability. Guidance on the required stiffness and strength for story bracing for frames is not precise but rather indefinite. This task is generally left to the designer. For instance, a steel braced frame, according to AISC-LRFD-C2.1, is one in which "lateral stability is provided by diagonal bracing, shear walls or equivalent means." The vertical bracing system must be "adequate, as determined by structural analysis, to prevent buckling of the structure and maintain the lateral stability of the structure, including overturning effects of drift, ...". Details on diagonal bracing under tension and compression are given in Volume II (Connections) of the AISC-LRFD Manual (1994, Section 11). In reinforced concrete buildings, frames can be considered braced "when the stability index ..... for a story is not greater than 0.04, the $P-\Delta$ moments should not exceed 5 percent of the first-order moments ...." or "alternatively, ... if bracing elements (shearwalls, shear trusses, or other types of lateral bracing) have a total stiffness at least six times the sum of the stiffnesses of all the columns within the story." (ACI code version 318R-89 revised 1992, R10.11.2).

In addition, construction codes base the "lateral" stability design on simplified 2-D analyses (i. e., models obtained by breaking the structure into vertical plane frames) ignoring the real 3-D stability behavior. The two major effects on the stability behavior of framed structures, namely, the overall torsional-flexural coupling and coupling among the columns at a story level are ignored by most codes. Except for totally symmetrical frames, torsional-flexural buckling must be considered in the design of 3-D framed structures since the buckling loads can be signifincantly below the 2-D buckling flexural loads, whereas column coupling becomes important in frames with columns of different heights or/and under different axial loads and boundary conditions (Aristizabal-Ochoa 1995a-b).

The only objective of this paper is to present, using the classical stability functions (Timoshenko and Gere 1961), a formulation for the minimum stiffness of bracing required by an entire story of an elastic framed structure to achieve fully "braced" conditions. Design recommendations for strength, ductility, fatigue, structural details of their connections, etc. of any particular bracing configuration and material are beyond the scope of this paper, the proposed formulation can be applied to plane and space multi-column frames with rigid, semirigid and simple connections, but it is limited to elastic structures with doubly symmetrical columns with their principal cross-sectional axes parallel to the global axes. The effects of axial deformations in all members are neglected.

2. Structural model

The models of an entire story of a 2-D and a 3-D framed structure are shown in Figs. 1a and 1b, respectively. In both models it is assumed that the floor diaphragms are rigid in their own planes. This allows condensation of the lateral degrees of freedom (DOF) into one DOF per floor level in 2-D frames, and three DOF per floor level (two horizontal translations and a rotation about the vertical axis at the stiffness or shear center) in 3-D frames.

The 2-D model shown in Fig. 1a is a linear elastic model consisting of n prismatic columns each one with different cross sectional properties (A_i, I_i), height (h_i), end flexural restraints (κ_{ai}, κ_{bi}), and under different axial load (P_i), all sharing the same lateral spring restraint S_Δ and sidesway Δ. A typical column element A_iB_i of the structural system (Fig. 2a) is made up of the column itself $A_i'B_i'$ and the end flexural restraints A_iA_i' and B_iB_i' with elastic stiffnesses κ_{ai}, κ_{bi} (whose units are in Moment/Radian) at the top and bottom ends, respectively, and represent the combined effects of both the flexural stiffness provided by the girders and those of the beam-to-column connections. It is assumed that a typical column A_i B_i is made of a homogeneous linear elastic material with: 1) a modulus of elasticity E_i; 2) straight line centroidal axis; and 3) buckling taking place in the plane of the frame about one of the principal axes of the cross section.

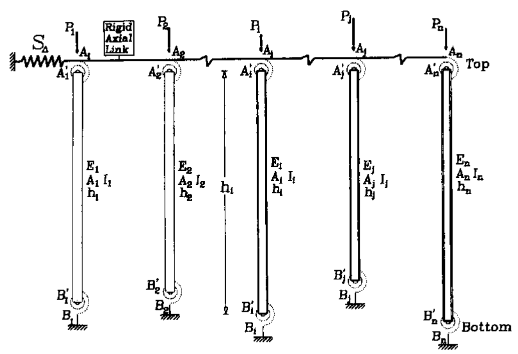
The ratios $R_{ai}=\kappa_{ai}/(E_i\ I_i/h_i)$ and $R_{bi}=\kappa_{bi}/(E_i\ I_i/h_i)$ will be denoted as the stiffness indices of the flexural restraints of column i. These indices vary from zero (i.e., $R_{ai}=R_{bi}=0$) for simple flexural restraints (i.e., hinged) to infinity (i.e., $R_{ai}=R_{bi}=\infty$) for fully rigid flexural restraints (i.e., infinitely rigid girders with beam-to-column connections). For convenience, the following two parameters are introduced (Wang 1983 and Cunningham 1990):

$$\rho_{ai} = \frac{1}{1+\dfrac{3}{R_{ai}}}$$

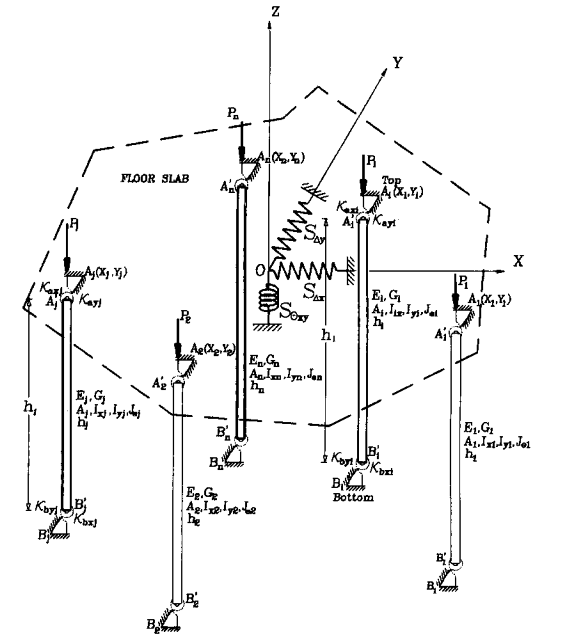
and $\rho_{bi} = \dfrac{1}{1+\dfrac{3}{R_{bi}}}$

where ρ_{ai} and ρ_{bi} are called the fixity factors at the top and bottom ends of column A_iB_i. For hinged flexural restraints, both the fixity factor ρ_i and the rigidity index R_i are zero; but for fully rigid flexural restraints, the fixity factor is 1 and the rigidity index is infinity. Since in real structures the fixity factors can only vary from 0 to 1 (while the rigidity index R_i may vary from 0 to ∞), they are more convenient to use in the analysis of framed structures with semirigid connections (Wang 1983, Cunningham 1990, Xu and Grierson 1993, and Aristizabal-Ochoa 1994a-c, 1995, 1996). It must be emphasized that in this formulation κ_{ai} and κ_{bi}, R_{ai} and R_{bi}, or ρ_{ai} and ρ_{bi} are stiffness parameters that include the effects of both the connections and the girders. These are not stiffness parameters of the beam-to-column connections alone.

The relationships between the fixity factors ρ_{ai}, ρ_{bi} and the alignment charts factors ψ_{ai} and ψ_{bi} [i.e., $\psi=\sum(EI/h)_c/\sum(EI/L)_g$ at the top and bottom ends, respectively] of column i in a symmetrical frame were presented by the writer (1994a). In symmetrical frames with sidesway buckling, these relationships are $\rho_{ai}=2/(2+\psi_{ai})$ and $\rho_{bi}=2/(2+\ \psi_{bi})$, when both ends A and B of the columns are rigidly connected at the top and bottom girders, respectively (it is assumed that an inflection point exists at the center span of each of the girders when story buckling occurs). Similarly, in symmetrical frames with sidesway buckling totally inhibited (i.e., "braced" frames), these relationships are $\rho_{ai}=2/(2+3\psi_{ai})$ and $\rho_{bi}=2/(2+3\psi_{bi})$ for rigid beam-to-column connections (it is assumed that the girders are subjected to uniform bending when buckling occurs). In frames that are unsymmetrical or irregular in loading or in geometry, the fixity factors can be calculated using structural principles as shown in Example 1. In frames with semirigid beam-to-column connections, the appropriate connection stiffness must be determined, however, this is outside of the scope of this paper. An alternative is to modify the fixity factors ρ_{ai} and ρ_{bi} as suggested by the writer elsewhere (1994a-c).



(a) Two-dimensional frame with braced buckling



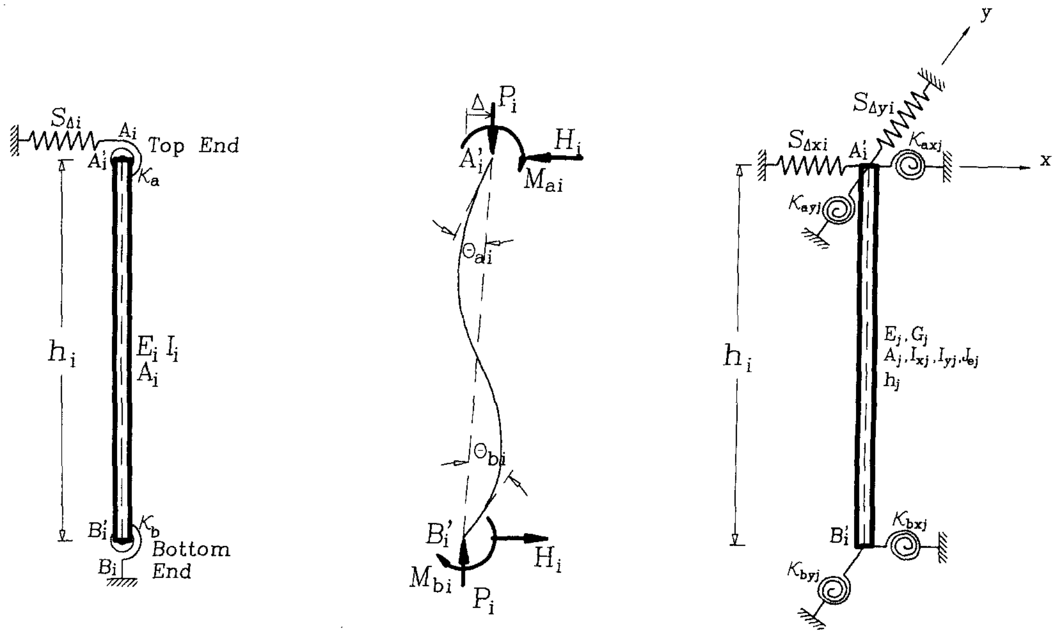
(b) Three-dimensional frame with braced buckling

Fig. 1 Structural models for a story with n different columns

In the case of a 3-D multi-column frame (Fig. 1b), the particular floor under consideration is on the XY plane with the origin O located at a convenient point (generally, at the shear center of the floor). Again, this is a linear elastic model consisting of n prismatic columns, with the centroid of column i located at point (X_i, Y_i) on the XY plane, under axial load (P_i) with individual properties including: cross area $(A)_i$; principal moments of inertia (I_{xi} and I_{yi}); effective polar moment of inertia (J_{ei}); height (h_i); end flexural connections (κ_{axi}, κ_{bxi} and κ_{ayi}, κ_{byi}). All n columns share the same lateral spring restraints $S_{\Delta x}$, $S_{\Delta y}$, $S_{\theta xy}$ and interstory sidesways Δ_x, Δ_y, and θ_{xy} (i.e., at each story, the top and bottom floors serve as rigid diaphragms allowing only 3-DOF per floor). All columns are assumed to be doubly symmetrical with their principal cross sectional axes parallel to the global axes X and Y (i.e., columns whose shear center and centroid coincide). Only three types of overalll-story buckling modes are considered: pure-translational sway flexural buckling, pure-torsional sway buckling, and combined flexural-torsional sway buckling. Individual column flexural buckling without overall story sway is also considered; but individual column torsional buckling is not.

A typical column element A_iB_i of the 3-D multi-column system (Fig. 2c) is made up of the column itself $A_i'B_i'$ and the end flexural restraints $A_{xi}A_{xi}'$, $B_{xi}B_{xi}'$ and $A_{yi}A_{yi}'$, $B_{yi}B_{yi}'$ located at the top and bottom ends and along the X and Y axes, respectively. It is assumed that a typical column A_iB_i is made of a homogeneous linear elastic material with moduli of elasticity E_i and G_i. The flexural restraints $A_{xi}A_{xi}'$ and $A_{yi}A_{yi}'$ at the top end A have stiffnesses κ_{axi}, κ_{ayi} [or stiffness ratios $R_{axi}=\kappa_{axi}/(E_iI_{xi}/h_i)$ and $R_{ayi}=\kappa_{ayi}/(E_iI_{yi}/h_i)$ along the X- and Y-axes, respectively. Similarly, flexural restraints $B_{xi}B_{xi}'$ and $B_{yi}B_{yi}'$ at the bottom end have stiffnesses κ_{bxi}, κ_{byi} and stiffness indices $R_{bxi}=\kappa_{bxi}/(E_iI_{xi}/h_i)$ and $R_{ayi}=\kappa_{byi}/(E_iI_{yi}/h_i)$.

Notice that the proposed algorithm can be utilized in inelastic framed structures when the non-linear behavior is concentrated at the connections. This can be carried out by modifying the flex-



(a) Two-dimensional model (b) End actions (forces, moments, deflections and rotations) (c) Three dimensional model

Fig. 2 Single column element.

ural stiffness of the end restraints AA' and BB'. Gerstle (1988) has indicated lower and upper bounds for κ_a and κ_b for plane frames. More recently, Xu and Grierson (1993) used these bounds in the design of plane frames with semirigid connection.

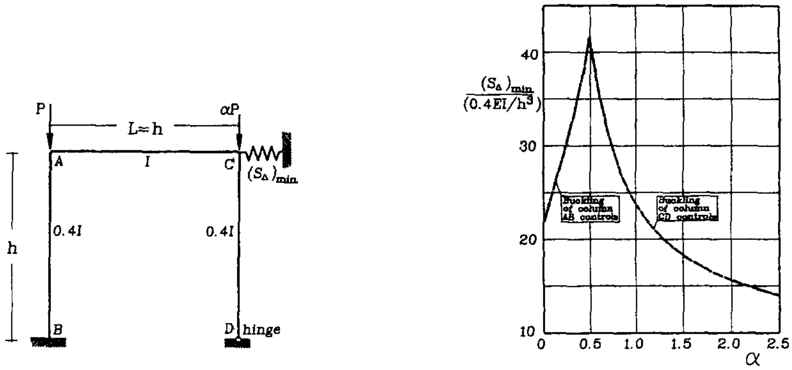
3. Criteria for minimum stiffness of bracing

Criteria to determine the minimum stiffness of bracing for 2-D and 3-D framed structures are given in this section. The proposed formulas formulas are presented next and derived in Appendix I.

3.1. For two-dimensional framed structures

The minimum lateral bracing S_Δ required to convert any story of a plane frame (Fig. 1a) with sidesway uninhibited or partially inhibited into a fully braced story can be determined from Eq. (1).

$$\phi_j^2 \sum_{i=1}^{n} \frac{\alpha_i}{\gamma_i}\left[1-\frac{3(\rho_{ai}+\rho_{bi}-2\rho_{ai}\rho_{bi})+9\rho_{ai}\rho_{bi}\mathrm{Tan}(\phi_i/2)/(\phi_i/2)}{\phi_i^2(1-\rho_{ai})(1-\rho_{bi})+3(\rho_{ai}+\rho_{bi}-2\rho_{ai}\rho_{bi})(1-\phi_i/\mathrm{Tan}\phi_i)+9\rho_{ai}\rho_{bi}[\mathrm{Tan}(\phi_i/2)/(\phi_i/2)-1]}\right]$$
$$=\frac{(S_\Delta)_{min.}}{(EI)_j/h_j^3} \qquad (1)$$



(a) Frame and loading

(b) Variation of minimum lateral bracing with α

Fig. 3 Example 1: simple plane frame.

where: $$\phi_j^2 = \frac{P_j}{(EI/h^2)_j} = (\pi/K_j)^2 \tag{2a}$$

$$\phi_i^2 = \frac{P_i}{(EI/h^2)_i} = (\pi/K_i)^2 = \phi_j^2 \alpha_i \gamma_i^2/\beta_i \tag{2b}$$

$\alpha_i = P_i/P_j$=ratio of axial load of column ***i*** to that of representative column ***j***
$\beta_i = (EI)_i/(EI)_j$=ratio of flexural stiffness of column ***i*** to that of representative column ***j***
$\gamma_i = h_i/h_j$=ratio of height of column ***i*** to that of representative column ***j***
K_j=K-factor of the representative column ***j*** calculated from Eq. (3) under braced conditions

$$(1-\rho_{aj})(1-\rho_{bj})(\pi/K_j)^2 + 3(\rho_{aj}+\rho_{bj}-2\rho_{aj}\rho_{bj})\left[1 - \frac{\pi/K_j}{\text{Tan}(\pi/K_j)}\right] + 9\rho_{aj}\rho_{bj}\left[\frac{\text{Tan}(\pi/2K_j)}{(\pi/2K_j)} - 1\right] = 0 \tag{3}$$

$(S_\Delta)_{min.}$ can be obtained following the four steps described below:

1) The representative ***j***-column is selected from the ***n***-column set. The column with the lowest critical axial load under braced conditions from Eq. (3) is generally recommended.

2) The fixity factors ρ_{ai} and ρ_{bi} for each column must be determined for both conditions, braced and unbraced-i.e., with sidesway buckling totally inhibited and uninhibited, respectively.

3) The effective length K-factor (or ϕ_j) of the representative column ***j*** is calculated from Eq. (3) utilizing the fixity factors ρ_{aj} and ρ_{bj} for braced conditions. For the rest of the columns, the braced K-factor (or ϕ_i) is determined from Eq. (2b). In this step is important to check that the K-factor of each column for "braced" conditions is less than that corresponding to "unbraced" conditions; otherwise, the ***j***-column initially selected must be changed to the one with the largest effective length K-factor among the ***n*** columns considering that the story level being analyzed is "braced".

4) The braced ϕ-factors (from step 3) and ρ_{ai} and ρ_{bi} for unbraced conditions (from step 2) are then substituted into Eq. (1) from which the required minimum bracings S_Δ can be calculated directly. The example that follows shows in detail the proposed procedure for plane frames.

3.1.1. Example 1: Simple plane frame

Utilizing Eqs. (1)-(3) and the procedure outlined above for plane frames, determine the minimum stiffness of the bracing required to convert the frame shown in Fig. 3a into a fully braced frame for any value of α. Assume $L=h=12.192$ m (40 ft), $I=41.6231\times10^{-4}\text{m}^4$ (10,000 in^4) and E =20,684,272 KPa (3000 Ksi). Neglect the effects of axial deformations in all members.

Solution: Steps 1-3: According to Eqs. (1) and (3) the fixity factors must be established first in order to determine the effective length K-factors and S_Δ. This can be carried out by applying a unit horizontal load at node A (or C) (i.e., 1 Kip=4.448 KN) and finding the moments and rotations at joints A and C of the frame with sidesway uninhibited (i.e., "unbraced" conditions). This was accomplished using a conventional computer program for plane frames with the following results:

1. For column AB-. For a unit horizontal load applied at node C of frame of Fig. 3a: Rotation at A=0.000624 radians; and Moment at A=19.30 KN-m (170.84 Kip-in). Therefore, the stiffness of the flexural restraint provided by the girder to column AB at A is κ_a=19.30/0.000624=30,931 KN-m/radian; and the stiffness index at A is $R_a=\kappa_a/(EI/h)_{AB}$=30,931/(20,684,272 $\times$ 0.4 $\times$ 0.00416231/12.192)=10.953; then the unbraced fixity factor at A becomes: $\rho_a=\dfrac{1}{1+3/10.953}=0.7850$; and the fixity factor at B: $\rho_b=\dfrac{1}{1+3/\infty}=1.0$
2. For column CD-. For a unit horizontal load applied at C of frame of Fig. 3a: Rotation at C= 0.000119 radians; and Moment at C=12.168 KN-m (107.70 Kip-in). Therefore, the stiffness of the flexural restraint provided by the girder to column CD at C is κ_c=12.168/0.000119= 102,251 KN-m/radian; and the stiffness index at C is $R_c=\kappa_c/(EI/h)_{CD}$=102,251/(20,684,272 $\times$ 0.4 $\times$ 0.00416231/12.192)=36.232; then the unbraced fixity factor at C becomes: $\rho_c=\dfrac{1}{1+3/36.232}=0.9235$; and the fixity factor at D: $\rho_d=\dfrac{1}{1+3/0}=0$

Similary, the fixity factors for fully "braced" conditions were determined by an additional structural analysis with a unit moment at C (1 Kip-in=0.11298 KN-m), and the frame restrained along the horizontal direction AC at the top, yielding the following results:

1. For column AB-. For a unit moment applied at C of frame of Fig. 3a: Rotation at A= 3.312×10^{-6} radians; and Moment at A=0.03742 KN-m (0.3312 Kip-in). Therefore, the stiffness of the flexural restraint provided by the girder to column AB at A is κ_a=0.03742/(3.312×10^{6})=11,298 KN-m/radian; and the stiffness index at A is $R_a=\kappa_a/(EI/h)_{AB}$=11,298/(20,684,272 $\times$ 0.4 $\times$ 0.00416231/12.192)=4; then the braced fixity factor at A becomes: $\rho_a=\dfrac{1}{1+3/4}=4/7$; and the braced fixity factor at B: $\rho_b=\dfrac{1}{1+3/\infty}=1.0$

   Therefore, using Eq. (3):
$$\left[1-\frac{\pi/K}{\text{Tan}(\pi/K)}\right]+4\left[\frac{\text{Tan}(\pi/2K)}{\pi/2K}-1\right]=0 \tag{4}$$
   Whose solution is: $(K_{AB})_{with\ sway\ totally\ inhibited}=0.5896$
2. For column CD-. For a unit moment applied at C of frame of Fig. 3a: Rotation at C= 1.2733×10^{-6} radians; and Moment at C=0.01079 KN-m (0.0955 Kip-in). Therefore, the stiffness of the flexural restraint provided by the girder to column CD at C is K_c=0.01079/(1.2733×10^{-6})=8,474.1 KN-m/radian; and the stiffness index at C is $R_c=\kappa_c/(EI/h)_{CD}$= 847.410/(20,684,272 $\times$ 0.4 $\times$ 0.00416231/12.192)=3.0; then the braced fixity factor at C becomes: $\rho_c=\dfrac{1}{1+3/3}=0.5$; and the braced fixity factor at D: $\rho_d=\dfrac{1}{1+3.0/0}=0$

Therefore, using Eq. (3):

$$(\pi/K)^2 + 3\left[1 - \frac{\pi/K}{\mathrm{Tan}(\pi/K)}\right] = 0 \tag{5}$$

Whose solution is: $(K_{CD})_{with\ sway\ totally\ inhibited}=0.8431$

Step 3: Taking into consideration that: 1) $\rho_a=0.785$ and $\rho_b=1$ for column AB, and $\rho_c=0.9235$ and $\rho_d=0$ for column CD, both under unbraced conditions; and 2) $K_{braced}=0.5896$ and 0.8431 for columns AB and CD, respectively, then the required minimum bracing $(S_\Delta)_{min.}$ can be determined from Eq. (1) appling the following conditions:

A) Assuming that column AB is the first to buckle with the frame under "braced" conditions and selecting column AB as the j-column, then $K_j=K_1=0.58955$; $\phi_j=\phi_1=\pi/0.58955=5.3288$; $\phi_2=5.3288\sqrt{\alpha}$; $\rho_{a1}=0.7850$, $\rho_{b1}=1$; $\rho_{a2}=0.9235$, $\rho_{b2}=0$; $\alpha_1=1$, $\alpha_2=\alpha$; $\beta_1=\beta_2=1$.
$(S_\Delta)_{min.}$ can be obtained after substituting these values into Eq. (1) as follows:

$$5.3288^2\{[1-\frac{3\times 0.215+9\times 0.785(\mathrm{Tan}\ 2.6644)/(2.6644)}{3\times 0.251(1-5.3288/\mathrm{Tan}\ 5.3288)+9\times 0.785\ [(\mathrm{Tan}\ 2.6644)/(2.6644)-1]}]$$

$$+\ \alpha[1-\frac{3\times 0.9235}{(5.3288)^2\alpha\times 0.0765+3\times 0.9235[1-5.3288\sqrt{\alpha}/(\mathrm{Tan}\ 5.3288\sqrt{\alpha})]}\} = \frac{(S_\Delta)_{min.}}{(EI)_1/h_1^3} \tag{6a}$$

Therefore:

$$\frac{(S_\Delta)_{min.}}{0.4(EI)/h} = 24.5468 + 28.3961\alpha[1-\frac{1}{0.78408\alpha - 5.3288\sqrt{\alpha}/\mathrm{Tan}(5.3288\sqrt{\alpha})}] \tag{6b}$$

B) Assuming the column CD is the first to buckle with the frame under "braced" conditions and selecting column AB as the j-column, then $K_2=0.84310$; $\phi_2=\pi/0.84310=3.72624$; $\phi_j=\phi_1=3.72624/\sqrt{\alpha}$; $\rho_{a1}=0.785$, $\rho_{b1}=1$, $\rho_{a2}=0.9235$, $\rho_{b2}=0$, $\alpha_1=1$, $\alpha_2=\alpha$; $\beta_1=\beta_2=1$.
$(S_\Delta)_{min.}$ can be obtained after substituting these values into Eq. (1) as follows:

$$\frac{3.72624^2}{\alpha}$$

$$\{[1-\frac{3\times 0.215+9\times 0.785(\mathrm{Tan}\frac{1.86312}{\sqrt{\alpha}})/(\frac{1.86312}{\sqrt{\alpha}})}{3\times 0.215[1-\frac{3.72624}{\sqrt{\alpha}}/(\mathrm{Tan}\frac{3.72624}{\sqrt{\alpha}})]+9\times 0.785[(\mathrm{Tan}\frac{1.86312}{\sqrt{\alpha}})/(\frac{1.86312}{\sqrt{\alpha}})-1]}]+$$

$$\alpha[1-\frac{3\times 0.9235}{(3.72624)^2\times 0.0765+3\times 0.9235(1-3.72625/\mathrm{Tan}\ 3.72625)}\} = \frac{(S_\Delta)_{min.}}{(EI)_1/h_1^3} \tag{7a}$$

Therefore:

$$\frac{(S_\Delta)_{min.}}{0.4(EI)/h} = \frac{13.8849}{\alpha}[1-\frac{1+5.87911\sqrt{\alpha}(\mathrm{Tan}\ \frac{1.86312}{\sqrt{\alpha}})}{5.87911\ \mathrm{Tan}\ \frac{1.86312}{\sqrt{\alpha}} - \frac{3.72624}{\sqrt{\alpha}}/(\mathrm{Tan}\ \frac{3.72624}{\sqrt{\alpha}}) - 9.95349}] + 17.1544 \tag{7b}$$

The minimum bracing required by the frame of Fig. 3a can be obtained directly from Eqs. (6b) and (7b). The variation of $(S_{\Delta})_{min.}/0.4(EI/h^3)$ with α as indicated by Eqs. (6b) and (7b) is shown in Fig. 3b.

Conclusion: Eq. (1) indicated the required lateral bracing at a given story level of a plane frame is a function of the degrees of fixity of the columns (ρ's), the load distribution (α), the ratio of the columns' flexural stiffness (β), height (γ), and the effective length K-factor of the column that first buckles under braced conditions. For instance, for the frame of Fig. 3, from α= 0 to 0.489 the buckling under braced conditions is controlled by buckling of column AB, and for α>0.489 by buckling of column CD. To guarantee braced buckling under any axial loading combination (i.e., for any α), then $(S_{\Delta})_{min.}/0.4(EI/h^3)$ must be greater than 41.7011, which is the minimum lateral bracing required to achieve simultaneous buckling in columns AB and CD [this occurs at $\alpha=(K_1/K_2)^2=(0.5896/0.8431)^2=0.489$]. Notice that the bracing indicated by the ACI Code-Version 1992 to 6 times the lateral stiffness of the frame is 1.777 times the value obtained from this analysis [i.e., $\dfrac{(S_{\Delta})_{min.}}{0.4(EI/h^3)}=74.085$].

3.2. For three-dimensional framed structures

The minimum stiffness of the lateral bracings $(S_{\Delta x})_{min.}$, $(S_{\Delta y})_{min.}$ and torsional bracing $(S_{\theta xy})_{min.}$ required to convert any given story of a 3-D framed structure with sidesway uninhibited or partially inhibited (Fig. 1b) into a full "braced" story can be determined from Eqs. (8), (9) and (10), respectively, as follows:

$$(S_{\Delta x})_{min.} = -\sum_{i=1}^{n} S_{xi} \tag{8}$$

$$(S_{\Delta y})_{min.} = -\sum_{i=1}^{n} S_{yi} \tag{9}$$

$$S_{\theta xy} = -\sum_{i=1}^{n} (X_i{}^2 S_{yi} + Y_i{}^2 S_{xi} + G_i J_{ei}/h_i) \tag{10}$$

where the lateral stiffness coefficients S_{xi} and S_{yi} are given by Eqs. (11a) and (11-b).

$$S_{xi} = \left[\frac{3(\rho_{axi}+\rho_{bxi}-2\rho_{axi}\rho_{bxi})+9\rho_{axi}\rho_{bxi}\,\mathrm{Tan}(\phi_{xi}/2)/(\phi_{xi}/2)}{\phi_{xi}^2(1-\rho_{axi}(1-\rho_{bxi})+3(\rho_{axi}+\rho_{bxi}-2\rho_{axi}\rho_{bxi})(\frac{1-\phi_{xi}}{\mathrm{Tan}\phi_{xi}})+9\rho_{axi}\rho_{bxi}[\mathrm{Tan}(\frac{\phi_{xi}}{2})/(\frac{\phi_{xi}}{2})-1]}-1\right]\frac{\alpha_i}{\gamma_i}\phi_{xj}^2(E_j I_{xj})/h_j^3 \tag{11a}$$

$$S_{yi} = \left[\frac{3(\rho_{ayi}+\rho_{byi}-2\rho_{ayi}\rho_{byi})+9\rho_{ayi}\rho_{byi}\,\mathrm{Tan}(\phi_{yi}/2)/(\phi_{yi}/2)}{\phi_{yi}^2(1-\rho_{ayi}(1-\rho_{byi})+3(\rho_{ayi}+\rho_{byi}-2\rho_{ayi}\rho_{byi})(\frac{1-\phi_{iy}}{\mathrm{Tan}\phi_{yi}})+9\rho_{ayi}\rho_{byi}[\mathrm{Tan}(\frac{\phi_{yi}}{2})/(\frac{\phi_{yi}}{2})-1]}-1\right]\frac{\alpha_i}{\gamma_i}\phi_{yj}^2(E_j I_{yj})h_j^3 \tag{11b}$$

and

$$\phi_{xj}^2 = \frac{P_j}{(EI/h^2)_{xj}} = (\pi/K_{xj})^2 \text{ obtained from Eq. (14a)} \tag{12a}$$

$$\phi_{xi}^2 = [\alpha_i \gamma_i^2/\beta_{xi}]\phi_{xj}^2 \tag{12b}$$

$$\phi_{yj}^2 = \frac{P_j}{(EI/h^2)_{yj}} = (\pi/K_{yj})^2 \text{ obtained from Eq. (14b)} \tag{12c}$$

$$\phi_{yj}^2 = [\alpha_i \gamma_i^2/\beta_{yi}]\phi_{yj}^2 \tag{12d}$$

$$K_{xi}^2 = [\beta_{xi}/(\alpha_i \gamma_i^2)]K_{xj}^2 \tag{13a}$$

$$K_{yi}^2 = [\beta_{yi}/(\alpha_i \gamma_i^2)]K_{yj}^2 \tag{13b}$$

where

$\alpha_i = P_i/P_j$=ratio of axial load of column i to that of representative column j
$\gamma_i = h_i/h_j$=ratio of height of column i to that of representative column j
X_i and Y_i=XY coordinates of column i with respect to center O (Fig. 1b)
K_{xj}=effective length K-factor of column-j corresponding to buckling along the X-axis
K_{yj}=effective length K-factor of column-j corresponding to buckling along the Y-axis

The stability equations for "braced" conditions along the X- and Y- directions are given by Eqs. (14a) and (14b).

$$(1-\rho_{axj})(1-\rho_{bxj})\phi_{xj}^2+3(\rho_{axj}+\rho_{bxj}-2\rho_{axj}\rho_{bxj})[1-\frac{\phi_{xj}}{\operatorname{Tan}\phi_{xj}}]+9\rho_{axj}\rho_{bxj}[\frac{\operatorname{Tan}(\phi_{xj}/2)}{\phi_{xj}/2}-1]=0 \tag{14a}$$

$$(1-\rho_{ayj})(1-\rho_{byj})\phi_{yj}^2+3(\rho_{ayj}+\rho_{byj}-2\rho_{ayj}\rho_{byj})[1-\frac{\phi_{yj}}{\operatorname{Tan}\phi_{yj}}]+9\rho_{ayj}\rho_{byj}[\frac{\operatorname{Tan}(\phi_{yj}/2)}{\phi_{yj}/2}-1]=0 \tag{14b}$$

The stiffnesses $(S_{\Delta x})_{min.}$, $(S_{\Delta y})_{min.}$ and $(S_{\theta xy})_{min.}$ can be obtained following the four steps described below:

1) Similar to 2-D frames, a representative j-column is selected from the n columns within the story level under consideration (Fig. 1b). The column with the lowest critical axial load under braced conditions is generally recommended.

2) The fixity factors ρ_{ai} and ρ_{bi} must be determined in the X- and Y-directions for both conditions, braced and unbraced;

3) The braced K_{xj} and K_{yj} factors (or ϕ_{xj} and ϕ_{yj}) of the representative column j are calculated from Eqs. (14a) and (14b) utilizing the fixity factors ρ_{axj}, ρ_{ayj}, ρ_{bxj} and ρ_{byj} for the braced case. For the rest of the columns the braced K-factors (or ϕ_{xi} and ϕ_{yi}) can be determined from Eqs. (13a) and (13b) or Eqs. (12b) and (12d). At this step is important to check that the effective length K-factor of each column for "braced" conditions is less than that corresponding to "unbraced" conditions; otherwise, the j-column initially selected must be changed to the column with the largest K-factor among the n columns considering that story level fully "braced".

4) The braced ϕ-factors (from step 2), the fixity factors ρ_{ai} and ρ_{bi} in X- and Y-directions for unbraced conditions (from step 1), the XY coordinates of each column, and the individual column torsional stiffness $(G_i J_{ej}/h_i)$ are then substituted into Eqs. (8), (9) and (10) from which the required minimum bracings $(S_{\Delta x})_{min.}$, $(S_{\Delta y})_{min.}$ and $(S_{\theta xy})_{min.}$ can be calculated directly. The two examples that follow show the proposed procedure for 3-D framed structures.

3.2.1. Example 2: Simple space frame

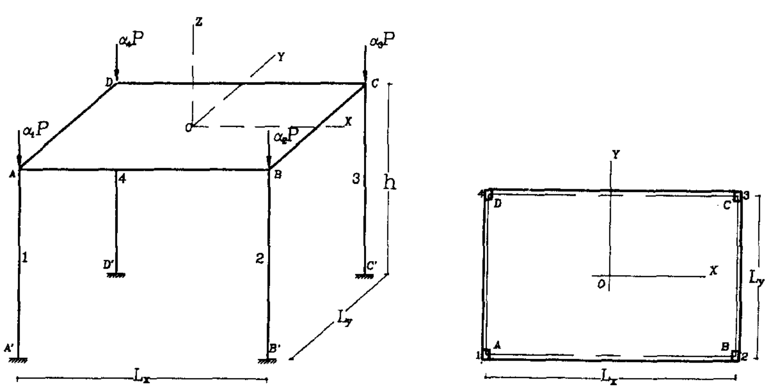
Utilizing Eqs. (8)-(13) and the procedure outlined above, determine the minimum bracings $(S_{\Delta x})_{min.}$, $(S_{\Delta y})_{min.}$ and $(S_{\theta xy})_{min.}$ required to convert the simple space frame of Fig. 4 into a fully "braced" frame for $\alpha_1=\alpha_2=\alpha_3=\alpha_4=1$. Assume $GJ_{ei}=EI_x$, $EI_y=2EI_x$, $L_x=1.5h$, $L_y=h$. Vary the flexural restraints in the X- and Y-directions at the top and bottom of all columns from hinge ($\psi_{ai}=\psi_{bi}=\infty$) to clamped conditions ($\psi_{ai}=\psi_{bi}=0$).

Solution: This is a doubly symmetrical frame about the X- and Y-axes. Therefore, the relationships between the ψ ratios and the fixity factors, namely $\rho_{braced}=2/(2+3\psi)$ and $\rho_{unbraced}=2/(2+\psi)$, will be utilized in this analysis. Table 1 shows the variation of ρ_{braced}, $\rho_{unbraced}$, the effective length K-factor, and the required lateral bracings (normalized with respect to π^2EI_x/h^3) and the torsional bracing (normalized with respect to π^2EI_x/h) with different ψ ratios.

Table 1 indicates that the required bracings vary very little for ψ ratios between ∞ and 2 (or $0\leq\rho_{unbraced}<0.5$) which correspond to simple connections, but they increase by a factor of four for completely clamped end conditions. This increase in the required bracings is also accompanied by an indentical increases in the total axial critical load from $4\pi^2EI_x/h^3$ for hinged end conditions in all four columns to $16\pi^2EI_x/h^3$ for clamped ends in all columns. In the latter case, the bracing indicated by the ACI Code (Version 1992) of $6\times4\times12EI_x/h^3=288EI_x/h^3$ is 1.824 times the value obtained from this analysis.

3.2.2. Example 3: Multi-story space frame

Utilizing Eqs. (8)-(13), determine the minimum bracings $(S_{\Delta x})_{min.}$, $(S_{\Delta y})_{min.}$, and $(S_{\theta xy})_{min.}$ required to convert the first story level of the space frame shown in Fig. 5a into a fully "braced" story for the values of α_i listed in Table 2. Assume that all columns are rigidly connected to the ground (i.e., clamped with $\rho_{xi}=\rho_{yi}=1$) and that each floor slab acts as a rigid diaphragm with the flexural stiffness provided by the girders about the X- and Y-axes. A 20" $\times$ 20" (508 mm $\times$ 508 mm) section was utilized in all girders and columns with following properties: $I_x=I_y=13{,}333.33$ in^4 (554,975 cm^4); A=400 in^2 (2580.64 cm^2); pure torsional stiffness=$GJ_e=3.536\times10^7$ Kip-in^2



(a) Space frame and loading (b) Column layout and cross section

Fig. 4 Example 2: simple space frame.

Table 1 Minimum bracings $S_{\Delta x}$, $S_{\Delta y}$ and $S_{\theta xy}$ for the simple 3-D frame of Example 2

ψ	ρ_{braced}	$\rho_{unbraced}$	K_{braced}	$S_{\Delta x} \div \pi^2 EI_x/h^3$	$S_{\Delta y} \div \pi^2 EI_x/h^3$	$S_{\upsilon xy} \div \pi^2 EI_x/h$
∞	0	0	1.0	4.0000	8.0000	5.0947
6	0.10	0.25	0.9450	3.8522	7.7043	4.8914
2	0.25	0.50	0.8553	4.0096	8.0191	5.1078
1	2/5	2/3	0.7743	4.6936	9.3872	6.0484
2/3	0.50	0.75	0.7223	5.5074	11.0147	7.1673
0	1.0	1.0	0.5	16.0000	32.0000	21.5947

(101,473 kN-m^2); elastic moduli E=3,605 ksi (24,856 MPa) and G=1,567.4 ksi (10,807.2 MPa).

Solution: Taking the origin of the XY axes at the centroid of the floor columns as shown in Fig. 4b, the fixity factors for each column are estimated as follows:

a) Top ends-.

For columns 1, 3, 6, 7 and 8: $\psi_{ax}=\psi_{ay}=\Sigma(EI/h)_c/\Sigma(EI/L)_g=(1/15+1/12)/(1/24)=3.6$

Therefore, the fixity factors for unbraced conditions become: $\rho_{ax}=\rho_{ay}=2/(2+\psi_{ax})=0.3571$; and the fixity factors for braced conditions: $\rho_{ax}=\rho_{ay}=2/(2+3\psi_{ax})=0.15625$

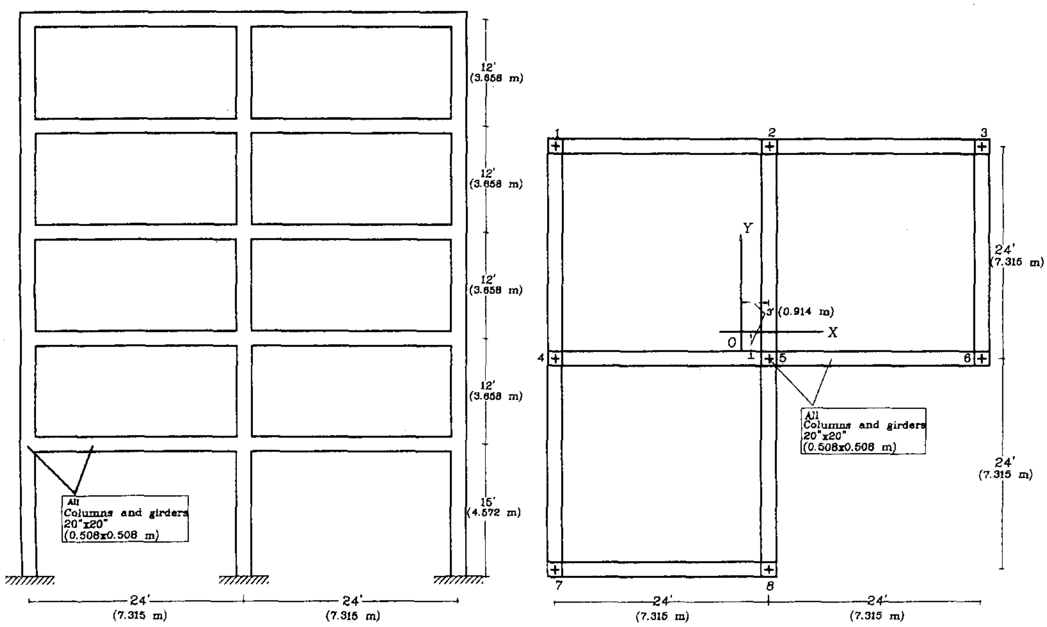
For column 2: $\psi_{ax}=\Sigma(EI/h)_c/\Sigma(EI/L)_g=(1/15+1/12)/(2/24)=1.8$ and $\psi_{ay}=\Sigma(EI/h)_c/\Sigma(EI/L)_g=(1/15+1/12)/(1/24)=3.6$

Therefore, the fixity factors for unbraced conditions become:

$\rho_{ax}=2/(2+\psi_{ax})=2/(2+1.8)=0.5263$; and $\rho_{ay}=2/(2+\psi_{ay})=2/(2+3.6)=0.3571$

and the fixity factors for braced conditions:

$\rho_{ax}=2/(2+3\psi_{ax})=2/(2+3\times1.8)=0.27027$; and $\rho_{ay}=2/(2+3\psi_{ay})=2/(2+3\times3.6)=0.15625$



(a) Elevation view of the space frame

(b) Column layout and dimensions (origin located at the columns' centroid)

Fig. 5 Example 3: multi-strory multi-bay space framed structure.

Table 2 Column load factors and minimum bracings for frame of example 3 (K_5=0.65168 and assuming ρ_{bx}= ρ_{by}=1.0 for all columns at the bottom)

Frame case	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	$S_{\Delta x} \div EI/h^3$	$S_{\Delta x} \div EI/h^3$	$S_{\theta xy} \div EI/h$
E2-1	1/3	2/3	1/3	2/3	1	1/3	1/3	1/3	62.080	62.080	109.297
E2-2	7/15	4/5	7/15	2/3	1	1/3	2/15	2/15	62.575	63.358	105.139
E2-3	1/5	8/15	1/5	2/3	1	1/3	8/15	8/15	62.424	62.083	116.757
E2-4	5/9	8/9	5/9	2/3	1	1/3	0	0	63.398	65.538	104.193
E2-5	1/9	4/9	1/9	2/3	1	1/3	2/3	2/3	63.347	62.875	124.590
E2-6	1/3	5/6	2/3	1/2	1	1/2	0	1/6	62.849	64.396	114.300
E2-7	0.9386	0.9386	0.9386	0.9386	1	0.9386	0.9386	0.9386	192.26	192.26	600.839

For column 4: $\psi_{ax}=\sum(EI/h)_c/\sum(EI/L)_g=(1/15+1/12)/(1/24)=3.6$ and $\psi_{ay}=\sum(EI/h)_c/\sum(EI/L)_g=(1/15+1/12)/(2/24)=1.8$

Therefore, the fixity factors for unbraced conditions become:

$\rho_{ax}=2/(2+\psi_{ax})=2/(2+3.6)=0.3571$; and $\rho_{ay}=2/(2+\psi_{ay})=2/(2+1.8)=0.5263$ and the fixity factors for braced conditions:

$\rho_{ax}=2/(2+3\psi_{ax})=2/(2+3\times 3.6)=0.15625$; and $\rho_{ay}=2/(2+3\psi_{ax})=2/(2+3\times 1.8)=0.27027$

For column 5: $\psi_{ax}=\psi_{ay}=\sum(EI/h)_c/\sum(EI/L)_g=(1/15+1/12)/(2/24)=1.8$

Therefore, the fixity factors for unbraced conditions become: $\rho_{ax}=\rho_{ay}=2/(2+\psi_{ax})=0.5263$ and the fixity factors for braced conditions: $\rho_{ax}=\rho_{ay}=2/(2+3\psi_{ax})=0.27027$

b) Bottom ends-.

For all columns $\psi_{bx}=\psi_{by}=0$; therefore, $\rho_{bx}=\rho_{by}=1$ for any type of buckling.

Seven different load combinations were studied and are designated by the symbols E2-1 through E2-7 in Table 2. The first six cases have a total axial load of $(\sum\alpha_i)P_5=4P_5$ and $(K_5)_{braced}=0.65168$. Case E2-7 has a load pattern corresponding to simultaneous braced buckling in all eight columns with a total axial load of $(\sum\alpha_i)P_5=7.57P_5$ and $(K_1=K_2=K_3=K_4=K_6=K_7=K_8)_{braced}=0.67266$. Notice that all columns, except column 5, have the same critical load under braced conditions. Column 5 has two restraining girders in both directions at the top end. Table 2 summarizes the load combination factors and the calculated results according to Eqs. (8)-(13).

Table 3 shows the solution to the same frame of Fig. 5 but with all columns partially restrained at the fundation level with $\psi_{bx}=\psi_{by}=1$ (this is the value commonly recommended by most construction codes for columns with foundations consisting of properly designed isolated footings); therefore, the fixity factors for unbraced conditions become $\rho_{bx}=\rho_{by}=2/(2+1)=2/3$, and for braced conditions $\rho_{bx}=\rho_{by}=2/(2+3)=2/5$. As expected, by reducing the flexural restraints at the foundation level of the columns the required bracings are also by reduced. The results shown in Tables 2 ans 3 also indicate that the required bracings are not very sensitive to the load pattern as long as the total vertical load on the story level remains constant. As the load distribution among the columns reaches that corresponding to simultaneous buckling of all columns under braced conditions (i.e., with story sidesway totally inhibited in all directions), the minimum bracings increase substantially. For instance, the minimum bacings required by the frame shown in Fig. 5 to achieve braced buckling for any load pattern are those listed for case E2-7, which are much larger than those for cases E2-1 through E2-6. The required torsional bracing is increased by a factor larger than that for the lateral bracings (compare the values in the last column in Tables 2 and 3).

Table 3 Column load factors and minimum bracings for frame of example 3 (K_5=0.7348 and assuming $\rho_{bx}=\rho_{by}$=2/3 for all columns at the bottom)

Frame case	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	$S_{\Delta x} \div EI/h^3$	$S_{\Delta y} \div EI/h^3$	$S_{\theta xy} \div EI/h$
E2-1	1/3	2/3	1/3	2/3	1	1/3	1/3	1/3	43.479	43.479	76.872
E2-2	7/15	4/5	7/15	2/3	1	1/3	2/15	2/15	43.617	43.894	73.228
E2-3	1/5	8/15	1/5	2/3	1	1/3	8/15	8/15	43.600	43.481	81.566
E2-4	5/9	8/9	5/9	2/3	1	1/3	0	0	43.848	44.595	71.354
E2-5	1/9	4/9	1/9	2/3	1	1/3	2/3	2/3	43.902	43.739	85.618
E2-6	1/3	5/6	2/3	1/2	1	1/2	0	1/6	43.695	44.239	78.455
E2-7	0.935	0.935	0.935	0.935	1	0.935	0.935	0.935	122.27	122.27	377.23

A computer program indicating the different steps that are required for the calculation of the minimum bracings for a 3-D framed structure is presented next.

4. Computer program

The following coomputer program written in Microsoft QUICKBASIC indicates the necessary steps in the calculation of $(S_{\Delta x})_{min.}$, $(S_{\Delta y})_{min.}$ and $(S_{\theta xy})_{min.}$. Explanations and variable identification are all included in the attached listing. The particular problem being analyzed is the 3-D frame of Example 3 (Case E2-6).

```
CLS
DEFDBL A-Z
PRINT "Bracing Analysis of 3-D Framed Structures for Stability under Vertical Loads."
PRINT "Published in the Int. Jour. of Structural and Engineering Mechanics"
PRINT "Author: Dr. J. Dario Aristizable-Ochoa Data: Feb. 1997"
DEFDBL A-Z
pi=3.1415927#
PRINT
FOR I=1 TO 8              '8=Number of columns (Example Problem 3, case E2-6)
  Tor(I)=196447.5         'Torsional Stiffness for all columns
  H(I)=180                'Column height for all columns
  rax(I)=2/(2+3.6)        'Fixity factors for all columns at the toop (ρaxi and ρayi)
  ray(I)=2/(2+3.6)
  rbx(I)=1                'Fixity factors for all columns at the bottom (ρbxi and ρbyi)
  rby(I)=1
  gam(I)=1                'Height ratios for all columns (γi)
  bx(I)=1                 'EI ratios for all columns (βi)
  by(I)=1
NEXT I
rax(2)=2/(2+1.8)          'Fixity factors for columns 2, 4 and 5 (corrected values)
rax(4)=2/(2+1.8)
rax(5)=2/(2+1.8)
al(1)=1/3                 'Load ratios (αi=Pi/Pj)
```

```
al(2)=5/6
al(3)=2/3
al(4)=1/2
al(5)=1
al(6)=1/2
al(7)=1E-12
al(8)=1/6
Hj=180                    'Values of Representative J-Column (i=5): hj, EIxj, and EIyj
EIxj=4.80666E+07
EIyj=EIxj
Kj=0.6516759#             'Braced K-factor of J-column from Eqs. (14a) and (14b)
X(1)=-288+36              'X-Y-coordinates of each Column (with respect to O or shear center)
Y(1)=288-36
X(2)=36
Y(2)=288-36
X(3)=288+36
Y(3)=288-36
X(4)=-288+36
Y(4)=-36
X(5)=36
Y(5)=-36
X(6)=288+36
Y(6)=-36
X(7)=-288+36
Y(7)=288+36
X(8)=36
X(8)=288+36
A=0                       'A, B and C are Stiffness Accumulators being initialized to zero
B=0
C=0
tot=0                     'Accumulator of torsional stiffness GjJei/hi of the columns, see Eq. (10)
FOR I=1 TO 8              'Calculation of Minimum Stiffness of Bracing according to Eqs. (8)-(10)
gam=gam(I)
al=al(I)
bx=bx(I)
by=by(I)
Tor=Tor(I)
rax=rax(I)
rbx=rbx(I)
ray=ray(I)
rby=rby(I)
X=X(I)
Y=Y(I)
fjx=pi/Kj
fjy=fjx*SQR(EIxj/EIyj)
mjx=EIxj/Hj ^ 3
```

```
mjy=EIyj/Hj ^ 3
fx=fjx*gam*SQR(al/bx)
fy=fjy*gam*SQR(al/by)
f1=3*(rax+rbx-2*rax*rbx)+9*rax*rbx*TAN(fx/2)/(fx/2)
f1=f1/fx  ^  2  *  (1-rax)*(1-rbx)+3*(rax+rbx-2*rax*rbx)*(1-fx/TAN(fx))+9*rax*rbx*(TAN(fx/2)/
   (fx/2)-1))
f1=(1-f1)*al/gam*mjx*fjx ^ 2                'Calculation of -S_xi according to Eq. (11a)
A=A+f1                                      'Calculation of (S_Δx)_min. according to Eq. (9)
f2=3*(ray+rby-2*ray*rby)+9*ray*rby*TAN(fy/2)/(fy/2)
f2=f2/(fy  ^  2*(1-ray)*(1-rby)+3*(ray+rby-2*ray*rby)*(1-fy/TAN(fy))+9*ray*rby*(TAN(fy/2)/(fy/
   2)-1))
f2=(1-f2)*al/gam*mjy*fjy^2                  'Calculation of -S_yi according to Eq. (11b)
B=B+f2                                      'Calculation of (S_Δy)_min. according to Eq. (9)
C=C+Y ^ 2*fl+X ^ 2*f2                       'Calculation of (S_θxy)_min. according to Eq. (10)
tot=tot+Tor
NEXT I
C=C-tot                                     '(S_θxy)_min. according to Eq. (10)
PRINT "Results of Minimum Stiffness of Bracing:"
PRINT "SDeltax/(EI/h^3)xj=" ; A/mjx         'Min. X-Bracing ratio S_Δx/(EI/h^3)xj
PRINT "SDeltax/(EI/h^3)yj=" ; A/mjx         'Min. X-Bracing ratio S_Δy/(EI/h^3)yj
PRINT "SThetaxy/(EI/h)=" ; C/mjx/Hj ^ 2     'Min. Rotational Bracing ratio S_θxy/(EI/h^3)xj
PRINT "Eng of Calculations. Check the answers and data!. Results obtained from this program
         and then utilized in any design is your whole responsibility."
END
```

5. Summary and conclusions

Definite criterion for minimum stiffness of bracing for 2-D and 3-D elastic framed structures is presented and the corresponding equations are derived using the classical stability functions. A condensed approach that determines the minimum stiffness of story bracing required by plane and space framed structures to achieve non-sway buckling conditions is proposed. The proposed approach and corresponding equations are applicable to multi-column frames with rigid, semirigid, and simple connections. The proposed method is only applicable to elastic framed structures with the following limitations: 1) the floor diaphragms including the ground floor of the framed structure are assumed to be rigid in their own planes; and 2) all columns are assumed to the doubly symmertrical with their principal cross-sectional axes parallel to the *XY* global axes (i.e., columns whose shear center and centroid coincide). The effects of axial deformations in the beam and columns are neglected. As a consequence, overall story flexural buckling occurring along the *X*- or and *Y*-axes and overall flexural-torsional buckling occurring in the *XY* plane and about the *Z* axis are considered in 3-D frames. Overall story flexural buckling occurring in the plane of the frame is only considered in the bracing analysis of 2-D frams. Pure torsional buckling in a single column is not considered herein.

The proposed algorithm can also be utilized in framed structures buckling in the inelastic range when the nonlinear behavior is concentrated at the end connections. This can be carried out by modifying the flexural stiffness of the end restraints of each column. This is of particular importance, since usual civil engineering structures are of such proportions as to fail in the inelastic range.

The proposed approach is not only more accurate and general than any other method available, but allows the designer to investigate the effects of semirigid connections, flexural hinges, load patterns, members' properties (span, cross sectional geometry, and elastic moduli), and column layout on the minimum bracing requirements for plane and space framed structures.

Analytical results indicate the required stiffness of lateral bracing at a given story level of a plane frame is a function of the degrees of fixity of the columns (ρ's), the load distribution (α), the ratio of the columns' flexural stiffness (β), hight (γ), and the effective length K-factor of the column that first buckles under braced conditions. For instance, for the plane frame of Fig. 3, from α=0 to 0.489 the buckling under braced conditions is controlled by the buckling of column AB, and for α>0.489 by the buckling of column CD. To guarantee buckling with no lateral sway under any axial loading combination (i.e., for any α) in this particular frame, then $\frac{(S_{\Delta})_{min.}}{0.4(EI/h^3)}$ must be larger than 41.7011, which is the minimum stiffness of the interstory lateral bracing required to achieve simultaneous buckling in both columns [this occurs when $\alpha=(K_1/K_2)^2=0.489$]. Notice that the bracing indicated by the ACI Code of 6 times the lateral stiffness of this frame is 1.777 times the value obtained using the proposed approach.

Analytical studies also indicated that for 3-D framed structures, the minimum bracing depends on the column layout in addition to the axial load distribution and the columns's properties including their end flexural restraints. As expected, by reducing the flexural restraints at the foundation level of the columns in a framed structure the required bracing is also reduced. Moreover, the required bracing is not very sensitive to the load pattern, as long as the total vertical load on the frams remains constant. However, the minimum bracing increases substantially when each one of the columns are loaded to their maximum axial capacity under braced conditions (i.e., under simultaneous braced buckling conditons). In 3-D frames, the required torsional bracing is increased by a factor larger than that corresponding to the lateral bracings. For instance, for the 3-D frame of Fig. 4, the required bracing at the first story varies very little for ψ ratios between ∞ and 2 (or $0 \leq \rho_{unbraced} < 0.5$, values that correspond to simple connections), but it increases by a factor of four for completely clamped end conditions. This increase in the required bracing is also accompanied by an identical increase in the total axial critical load. Similar behavior was observed in the 3-D frame of Fig. 5, Tables 2 and 3 show that the bracing stiffness demand stays almost constant irrespectively of the load distribution (as long as the total applied load remains unchanged, as shown by Cases E2-1 through E2-6). However, in Case E2-7, when all columns buckle simultaneously, the required bracings increase radically as well as the total critical load.

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Appendix I-. Derivation of the bracing equations

Determination of lateral stiffness coefficients S_{xi} and S_{yi}-. The flexural stability coefficients for a typical column i with sidesway in one of the principal planes of its cross section (Fig. 2a) can be formulated in a classical manner using the flexibility coefficients (Salmon and Johnson 1980, page 849) as follows:

$$\Theta'_{ai} = \frac{M_{ai}}{(E_i I_i/h_i)} \frac{\mathrm{Sin}\phi_i - \phi_i \mathrm{Cos}\phi_i}{\phi_i^2 \mathrm{Sin}\phi_i} + \frac{M_{bi}}{(E_i I_i/h_i)} \frac{\mathrm{Sin}\phi_i - \phi_i}{\phi_i^2 \mathrm{Sin}\phi_i} = -\frac{\Delta}{h_i} - \frac{M_{ai}}{\kappa_{ai}} \tag{15a}$$

$$\Theta'_{bi} = \frac{M_{bi}}{(EI/h)_i} \frac{\mathrm{Sin}\phi_i - \phi_i \mathrm{Cos}\phi_i}{\phi_i^2 \mathrm{Sin}\phi_i} + \frac{M_{ai}}{(EI/h)_i} \frac{\mathrm{Sin}\phi_i - \phi_i}{\phi_i^2 \mathrm{Sin}\phi_i} = -\frac{\Delta}{h_i} - \frac{M_{bi}}{\kappa_{bi}} \tag{15b}$$

Where: Θ'_{ai} and Θ'_{bi}=rotations at A_i' and B_i' with respect to cord $A_i'B_i'$ respectively (Fig. 2b); Δ=lateral sway between ends A and B;

$$\phi_i = \frac{\pi}{K_i}$$

Since three unknows (M_{ai}, M_{bi}, and Δ) are involved, one more equation is required at the element level. This equation can be obtained applying static equilibrium in the plane under consideration (rotational equilibrium of column AB in Fig. 2b) as follows:

$$M_{ai} + M_{bi} + P_i \Delta - H_i h_i = 0 \tag{16a}$$

or simply

$$\frac{M_{ai}}{h_i} + \frac{M_{bi}}{h_i} + \frac{P_i}{h_i}\Delta = H_i \tag{16b}$$

From Eqs. (15a) and (15b) the end moments M_{ai} and M_{bi} can be expressed in terms of the interstory drift $\frac{\Delta}{h_i}$ as follows:

$$M_{ai} = -\frac{\left[\frac{1}{R_{bi}} + \frac{1-\mathrm{Cos}\,\phi_i}{\phi_i \mathrm{Sin}\,\phi_i}\right]\phi_i^2}{\frac{\phi_i^2}{R_{ai}R_{bi}} + \left(\frac{1}{R_{ai}} + \frac{1}{R_{bi}}\right)\left(1 - \frac{\phi_i}{\mathrm{Tan}\,\phi_i}\right) + \frac{\mathrm{Tan}(\phi_i/2)}{\phi_i/2} - 1} \frac{(EI)_i}{h_i} \frac{\Delta}{h_i} \tag{17a}$$

and

$$M_{bi} = -\frac{\left[\frac{1}{R_{ai}} + \frac{1-\mathrm{Cos}\phi_i}{\phi_i \mathrm{Sin}\phi_i}\right]\phi_i^2}{\frac{\phi_i^2}{R_{ai}R_{bi}} + \left(\frac{1}{R_{ai}} + \frac{1}{R_{bi}}\right)\left(1 - \frac{\phi_i}{\mathrm{Tan}\phi_i}\right) + \frac{\mathrm{Tan}(\phi_i/2)}{\phi_i/2} - 1} \frac{(EI)_i}{h_i} \frac{\Delta}{h_i} \tag{17b}$$

Substituting (17a) and (17b) into (16b)

$$\left\{-\frac{[(1/R_{ai} + 1/R_{bi}) + \mathrm{Tan}(\phi_i/2)/(\phi_i/2)]\phi_i^2}{\frac{\phi_i^2}{R_{ai}R_{bi}} + \left(\frac{1}{R_{ai}} + \frac{1}{R_{bi}}\right)\left(1 - \frac{\phi_i}{\mathrm{Tan}\phi_i}\right) + \frac{\mathrm{Tan}(\phi_i/2)}{(\phi_i/2)} - 1} \frac{(EI)_i}{h_i^3} + \frac{P_i}{h_i}\right\} = H_i/\Delta \tag{18}$$

The left term in Eq. (18) represent the lateral stiffness (including the effects of the compressive axial load P_i) provided by column i to the floor system in the direction of buckling. Eq. (18) can be expressed in terms of the fixity factors ρ_{ai} and ρ_{bi} and ratios α_i, γ_i as follows:

$$\{\frac{3(\rho_{ai}+\rho_{bi}-2\rho_{ai}\rho_{bi})+9\rho_{ai}\rho_{bi}\mathrm{Tan}(\phi_i/2)/(\phi_i/2)}{\phi_i^2(1-\rho_{ai})(1-\rho_{bi})+3(\rho_{ai}+\rho_{bi}-2\rho_{ai}\rho_{bi})(1-\phi_i/\mathrm{Tan}\phi_i\, i)+9\rho_{ai}\rho_{bi}[\mathrm{Tan}(\phi_i/2)/(\phi_i/2)-1]}+1\}$$

$$\frac{\alpha_i}{\gamma_i}(EI)_j/h_j^3=H_i/\Delta \tag{19}$$

The left term of Eq. (19) when applied in the X- and Y-directions becomes the stiffness coefficients $-S_{xi}$ and $-S_{yi}$ given by Eqs. (11a) and (11b), respectively, and also utilized in Eq. (1).

Considering static equilibrium of the top rigid link in 2-D frames (Fig. 1a) or top diaphragm (Fig. 1b) along the X axis and assuming that the floor is braced along the Y axis and restrained to rotate around the Z axis for 3-D frames, then:

$$\{[\sum_{i=1}^{n} S_{xi}]+S_{\Delta x}\}\Delta_x=0 \tag{20}$$

Now, considering static equilibrium about the Y-axis and about the Z-axis independently (Fig. 1b), and in a similar fashion as it was done along the X-axis, the following two equations can be obtained:

$$\{[\sum_{i=1}^{n} S_{yi}]+S_{\Delta y}\}\Delta_y=0 \tag{21}$$

$$\{[\sum_{i=1}^{n}(X_i^2S_{yi}+Y_i^2S_{xi}+G_iJ_{ei}/h_i)+S_{\theta xy}\}\theta_{xy}=0 \tag{22}$$

Eqs. (20)-(22) have the trivial soultion (i.e., $\Delta_x=\Delta_y=\theta_{xy}=0$) indicating that equilibrium is possible at any axial loadings provided that all columns remains straight. There is also a nontrivial solution possible when the terms within the brackets in Eqs. (20), (21) and (22) vanish. This latter solution is the one that corresponds to the minimum bracing conditions. Therefore, Eq. (1) for 2-D frames and Eqs. (8)-(10) for 3D-frames correspond to the minimum bracing conditions. These are also the conditions for "braced" buckling in all directions.

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Notation

The following symbols are used in this paper:

E	Elastic modulus of the material;
G	Shear modulus of the material;
h_i	height of column i;
i	subscript indication column-i of the 3-D multi-column system;
j	subscript indication the representative column-j of the 3-D multi-column system;
I_g	girder moment of inertia;
I_i or I_c	moment of the inertia of column i;
J_{ei}	effective polar moment of inertia of column i;
κ_{ai} and κ_{bi}	the flexural stiffness of the end restraints at A_i and B_i, respectively;
K_j	effective length K-factor of the representative column j;
L_g	girder span;
$(P_{cr})_j$	buckling load of representative column $j[=\pi^2 E_j I_j/(K_j h_j)^2]$;
$(P_{cr})_i$	buckling load of column $i[=\alpha_i (P_{cr})_j]$;
$(P_{cr})_{Total}$	critical load of the entire story$[=(P_{cr})_j \sum_{i=1}^{n} \alpha_i]$;
n	total number of columns in the story system;
$S_{\Delta x}$	interstory lateral stiffness or bracing provided to the story system along the X-axis;
$S_{\Delta y}$	interstory lateral stiffness or bracing provided to the story system along the Y-axis;
$S_{\theta xy}$	interstory torsional stiffness or bracing provided to the story system about the Z-axis;
R_{ai}	stiffness index of the flexural restraint at $A_i[=\kappa_{ai}/(EI_i/h_i)]$;
R_{bi}	stiffness index of the flexural restraint at $B_i[=\kappa_{bi}/(EI_i/h_i)]$;
X_i	X-coordinate of column i with respect to origin O;
Y_i	Y-coordinate of column i with respect to origin O;
x	subscript that indicates that the calculation is in the global X-direction;
y	subscript that indicates that the calculation is in the global Y-direction;
α_i	ratio of axial load of column i to that of representative column j=$[P_i/P_j]$;
β_i	ratio of flexural stiffness of column i to that of representative column $j[=(E_i I_i)/(E_j I_j)]$;

γ_i ratio of height of column i to that of representative column $j[=(h_i h_j)]$;
Δ interstory drift;
ρ_{ai} and ρ_{bi} fixity factors at A_i and B_i, respectively;
ϕ_i π/K_i;
ψ_{ai} and ψ_{bi} ratios $\sum(EI/h)_c/\sum(EI/L)_g$ at ends A_i and B_i, respectively;
θ interstory angle of twist of the story floor;
Θ_{ai} and Θ_{bi} rotations of column i at A_i' and B_i' with respect to column's cord, respectively.