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Boundary stress resolution and its application to adaptive finite element analysis

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Abstract. A novel boundary stress resolution method is suggested in this paper, which is based upon the displacements of finite element analysis and of high precision with stress boundary condition strictly satisfied. The method is used to modify the Zienkiewicz-Zhu (Z^2) *a posteriori* error estimator and for the h-version adaptive finite element analysis of crack problems. Successful results are obtained.

Key words: boundary stress resolution; Z^2 error estimator; h-version adaptive finite element analysis; crack problem.

1. Introduction

Discretization should be carried out on the domain to be solved while using finite element method (FEM) to solve an engineering problem. This dicretization procedure, however, makes the numerical solution of FEM approximate to the true solution only to a limited extent, viz., there exists discretization error. The purpose of adaptive FEM is to estimate the magnitude of the dicretization error quantitatively and feedback the error information to the mesh parameters, then gain specified computation precision economically and effectively through continuous adjustment and optimization of meshes.

The adaptive finite element method was introduced in the early 1970's. Due to the creative work of Babuska, Zienkiewicz and Zhu, etc., the method has been developing rapidly in recent years. Briefly, the method comprises three basic aspects:

1) estimation of the magnitude of discretization error quantitatively (*a posteriori* error estimator);
2) feedback of the error information to mesh parameters (adaptive strategy) and
3) implementation of adaptive strategy.

Among them *a posteriori* error estimator is of paramount importance, its precision is closely related both to the optimizing extent of meshes and to the amount of computation work. This paper suggests a method for resolving boundary stresses, which strictly meets stress boundary condition with resolved stresses of high precision. The method is used to modify the Zienkiewicz-Zhu (Z^2) *a posteriori* error estimator. h-version adaptive finite element analysis is carried out using both modified Z^2 error estimator and the original one. Numerical results show that the modified estimator is more accurate than the original one, so faster convergence rate and better

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optimized mesh can be obtained. The technique is also used for the h-version adaptive finite element analysis of crack problems with satisfactory results achieved.

2. Boundary stress resolution

The stress boundary condition in terms of displacements at stress boundary Γ_σ is

$$[G(u_{i,j}+u_{j,i})+\lambda\,\delta_{ij}u_{k,k}]\,n_j=t_i \tag{1}$$

In addition, at Γ_σ there exists the following identical equation

$$\frac{\partial u_i}{\partial \xi_j}\equiv u_{i,k}\frac{\partial x_k}{\partial \xi_j} \tag{2}$$

where G and λ are Lame constants; t_i the force acting on the boundary Γ_σ; δ_{ij} Chronecker dalta, $\delta_{ij}=\begin{cases}1 & i=j\\ 0 & i\neq j\end{cases}$; ξ local coordinate; $u_{i,j}=\dfrac{\partial u_i}{\partial x_j}$.

For 2-dimensional problems, each of Eqs. (1) and (2) contains two equations, $u_{i,j}$ includes four unknowns, while for 3-dimensional problems, Eqs. (1) and (2) contain three and six equations respectively and $u_{i,j}$ includes nine unknowns. Consequently $u_{i,j}$ can be solved uniquely. Taking plane stress problem as an example, for four node quadrilateral element, the above equation system can be written in matrix form as

$$[A]\{u'\}=\{f\} \tag{3}$$

where

$$[A]=\begin{bmatrix} n_2 & -n_1 & 0 & 0\\ 0 & 0 & n_2 & -n_1\\ n_1D_0 & n_2G & n_2G & n_1D_0\mu\\ n_2D_0\mu & n_1G & n_1G & n_2D_0 \end{bmatrix}$$

$$\{u'\}=\left\{\frac{\partial u}{\partial x}\ \ \frac{\partial u}{\partial y}\ \ \frac{\partial v}{\partial x}\ \ \frac{\partial v}{\partial y}\right\}^T$$

$$\{f\}=\left\{\frac{u_2-u_1}{l}\ \ \frac{v_2-v_1}{l}\ \ t_x\ t_y\right\}^T$$

$$D_0=E/(1-\mu^2)$$

E, μ are Young's module and Poisson's ratio respectively; u_1, u_2, v_1, v_2, l are the displacements of finite element analysis of any two adjacent nodes at stress boundary Γ_σ and the length of them; n_1, n_2, are outer normal directional cosine of the stress boundary.

For $|A|\equiv -D_0G\neq 0$, $u_{i,j}$ exists uniquely. As for node stresses at boundary, they can be calculated from geometrical and physical equations.

3. Z^2 error estimator and its modification

3.1. Z^2 error estimator

For a linear elastic problem, it follows the basic equation expressed by Eq. (4)

$$Lu-b=S^TDSu-b=0 \tag{4}$$

where S is the matrix of strain derivatives; D the matrix of elastic coefficients; b vector of body forces and L the linear operator, $L=S^TDS$

The system equation of finite element method with displacements as basic variables is generally established based on the theorem of minimum energy. Provided u and $\hat{u}$ are the accurate and the FEM solutions of displacements respectively, the accurate and FEM solution of stresses σ, $\hat{\sigma}$ can be obtained from Eq. (5)

$$\sigma=DSu,\ \hat{\sigma}=DS\hat{u} \tag{5}$$

Define local errors of displacements and stresses as

$$e_u=u-\hat{u},\ e_\sigma=\sigma-\hat{\sigma} \tag{6}$$

and error of the discrete system in energy norm as

$$\|e\|=\left[\int_\Omega e_u{}^T Le_u d\Omega\right]^{1/2}=\left[\int_\Omega e_\sigma{}^T D^{-1}e_\sigma d\Omega\right]^{1/2} \tag{7}$$

Generally, accurate stresses can not be obtained, except for simple problems. Errors can be approximately calculated from relatively accurate modified stress, σ^*, viz.

$$\|e^*\|=\left[\int_\Omega (\sigma^*-\hat{\sigma})^T D^{-1}(\sigma^*-\hat{\sigma})d\Omega\right]^{1/2} \tag{8}$$

The relative error of the system η and the effective index of error estimator θ are defined as

$$\eta=\frac{\|e^*\|}{[\|\hat{\sigma}\|^2+[\|e^*\|^2]^{1/2}} \tag{9}$$

$$\theta=\frac{\|e^*\|}{\|e\|} \tag{10}$$

where

$$\|\hat{\sigma}\|^2=\int_\Omega \hat{\sigma}^T D^{-1}\hat{\sigma}d\Omega$$

Zienkiewicz and Zhu, based upon the following intuition, proposed the error estimator: for the displacement approximation of C_0 order, stresses are neither continuous nor accurate in the domain. If stresses are so treated through stress smoothing that they could be as continuous in the domain as displacements, then the modified stresses are more accurate than those from FEM analysis. Assuming that stresses can be expressed by the same shape function as that of displacements, viz.,

$$\sigma^* = N\bar{\sigma}^* \tag{11}$$

then the modified node stresses can be obtained through Eq. (12b) according to Eq. (12a) established by least square method.

$$\int_\Omega N^T(\sigma^* - \hat{\sigma})\,d\Omega = 0 \tag{12a}$$

$$\bar{\sigma}^* = B^{-1}\int_\Omega N^T\hat{\sigma}\,d\Omega \tag{12b}$$

where

$$B = \int_\Omega N^T N\,d\Omega$$

When the modified stresses are calculated, the errors defined as in Eqs. (8) and (9) can be obtained. if the domain for integration in Eq. (7) to (9) is changed into elements, then the errors of elements can also be obtained.

3.2. Modified Z^2 error estimator

Z^2 error estimator has been proved right in theory and numerical experiments have shown that it is quite effective for linear elastic problems. It should be pointed out, however, that the modified stresses obtained from Eq. (12) only meet the requirement of stress continuity in the domain, stress boundary condition is still not satisfied. If we could make the modified stresses meet stress boundary condition, the precision of error estimation should be increased. The method described in Eqs. (1) and (2) can provide more accurate boundary stresses, so a modified scheme of Z^2 error estimator is suggested as follows.

(1) Calculate the modified node stresses of system using Eq. (12);
(2) Remodify node stresses at stress boundary, Γ_σ, according to Eqs. (1) and (2);
(3) Force shear stresses of the nodes at sliding displacement boundaries to be zero;
(4) Calculate the errors of both elements and system using Eqs. (8) and (9):

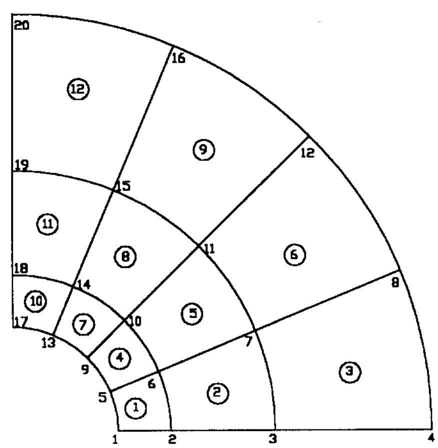


Fig. 1 Mesh of a think-wall cylinder (quarterly).

Table 1 Comparison of boundary stresses

nodes		accurate stress values			Modified values of Z^2			Modified values in this paper		
		σ_x	σ_y	τ_{xy}	σ_x	σ_y	τ_{xy}	σ_x	σ_y	τ_{xy}
nodes at inner boundary	1	−10.00	11.33	0.00	−4.62	11.61	−1.62	−9.78	10.45	0.00
	5	−6.88	8.21	−7.54	−2.25	9.22	−5.74	−6.82	7.49	−7.15
	9	0.67	0.67	−10.67	3.49	3.49	−8.11	0.33	0.33	−10.12
	13	8.21	−6.88	−7.54	9.22	−2.25	−5.74	7.49	−6.82	−7.15
	17	11.33	−10.00	0.00	11.61	−4.62	−1.62	10.45	−9.78	0.00
nodes at outer boundary	4	0.00	1.33	0.00	−0.55	1.01	−0.15	0.05	1.22	0.00
	8	0.20	1.14	−0.47	−0.32	0.78	−0.55	0.22	1.05	−0.42
	12	0.67	0.67	−0.67	0.23	0.23	−0.78	0.64	0.64	−0.59
	16	1.14	0.20	−0.47	0.78	−0.32	−0.55	1.05	0.22	−0.42
	20	1.33	0.00	0.00	1.01	−0.55	−0.15	1.22	0.05	0.00

Table 2 Comparison of relative errors and effective indices

No. of elements	accurate error	estimated value by Z^2		estimated value by modified scheme	
		error(%)	effective index	error(%)	effective index
1	36.47	29.66	0.81	34.27	0.94
3	17.46	24.39	1.40	21.21	1.21
4	34.46	32.31	0.89	34.28	0.94
6	17.50	23.70	1.35	21.27	1.22
7	36.46	32.31	0.89	34.28	0.94
9	17.50	23.70	1.35	21.27	1.22
10	36.47	29.66	0.81	34.27	0.94
12	17.46	24.39	1.40	21.21	1.21
system	28.64	28.39	0.99	28.69	1.00

Example 1.

Shown in Fig. 1 is a thick-wall cylinder with an inner diameter of 2 m, an outer one of 8 m and bearing an inner pressure of 10 MPa. Table 1 lists the node stresses at boundaries and Table 2 lists the relative errors of boundary elements and the effective indices of error estimation. It can be seen from the Tables, that the modified scheme proposed above can give a high accuracy of boundary stresses and make more accurate error estimation.

4. *h*-version adaptive finite element analysis

4.1. Adaptive strategy

The aim of adaptive analysis is to make the relative system error, η, less than or equal to a given value, η_0 (5~10% in general), viz.,

$$\eta \leq \eta_0 \tag{13}$$

If $\eta > \eta_0$, then the meshes need to be enriched. The enriching index, ξ_i, of elements is defined as

$$\xi_i = \eta_i / \eta_0 \tag{14}$$

So, the elements to be enriched are those with $\xi_i > 1$, η_i is the relative error of the element i. The adaptive analysis requires that all elements have the same precision or error, viz., optimized mesh, so the allowable error of element i for the next analysis can be obtained from Eq. (15)

$$\|e^*\| = \eta_0 \left[\frac{\|e^*\|^2 + \|\hat{\sigma}\|^2}{NE} \right]^{1/2} \tag{15}$$

Where NE is the number of elements in the system. According to the theory of FEM,

$$\|e\| \propto h^{\min(P,\lambda)} \tag{16}$$

where h is the element size, P the order of interpolation shape function, λ stress singularity index of the problem to be solved ($\lambda = 0.5 \sim 0.711$).

Combine Eqs. (14), (15) and (16), the element size for the next analysis can be obtained as follows

$$h = \xi_i^{-1/P} h_i \tag{17}$$

and around the singularity,

$$h = \xi_i^{-1/\lambda} h_i \tag{18}$$

where h_i is the size of element i being under computation.

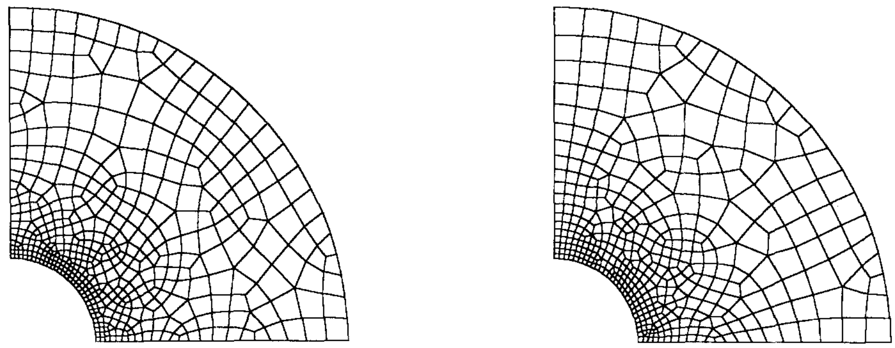
4.2. Implementation

The adaptive strategy described above can be implemented by adaptive remeshing. The reader is refered to references for detailed mesh generation technique. The procedure of adaptive remeshing is as follows.

(1) Preparation of initial background mesh and mesh parameters (viz. element size);
(2) Auto-generation of meshed and finite element computation;
(3) Estimation of analysis precision
goto (5), if precision is reached,
goto (4), if precision not reached;
(4) Take the mesh of current analysis as background mesh (according to principle of maximum interior angle, each quadrilateral element is divided into two triangles), and then calculate background mesh parameters from Eq. (17) or Eq. (18), goto (2);
(5) end.

Example 2. Thick-wall cylinder

Take the mesh shown in Fig. 1 as one for initial analysis and employ Z^2 error estimator and modified Z^2 estimator respectively to perform adaptive analysis. Let $\eta_0 = 5\%$. The results of adaptive analysis for two times are all convergent to the given precision. In Fig. 2(a), shown is the final mesh using Z^2 error estimator, having 501 nodes ($M = 501$) and 450 elements ($NE = 450$), and $\eta_{Z^2} = 4.7\%$; in Fig. 2(b), shown is the final mesh using the modified estimator, $M = 439$,



(a) Z^2 eror estimator (b) modified Z^2 estimator

Fig. 2 Final mesh of the adaptive analysis of a thick-wall cylinder.

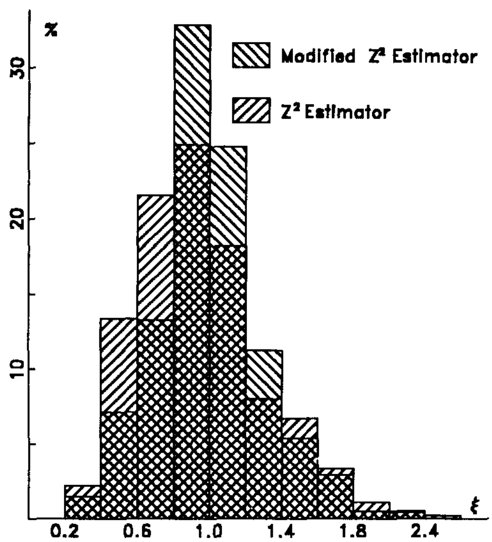


Fig. 3 Bar diagram of the distribution of effective enriching index.

$NE=393$, $\eta_{MZ^2}=5.0\%$.

There exists a theoretical solution to this example, so errors can be obtained accurately to compare the optimization extent and the convergence rate of the two error estimators. Fig. 3 gives the bar diagram of the effective enriching index distribution of two sets of meshes and Fig. 4 shows the convergence rate of two error estimators, viz., $\log NDF - \log \eta_e$ curve. *NDF* is the number of degrees of the system and η_e is the accurate value of relative error. From the figures, it can be seen that compared with Z^2 error estimator, the modified estimator achieves more optimized mesh and faster convergence rate.

5. Crack problem under compression and shearing

Goodman joint model is employed to simulate cracks. As the stresses of a Goodman element

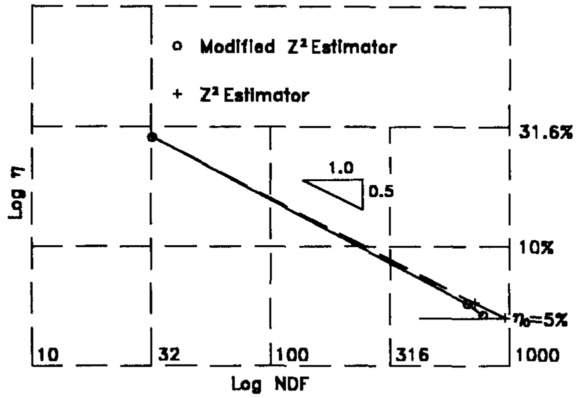
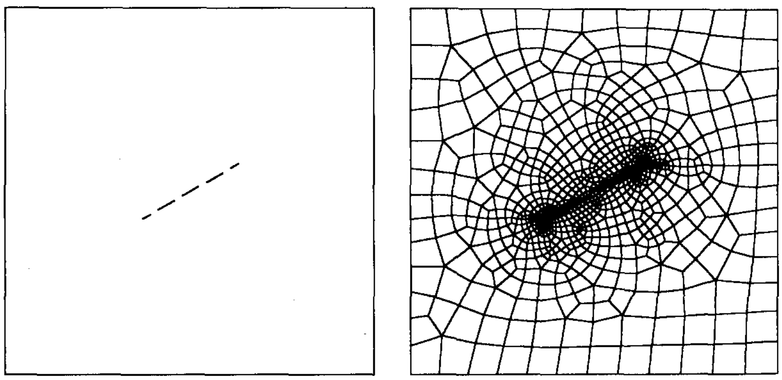
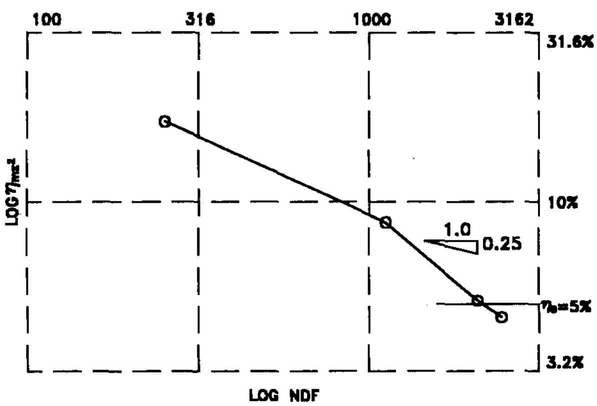


Fig. 4 Diagram of convergence rate.



(a) (b)



(c)

Fig. 5 Adaptive analysis for a crack problem.

are a linear function of its displacements, its stresses and displacements have the same order of precision. From Eq. (5), it can be known that the stress precision of general elements is one order lower than that of displacements, that is, the stress precision of joint elements is one order higher than that of general elements. Consequently, it is reasonable to consider that compared with the stresses of general elements, the stresses of joint elements can be assumed to be accurate, so the errors of them are zero. In the adaptive process, cracks are taken as internal stress boundary and the stresses of general elements at the upper and lower banks of cracks are corrected using joint element stresses according to Eqs. (1) and (2). The enrichment of joint elements depends upon the mesh compatability required by the enrichment of general elements at the two banks of cracks.

Example 3. Block with a single crack

Shown in Fig. 5(a) is a block (20×20 cm in size) with a single crack of 6 cm long at an angle of 30° to horizon. Parameters for the analysis are as follows: for the block, young's module, $E=15000$ MPa, Poison's ratio, $\mu=0.24$; and for the crack, normal stiffness, $K_n=25$ MPa/M, shearing stiffness, $K_s=10$ MPa/M. The problem is of plane stress with weight neglected. The aim of adaptive analysis is $\eta_0=5\%$. The optimized mesh after adaptive analysis for three times is shown in Fig. 5(b), having 1247 nodes, 1222 elements and $\eta_{mz^2}=4.5\%$, being convergent to the given precision. Fig. 5(c) is the diagram of convergence rate. The example has a singularity index of $\lambda=0.5$, a higher convergence rate is still achieved.

6. Conclusions

This paper suggests a method for boundary stress resolution, which strictly meets stress boundary condition and has a high precision. When used for the modification of Z^2 error estimator, the method can increase the precision of error estimator, making convergence rate higher and adaptive mesh more optimum with a little calculation added. In addition, the method, when used for crack problems, can widen the appliction scope of adaptive approches, viz., to the adaptive analysis of discontinuous media.

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