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Methods of pairwise comparisons and fuzzy global criterion for multiobjective optimization in structural engineering

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Abstract. The method of pairwise comparison inherently contains information of ambiguity, fuzziness and conflict in design goals for a multiobjective structural design, This paper applies the principle of paired comparison so that the vaguely formulated problem can be modified and a set of numerically acceptable weight would reflect the relatively important degree of multiple objectives. This paper also presents a fuzzy global criterion method ($FGCM_\lambda$) included fuzzy constraints that coupled with the objective weighting rank obtained from the modified pairwise comparisons for fuzzy multiobjective optimization problems. Descriptions in sequence of this combined method and problem solving experiences are given in the current article. Multiobjective design examples of truss and mechanical spring structures illustrate this optimization process containing the revising judgement techniques.

Key words: pairwise comparison; fuzzy global criterion method; global criterion method; multiobjective fuzzy optimization; weighting coefficient; structural design; consistent matrix.

1. Introduction

The ε-constraint method (Carmichael 1980), the weighting method (Cohon 1978), and the min-max approach with weighting strategy (Osyczka 1985) are primary techniques conventionally to generate a noninferior set for a multiobjective engineering optimization. A designer usually decides a set of absolute and precise weighting values represented the degrees of importance to multiple design objectives. Actually such weighting values in nature contain amibiguity, conflict, and fuzzy information. Shih and Yu (1995) had presented both crisp and fuzzy weighting strategies in pure continuous or mixed design space with crisp and fuzzy constraints. However, the relative weights among objectives are difficult to get a total idea by the human comparison or judgement. That is true of increasing the problem complication particularly when many objectives exist. Saaty (1980) suggested that it is simpler and better to compare the design objectives in pairs, than trying to compare all objectives at once, based upon the experience, experiment, and/or

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the design requirement. Subsequently one can adopt a systematic "scale" for the pairwise comparisons which focuses on two objectives and their relation at a time. Thus, a reciprocal matrix can be built up at this stage.

However, this matrix captured from the paired comparisons cannot directly apply to the weighting method or min-max approach in the optimization process. This can be done by revising the matrix untill a set of satisfactory weights of the design objectives are obtained. This paper examines the above procedure based on the program developed by authors that modifies the inconsistent matrix to be consistent to result in a final weighting rank for individual objective. Beyond obtaining an acceptable weighting rank, this algorithm enables one to maintain the original information of paired comparisons as much as possible. We also show the principle and process of the revising judgement for obtaining a reasonable final weight. The computing process, some data and experiences are presented to share with readers who are interested in this subject.

The theory of pairwise comparisons in used popularly in the decision making concerning economic, social and management sciences. It is conceptually able to apply on structural engineering synthesis, however, very few literatures systematically reported on this kind of design process neither a really structural design problem. Such that Koski (1984) positively mentioned about this method in little without any example for structural design. The possible reasons are that the applied engineers lack the knowledge about the method of paired comparisons or lack the fuzzy perception embodying in the problems. The other reason may be the existing conventional design method that lacks a way to deal with the fuzzy information.

One can realize that a set of weighting coefficients containing fuzzy nature is best to fit into a design methodology also containing fuzzy nature. Because of this motivation, we present a fuzzy global ciriterion method ($FGCM_\lambda$) based on fuzzy logic (Zadeh 1965) that is a convenient way to deal with a problem consisting of fuzzy weights and other fuzzy information. This max-min approach of the fuzzy global criterion is a variant of λ-formulation method (Rao 1987a and 1987b) by adding additional constraints. The model has an optimum result that is closest to the ideal solution. The $FGCM_\lambda$ can deal with both objectives and constraints contained fuzzy information. That can generates a Pareto optimum set as well as the preferred solution. We also compare this method with another fuzzy global criterion method of β-formulation of min-max variant ($FGCM_\beta$) originated from the crisp global criterion approach (GCM) (Hwang and Masud 1979, Hajela and Shih 1990). The consistent weightings of design objectives which are obtained from pairwise comparisons method is then applied to this fuzzy global criterion of λ-formulation for solving a fuzzy multiobjective optimization problem.

The subsequent sections will introduce the analysis of pairwise comparisons, revising algorithm and judgement for yielding to the consistent matrix. The description in this paper is concise and helpful to a practitioner to understand and further apply the mentioned techniques. The methodologies of pairwise comparisons combined with fuzzy global criterion approach are illustrated by multiobjective mechanical and multiobjective structural design examples. The paper also contains some solution notes and experiences of this work. At the end we give a closing remark with discussion about the presented design strategy.

2. Analysis of reciprocal pairwise comparison matrices

It is known that the pairwise comparisons enables one to improve the cardinal consistency

Table 1 The scale and its definition

Intensity of importance	Definition
1	Equal importance
3	Weak importance of one over another
5	Essential or strong importance of one over another
7	Demonstrated importance of one over another
9	Absolute importance of one over another
2,4,6,8	Intermediate values between the two adjacent judgements

in the measurement of weights using as much information as possible. A commonly used scale system (Saaty 1980) shown in Table 1 suggests the intensity scale of comparative importance and its definitions for design objectives. In this scale system, the range of scale values is 1 to 9. One can use any value between the two intensities to describe the importance. Of course one can adopt another scale system for expressing the degree of importance. Saaty (1980) showed that this scale system is quite good compared to any other scale. It has the advantage of simplicity, and is appropriately quite natural.

In an optimization problem with m objectives, let ω_i ($i=1, \cdots, m$) represent the degree of importance of the ith design objective. We let a_{ij} has the following relation:

$$a_{ij}=\frac{\omega_i}{\omega_j}, \qquad i, j=1, \cdots, m \tag{1}$$

A matrix $\boldsymbol{A}$ can be constructed by the ratio of a_{ij} and $1/a_{ij}$, which is denoted as following:

$$\boldsymbol{A}=\begin{bmatrix} 1 & a_{12} & a_{13} & \cdots & a_{1m} \\ 1/a_{12} & 1 & a_{23} & \cdots & a_{2m} \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ \cdot & \cdot & \cdot & \cdots & \cdot \\ 1/a_{1m} & 1/a_{2m} & 1/a_{3m} & \cdots & 1 \end{bmatrix} \tag{2}$$

The above expression can be simplified as $\boldsymbol{A}=[a_{ij}]$. If the element satisfies the condition of $a_{ji}=1/a_{ij}$, we call such matrix $\boldsymbol{A}$ is a reciprocal matrix. If the judgement of the weighting coefficient ω_i for the ith objective is perfect in all comparisons, then $a_{ik}=a_{ij}\cdot a_{jk}$ for i, j, and k. Thus one defines the matrix $\boldsymbol{A}$ to be consistent.

We utilize Eq. (1) and obtain:

$$a_{ij}\frac{\omega_j}{\omega_i}=1, \qquad i, j=1, \cdots, m \tag{3}$$

The summation of the above equation is written as:

$$\sum_{j=1}^{m} a_{ij}\,\omega_j\frac{1}{\omega_i}=m, \qquad i=1, \cdots, m \tag{4}$$

or it can be expressed as:

$$\sum_{j=1}^{m} a_{ij}\,\omega_j = m\,\omega_i, \qquad i=1, \cdots, m \tag{5}$$

Eq. (5) can be equivalent to and represented by a matrix form in the following:

$$A\,\omega = m\,\omega \tag{6}$$

In Eq. (6), ω indicates the eigenvector of matrix A with eigenvalue m. This matrix equation has a nonzero solution if and only if m is an eigenvalue of the matrix A. For a unit rank matrix A, all of the eigenvalues α_i ($i=1, \cdots, m$) are zero, except one, which is denoted by α_{max} and is equivalent to m.

In a practical design case, the ratio of a_{ij} is obtained by fuzzy, conflicting, subjective, and/or objective measurements. Thus, a_{ij} will deviate from the consistent element ω_i/ω_j, and Eq. (6) cannot hold any more. Our problem is to find the priority vector ω that satisfies the following matrix representation:

$$A\,\omega = \alpha_{max}\,\omega \tag{7}$$

where

$$\sum_{i=1}^{m} \omega_i = 1 \tag{8}$$

The helpful matter is that one can change the eigenvalue α_{max} with a small amount, and by changing the input a_{ij} of the reciprocal matrix A with a small amount. The consistency index (CI) and the consistency ratio (CR) acts as indicators of closeness to the consistency. CI and CR can be expressed as:

$$CI = \frac{\alpha_{max} - m}{m-1} \tag{9}$$

$$CR = \frac{CI}{RI} \tag{10}$$

Where RI (random index) is called the consistency index of a randomly generated reciprocal matrix from the scale 1 to 9. The Table 2 gives the order of the matrix (first row) and the averaged value of RI (second row). Saaty (1980) suggested, in general, which CI value or CR value of 0.1 or less is considered acceptable for the consistency.

3. Revising judgements on the matrix

The processes of revising the matrix A in Eq. (2) which consists of the pairwise comparisons of design objectives is illustrated in Fig. 1. In the first step, we compute the principal eigenvalue α_{max} and the corresponding eigenvector ω_i ($i=1, \cdots, m$). In the second step, assuming the consistency index of the matrix A is large, one constructs another matrix A_a in terms of the relation of Eq. (1) and Eq. (2) as following:

Table 2 The averaged value of *RI* (Saaty 1980)

1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0.00	0.00	0.58	0.90	1.12	1.24	1.32	1.41	1.45	1.49	1.51	1.48	1.56	1.57	1.59

$$A_a = \begin{bmatrix} \omega_1/\omega_1 & \omega_1/\omega_2 & \cdots & \omega_1/\omega_m \\ \omega_2/\omega_1 & \omega_2/\omega_2 & \cdots & \omega_2/\omega_m \\ \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot \\ \omega_m/\omega_1 & \omega_m/\omega_2 & \cdots & \omega_m/\omega_m \end{bmatrix} \tag{11}$$

It is simpler to write the form as $A_a = [\omega_i/\omega_j]$. We revise the judgement on the ith row of matrix A in Eq. (2) by replacing a_{ij} ($j=1, \cdots, m$) with ω_i/ω_j ($j=1, \cdots, m$) in Eq. (11). based on the sum of the largest difference such as:

$$\text{row}_i = \max_i \left[\sum_{j=1}^{m} |(a_{ij}) - (\omega_i/\omega_j)| \right], \quad i=1, \cdots, m \tag{12}$$

If the relative importance of the ith design objective to the jth design objective is already and clearly had, then on can fix this associated element of $[a_{ij}]$. In other word, it is not necessary to modify or replace it at all. The reason for modifying the raw with the largest difference is that because of the following:

$$\omega_i = \frac{\omega_i}{\omega_1 + \omega_2 + \cdots + \omega_m} = \frac{\sum_{j=1}^{m} a_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{m} a_{ij}} \tag{13}$$

which shows that the sum of the ith row elements has the maximum influence on the weight of the ith objective. Therefore the modification of certain row associated to the largest row difference between $[a_{ij}]$ in Eq. (2) and $[\omega_i/\omega_j]$ in Eq. (11) is reasonable. Experience shows that the iteration will converge satisfactorily at a small ε (Fig. 1) of about one or less while satisfying both of the *CI* and *RI* are below 0.1.

4. Approach of fuzzy global criterion

In multiobjective optimization, a min-max variant of the global criterion method (GCM) with crisp constraints (Hajela and shih 1990) is expressed as

$$\text{minimize } \beta \tag{14}$$

$$\text{subject to } g_j(X) \le b_j, \qquad j=1, 2, \cdots, p \tag{15}$$

$$h_k(X) = 0, \qquad k=1, 2, \cdots, q \tag{16}$$

$$X_i^l \le X_i \le X_i^u \qquad i=1, 2, \cdots, n \tag{17}$$

$$X = [x_1, x_2, \cdots, x_n]^T$$

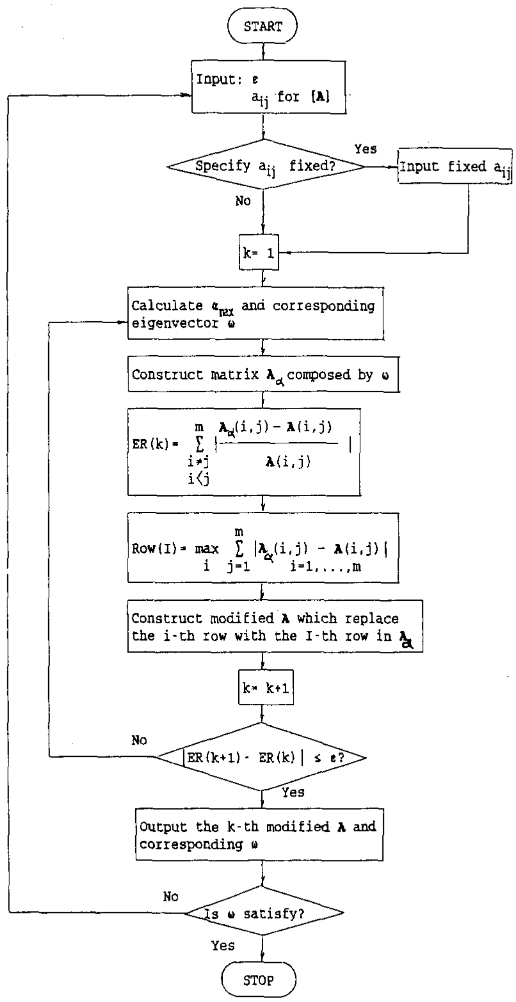

Fig. 1 Flow-diagram of revising judement for a reciprocal matrix.

and the following additional constraints:

$$\omega_i \left| \frac{f_i(X)-f_i^{id}(X)}{f_i^{id}(X)} \right| -\beta \leq 0, \qquad i=1, \cdots, m \tag{18}$$

$$\sum_{i=1}^{m} \omega_i = 1 \tag{19}$$

where β is a scalar treated as an additional design variable in the optimization process; $g_j(X)$ and h_k (X) represents the jth inequality and kth equality constrained functions, respectively; ω_i represents the design degree of importance corresponding to the ith objective; $f_i^{id}(X)$ indicates the ith ideal objective value defined by the designer or obtained by optimizing individual objective.

Very often the fuzziness or vague information exists in the design problems. The fuzzy objective function and contraints are characterized by objective membership functions μ_f and constraint membershiip functions μ_g, respectively. If the formulation is based on Eq. (14) to Eq. (18) included fuzzy constraints, then we call this method as fuzzy global criterion method of β-formulation

($FGCM_\beta$). If the linear type membership function is adopted, an approach modified from Rao (1987b) of fuzzy λ-formulation with the additions of Eq. (24) to Eq. (25) are presented in the following mathematical form:

$$\text{maximize} \quad \lambda \tag{20}$$

$$\text{subject to} \quad \lambda \leq \mu_{fi}(X) \tag{21}$$

$$\lambda \leq \mu_{gj}(X), \qquad j=1, \cdots, p \tag{22}$$

$$\lambda \leq \mu_{hk}(X), \qquad k=1, \cdots, q \tag{23}$$

$$\omega_i \left| \frac{f_i(X)-f_i^{id}(X)}{f_i^{id}(X)} \right| = \omega_{i+1} \left| \frac{f_{i+1}(X)-f_{i+1}{}^{id}(X)}{f_{i+1}{}^{id}(X)} \right| \tag{24}$$

$$\sum_{i=1}^{m} \omega_i = 1 \tag{25}$$

where

$$\mu_{gj}(X)=\begin{cases} 0, \text{ if } g_j(X)>b_j+d_j \\ 1, \text{ if } g_j(X)\leq d_j \\ 1-\left\{\dfrac{g_j(X)-b_j}{d_j}\right\}, & \text{if } b_j<g_j(X)\leq b_j+d_j \end{cases} \tag{26}$$

and

$$\mu_{fi}(X)=\begin{cases} 0, & \text{if } f_i(X)>f_i^{max} \\ 1, & \text{if } f_i(X)\leq f_i^{min} \\ \dfrac{f_i^{max}-f_i(X)}{f_i^{max}-f_i^{min}}, & \text{if } f_i^{min}<f_i(X)\leq f_i^{max} \end{cases} \tag{27}$$

Here d_j is an allowable fuzzy transition interval for the jth inequality constraint. The values of f_i^{max} and f_i^{min} are the largest and the smallest values obtained by the optimization in crisp and fuzzy feasible domains respectively. Since this fuzzy λ-mathematical formulation yields the same results as the global criterion method of minimizing β with fuzzy constraints ($FGCM_\beta$) (Shih and Lai 1994, Shih and Yu 1995), we call it fuzzy global criterion method ($FGCM_\lambda$). One can see a verification from the results of the three-bar structural design in the next paragraph. Because the global criterion method can generate a Pareto solution set, therefore we can predict that if the constraints contain fuzzy information, a set of Pareto optimal solution also can be obtained by the fuzzy global criterion approach.

4.1. Three-bar truss design of two-objective criteria

The algorithm of the fuzzy global criterion method of λ-formulation is implemented in a popular three-bar truss design shown in Fig. 2. The problem is to find design variables of x_1 and x_2 to minimize the structural weight of $W(X)$ and the loading deflection of $\delta(X)$. We assume a 20% tolerable fuzzy transition zone in the allowable stress and side constraints. We also adopt the linear membership function in this study. Other information is available to find in Rao's

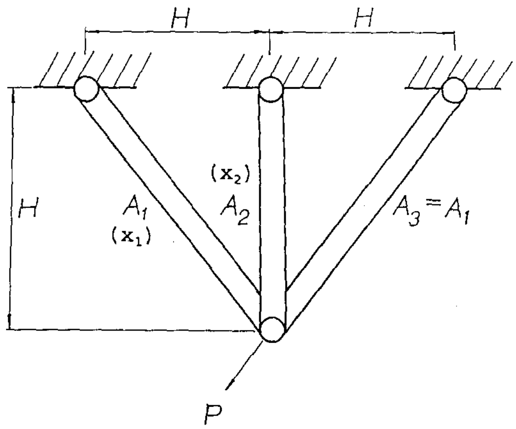

Fig. 2 Static three-bar truss ($P=20$, $H=1$, $\rho=1$).

Table 3 Optimum results of a 3-bar truss desgn by $FGCM_\lambda$ and $FGCM_\beta$.

Weights $(\omega_w, \omega_\delta)$	$FGCM_\lambda$ $(W^*(X), \delta^*(X))$	Calculated weights $(\omega_w, \omega_\delta)$
0.4, 0.6	5.9053, 3.0303	0.4002, 0.5998
0.5, 0.5	5.3984, 3.3965	0.5000, 0.5000
0.6, 0.4	4.9217, 3.8194	0.6003, 0.3997
0.7, 0.3	4.4567, 4.3372	0.7003, 0.2997
Weights $(\omega_w, \omega_\delta)$	$FGCM_\beta$ $(W^*(X), \delta^*(X))$	Calculated weights $(\omega_w, \omega_\delta)$
0.4, 0.6	5.8384, 3.0001	0.3998, 0.6002
0.5, 0.5	5.3860, 3.3851	0.4995, 0.5005
0.6, 0.4	4.8732, 3.7704	0.6000, 0.4000
0.7, 0.3	4.4327, 4.3229	0.7020, 0.2980

(1987b) paper.

The final designs of fuzzy optimization are listed in Table 3 that shows the results of $FGCM_\lambda$ as well as $FGCM_\beta$ from the same starting point by the numerical optimization. From the table, the optimal results for both methods are almost the same. Especially, one can confirm this point by computing the actual weighting values depicted in calculated weights in the last column. The experiences told us that the numerical iterations in the fuzzy global criterion approach ($FGCM_\lambda$) are fewer than with the global criterion strategy (GCM) when the constraints are crisp. When the design constraints are fuzzy, the procedure of the global criterion approach of β-formulation ($FGCM_\beta$) requires more work than the fuzzy global criterion approach of λ-formulation. Thus we recommend that the fuzzy global criterion method replace the global criterion method for solving multiobjective optimization problems with or without fuzzy information. In the next section of illustrative design examples, we only show the work and result of $FGCM_\lambda$.

5. Illustrative examples

A structural design of mechanical spring is presented that shows the techniques of paired comparisons with or without the restoration to the resulting matrix A. A dynamic truss design of maximizing the natural frequency that shows the techniques of paired comparisons having a role to reexamine the possible error or mistake in the initial paired judgement of corresponding weighting rank. Then we plug those final relative weights of design goals obtained from paired comparison technique into the FGCM_λ and solve the optimum design problem.

5.1. Design of a mechaincal compression spring

We know that mechanical springs are used in machines to exert force, to provide flexibility, and to store or absorb energy. A helical spring of round wire modified from Arora (1989) shown in Fig. 3 is assumed to be used for resisting a dynamic compressive load. Design goals are to minimize structural weight f_1, to minimize free length f_2, and to maximize the applied load f_3. The design variables are x_1, x_2, x_3, and x_4 that represent the number of active coils, the mean coil diameter, the wire diameter, and the applied load, respectively. The complete mathematical formulations are:

$$\text{minimize} \quad f_1(X)=\pi^2(x_1+N_q)x_2x_3^2\rho g/4 \text{ lb.} \tag{28}$$

$$\text{minimize} \quad f_2(X)=x_4/(Gx_3^4/8x_2^3x_1)+x_3(x_1+N_q+1) \text{ in.} \tag{29}$$

$$\text{maximize} \quad f_3(X)=x_4 \text{ lb}_f. \tag{30}$$

subject to the following constraints:

$$8x_1x_2^3x_4/GX_3^4 \geq 0.5 \text{ in (deflection)} \tag{31}$$

$$(8x_4x_2/\pi x_3^2)/[(4x_2-x_3)/(4x_2-4x_3)+(0.615x_3/x_2)] \leq 136400 \text{ psi (shear stress)} \tag{32}$$

$$\sqrt{G/2\rho}\; x_3/2\pi x_2^2x_1 \geq 100 \text{ Hz (natural frequency)} \tag{33}$$

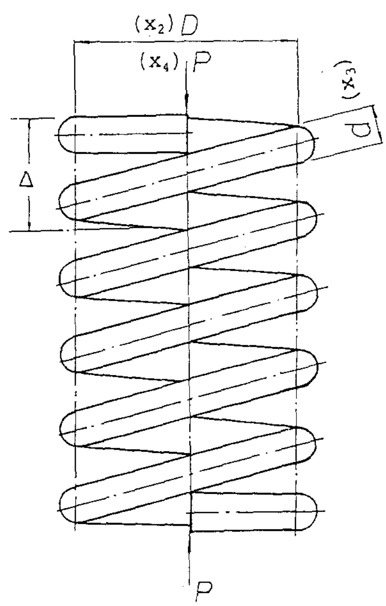

Fig. 3 A mechanical coil spring (Δ=0.5 inch).

$$(y^2/C_1^2+C_2D^2/\alpha^2)\,C_1/2y \geq f_2(X) \text{ in (buckling)} \tag{34}$$

$$x_2+x_3 \leq 1.5 \text{ in (limit on the outer diameter)} \tag{35}$$

$$2 \leq x_1 \leq 15 \tag{36}$$

$$0.25 \leq x_2 \leq 1.30 \tag{37}$$

$$0.05 \leq x_3 \leq 0.20 \tag{38}$$

where the paramters of y, C_1 and C_2 are given as:

$$y=x_4/(Gx_3^4/8x_2^3x_1) \tag{39}$$

$$C_1=0.5\,E/(E-G) \tag{40}$$

$$C_2=2\pi^2(E-G)/(2G+E) \tag{41}$$

where the number of inactive coils N_q is 2. The other useful information are $\rho=7.383E04$ lb-sec^2/in^4, $g=386$ in/sec^2, $G=1.15E07$ lb/in^2, and $E=30.E06$ lb/in^2. Optimum results of crisp optimization of indiviual objective functions are given in Table 4. It is no difficult to find the f_i^{max} and f_i^{min} corresponding to the ith design objective. The next paragraph introduces two design applications depending upon the using environment.

5.1.1. Application on a spacecraft

This spring is assumed to be a component on a space craft that requires the minimum weight that is the most critical than other design objectives. Thus, a paired comparison of three design objectives can be: f_1 is more important than f_2, the intensity is 2. f_1 is over f_3 intensity 7 and f_2 is over f_3 intensity 4. Using the above information, we construct a 3×3 reciprocal matrix:

$$[A]=\begin{bmatrix} 1 & 2 & 7 \\ 1/2 & 1 & 4 \\ 1/7 & 1/4 & 1 \end{bmatrix} \tag{42}$$

Solving for the associated maximum eigenvalue and eigenvector ω is $[0.6026, 0.3150, 0.0823]^T$. The resulting values of CI and CR are 0.0009 and 0.0015, respectively. Both of the ratios are far less than 0.1 that means a good consistency. This eigenvector is then the degree of importance associated with each objective in the design.

We formulate the linear membership functions of μ_{fi} ($i=1, 2, 3$) by Eq. (27). The completed mathematical formulation of the fuzzy global criterion method is described in Eqs. (20), (21), (24), (25), (31)-(38). The optimum results are $X^*=[6.567, 0.394 \text{ in}, 0.057 \text{ in}, 20.604 \text{ lb}_f]^T$ and $f(X^*)=[0.0077 \text{ lb}, 1.092 \text{ in}, 20.604 \text{ lb}_f]^T$.

Table 4 Results of optimizing individual design objective

	X^*				$f_1(X^*)$	$f_2(X^*)$	$f_3(X^*)$
min $f_1(X)$	9.924	0.299	0.052	20.	0.0067	1.1734	20.
min $f_2(X)$	2.824	0.595	0.064	20.44	0.0118	0.8735	20.44
max $f_3(X)$	15.	0.545	0.2	473.37	0.371	4.101	473.37

5.1.2. Application on a measuring device

This compression spring is assumed and used on a measuring device as a mechanical component. The loading capacity and the limit of mounting space are the most significant objectives. The pairwise comparisons between two objectives initially selected by designer are: f_1:f_2=1:2; f_1:f_3=1:9 and f_2:f_3=1:1. The associated eigenvector of the reciprocal matrix is obtained as $[0.1025, 0.3385, 0.5589]^T$. Both the CI and CR values are more than 0.1, and are 0.1283 and 0.2212, respectively. Based on the revising algorithm of Fig. 1, a modified matrix A_α is constructed as

$$A_\alpha=\begin{bmatrix} 1 & 0.5 & 0.1834 \\ 0.8254 & 1 & 0.6057 \\ 5.4564 & 1.6510 & 1 \end{bmatrix} \tag{43}$$

Solving for the associated maximum eigenvector, we get $\omega=[0.1254, 0.2964, 0.5783]^T$. Values of CI and CR are 0.014 and 0.024, respectively. This acceptable important rank is then substituted into the fuzzy global criterion approach; the final optimum results obtained are $X^*=[14.994, 0.414\text{ in}, 0.083\text{ in}, 56.664\text{ lb}_f]^T$ and $f(X^*)=[0.0344\text{ lb}, 2.373\text{ in}, 56.664\text{ lb}_f]^T$.

5.2. Design of a dynamic three-bar truss

A statically indeterminate three-bar structure also modified from Arora (1989) is shown in Fig. 4. This is to be designed for minimum volum $f_1(X)$, minimum deflection $f_2(X)$ of the 4th node, maximum of the lowest natural frequency $f_3(X)$, and mixmum applied loading $f_4(X)$ simultaneously. The design variables are:

x_1=Cross-sectional area of material for member 1 and 3 (m^2).
x_2=Cross-sectional area of material for member 2 (m^2).
x_3=Loading angle (degree).
x_4=Applied load (Newton).

The mathematical formulation is stated as:

$$\text{minimize} \quad f_1(X)=l(2\sqrt{2}x_1+x_2)\ (m^3) \tag{44}$$

$$\text{minimize} \quad f_2(X)=[\sqrt{2}lx_4\cos x_3/(x_1E))^2+(\sqrt{2}lx_4\sin x_3/E(x_1+\sqrt{2}x_2)))^2]^{0.5}\ (m) \tag{45}$$

$$\text{maximize} \quad f_3(X)=(3Ex_1/(\rho l^2(4x_1+\sqrt{2}x_2)))^{1/2}/2\pi \quad (\text{Hz}) \tag{46}$$

$$\text{maximize} \quad f_4(X)=x_4\ (\text{Newton}) \tag{47}$$

subject to the following constraints:

$$(x_4\cos x_3/x_1+x_4\sin x_3/(x_1+\sqrt{2}x_2)/\sqrt{2}\le 140\ (10^6)\ (\text{N/m}^2) \quad (\text{stress constraint}) \tag{48}$$

$$(\sqrt{2}x_4\sin x_3/(x_1+\sqrt{2}x_2)\le 140\ (10^6)\ (\text{N/m}^2) \quad (\text{stress constraint}) \tag{49}$$

$$\sqrt{2}lx_4\cos x_3/(x_1E)\le 0.0045\ (\text{m}) \quad (\text{limit of deflection}) \tag{50}$$

$$\sqrt{2}lx_4\sin x_3/(E(x_1+\sqrt{2}x_2))\le 0.0045\ (\text{m}) \quad (\text{limit of deflection}) \tag{51}$$

$$-[x_4\cos x_3/x_1+x_4\sin x_3/(x_1+\sqrt{2}x_2)]/\sqrt{2}\le \pi^2Ex_1/2l^2 \quad (\text{buckling constraint}) \tag{52}$$

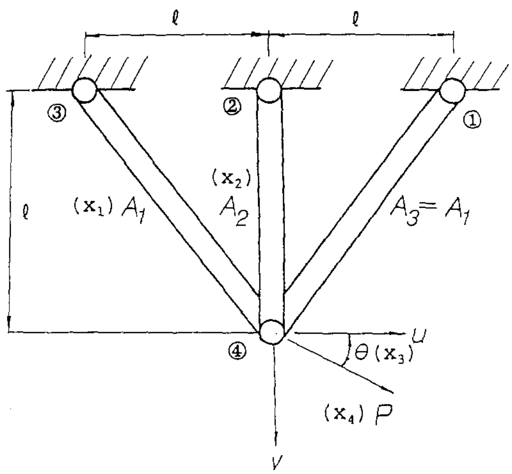

Fig. 4 Dynamic three-bar truss for the method of paired comparison.

Table 5 Results of optimizing individual objective for a dynamic three-bar structure

X_1^*	$f_1(X_1^*)$	$f_2(X_1^*)$	$f_3(X_1^*)$	$f_4(X_1^*)$
0.000498, 0.00999, 88.059, 1450000	0.01407	0.00283	242.093	1450000
X_2^*	$f_1(X_2^*)$	$f_2(X_2^*)$	$f_3(X_2^*)$	$f_4(X_2^*)$
0.01, 0.01, 90., 1450000	0.0383	0.00123	592.36	1450000
X_3^*	$f_1(X_3^*)$	$f_2(X_3^*)$	$f_3(X_3^*)$	$f_4(X_3^*)$
0.01, 0.0002, 32.0, 1450000	0.0285	0.00294	686.74	1450000
X_4^*	$f_1(X_4^*)$	$f_2(X_4^*)$	$f_3(X_4^*)$	$f_4(X_4^*)$
0.01, 0.01, 67.505, 2586859	0.0383	0.00282	592.36	2586859

$$-\sqrt{2}lx_4\sin x_3/(x_1+\sqrt{2}x_2)\leq \pi^2 Ex_2/l^3 \quad \text{(buckling constraint)} \tag{53}$$

$$-[x_4\sin_3/(x_1+\sqrt{2}x_2)-x_4\cos x_3/x_1]/\sqrt{2}\leq \pi^2 Ex_1/2l^2 \quad \text{(buckling constraint)} \tag{54}$$

$$0.00002\leq x_1\leq 0.01 \tag{55}$$

$$0.0002\leq x_2\leq 0.01 \tag{56}$$

$$0^\circ \leq x_3\leq 90^\circ \tag{57}$$

$$1450000\leq x_4 \tag{58}$$

where $E=70E09$ N/m^2, $\rho=2800$ kg/m^3, and $l=1$ m

Optimum results of the crisp optimization for individual objective function are solved and written in Table 5.

The paired comparisons of objective importance in this study are assumed as: $f_1:f_2=1:3$; $f_1:f_3=1:4$; $f_1:f_4=1:9$; $f_2:f_3=1:5$; $f_2;f_4=1:7$; and $f_3:f_4=1:6$. The associated maximum eigenvector of the reciprocal matrix is obtained as $[0.0456,\ 0.0814,\ 0.2085,\ 0.6644]^T$. CI and CR values are 0.1194 and 0.1327, respectively. Both are greater than 0.1 and unacceptable. Based on the revising algorithm in this paper, the modified matrix A_a is constructed as:

$$A_a=\begin{bmatrix} 1 & 1/3 & 1/4 & 0.0686 \\ 3 & 1 & 1/5 & 0.1225 \\ 4 & 5 & 1 & 0.3138 \\ 14.575 & 8.161 & 3.187 & 1 \end{bmatrix} \tag{59}$$

The associated maximum eigenvector ω is $[0.0413, 0.0792, 0.2411, 0.6384]^T$. Although the values of CI and CR are 0.0531 and 0.059, respectively, the element a_{41} (14.575) is still larger than 9. It is necessary to reexamine the original relation of the fourth objective to other objective. Consequently, the designer revised and concluded that f_4 is more important than f_2 with intensity 3, and f_4 is equal to f_3. The associated maximum eigenvector of the modified reciprocal matrix is obtained as $[0.0593, 0.1250, 0.4022, 0.4135]^T$. CI and CR values are 0.057 and 0.063, respectively. Both are less than than 0.1 and acceptable. This eigenvector is then the weighting rank associated with each design objective.

The linear membership functions of $\mu_{fi}(i=1, \cdots, 4)$ by Eq. (27) and $\mu_{gj}(j=1, \cdots, 11)$ from Eqs. (48~54) by Eq. (26) with assumed an allowable 20% fuzzy transition zone is then built up. The completed mathematical formulation of the fuzzy global criterion aproach can be described by the form of Eqs. (20-27). The final optimum results obtained is $X^*=[64.71\ \text{cm}^2, 10.30\ \text{cm}^2, 88.98°, 2082130\ \text{N}]^T$ and the associated optimum objective values are $f(X^*)=[28604\ \text{cm}^3, 0.2\ \text{cm}, 551.3\ \text{Hz}, 2082130\ \text{N}]^T$

6. Closing remarks

In structural engineering design, it is simpler to recognize the relative importance of two objectives at a time than that of all fuzzy objectives simultaneously. Thus, a fuzzy structural engineering optimization with unequal important objectives was derived from the method of paired comparisons, and was examined. Based on the principle of pairwise comparisons, we develop a program and process to construct and repeatedly revise the matrix structure that consequently yields to a reasonable weighting rank for design goals. To fulfill the role of unequal important rank in fuzzy optimization process, we present the fuzzy global criterion approach of λ-formulation (FGCM_λ) to accommodate this necessity. This weighting strategy allows the original fuzzy optimization can involve the weighting coefficients easily.

The first static three-bar truss design example shows that the FGCM_λ is equivalent to the FGCM_β. FGCM_λ is recommended when solving a multiobjective optimization problem with or without fuzzy constraints. The design example of a mechanical spring illustrates the paired comparison technique and optmization process with or without revising judgement. The last three-bar structural design with maximizing the structural natural frequency demonstrate the presented strategy that can help a designer to find the mistake on the initial paired comparison among objectives and make a further modification. That means that the presented strategy can suggest any inappropriate ranking of importance associated with design objectives selected by the designer. It also serves as a very useful instrument to designers for rechecking the design conditions and formulations. The applications of method of pairwise comparison seems to have the unexploited potential attraction and require further study in the field of fuzzy optimization.

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