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Nonlinear vibration of conservative oscillator's using analytical approaches

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Abstract. In this paper, a new analytical approach has been presented for solving nonlinear conservative oscillators. Variational approach leads us to high accurate solution with only one iteration. Two different high nonlinear examples are also presented to show the application and accuracy of the presented approach. The results are compared with numerical solution using runge-kutta algorithm in different figures and tables. It has been shown that the variatioanl approach doesn't need any small perturbation and is accurate for nonlinear conservative equations.

Keywords: analytical methods; nonlinear vibrations; numerical solution

1. Introduction

Nonlinear differential equations are not an easy task to solve analytically. Recently, a great has been attempted to prepare some new approximate analytical solutions to analysis high order nonlinear differential equations. To have better understanding the behavior of the system and the effects of the important parameters on the nonlinear response of the problem, it makes us to solve them analytically.

Numerical solutions are also available to solve high nonlinear vibration equations; we note that these methods need computational effort and a careful attention to the capability and stability of the numerical methods. Many asymptotic techniques including; Differential transformation method (Kuo and Lo 2009), Generalized differential transform method (Odibat *et al*. 2008), energy balance method (Jamshdi and Ganji 2010, Mehdipour *et al*. 2010).

Max-Min approach (Shen and Mo 2009); Adomian decomposition method (Wu 2011), Variational approach (Xu and Zhang 2009), Hamiltonian approach (He 2010), improved Amplitude-frequency Formulation (He 2008) and other analytical methods (Alicia *et al.* 2010, Bayat *et al*. 2011a, b, c, 2012a, b, c, 2013a, b, c, 2014a, b, c, e, d, Dehghan and Tatari 2008, Geng and Cai 2007, He 2004, 2007, Pakar 2011a, b, 2013a, b, Suna *et al*. 2007, Xu 2010, Zeng 2009, Bayat *et al*. 2015 a, b, c, Cveticanin 2012, 2015) were used to handle strongly nonlinear systems. In this work we aim to apply the He's Variational Approach Method to solve high nonlinear vibration problems in different examples. It has been shown that the methodis an easy to apply

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approach for nonlinear problems as indicated in this paper.

2. Basic idea of He's variational approach

He suggested a variational approach which is different from the known variational methods in open literature (He 2007). Hereby we give a brief introduction of the method

$$\ddot{u} + f(u) = 0 \tag{1}$$

Its variational principle can be easily established utilizing the semi-inverse method

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}\dot{u}^2 + F(u) \right) dt \tag{2}$$

Where T is period of the nonlinear oscillator, $\partial F / \partial u = f$. Assume that its solution can be expressed as

$$u(t) = A\cos(\omega t) \tag{3}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (3) into Eq. (2) results in

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4} \left(-\frac{1}{2} A^2 \omega^2 \sin^2 \omega t + F(A\cos\omega t) \right) dt \\ &= \frac{1}{\omega} \int_0^{\pi/2} \left(-\frac{1}{2} A^2 \omega^2 \sin^2 t + F(A\cos t) \right) dt \\ &= -\frac{1}{2} A^2 \omega \int_0^{\pi/2} \sin^2 t\, dt + \frac{1}{\omega} \int_0^{\pi/2} F(A\cos t) dt \end{aligned} \tag{4}$$

Applying the Ritz method, He require

$$\frac{\partial J}{\partial A} = 0 \tag{5}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{6}$$

But with a careful inspection, for most cases He fined that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2} A^2 \int_0^{\pi/2} \sin^2 t\, dt - \frac{1}{\omega^2} \int_0^{\pi/2} F(A\cos t)\, dt < 0 \tag{7}$$

Thus, He modify conditions Eq. (5) and Eq. (6) into a simpler form

$$\frac{\partial J}{\partial \omega} = 0 \tag{8}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

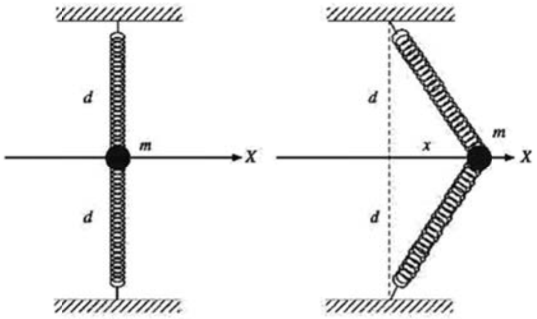


Fig. 1 Mass attached to a stretched wire

3. Application of He's variational approach

3.1 Example 1

A high nonlinear conservative oscillator has been presented here as an example which has an irrational elastic item. In dimensionless form the equation of motion of this system is (Suna *et al*. 2007)

$$u'' + u - \frac{\lambda\, u}{\sqrt{1+u^2}} = 0, \qquad 0 < \lambda \le 1 \tag{9}$$

With initial conditions

$$u(0) = A, \qquad u'(0) = 0, \tag{10}$$

This system oscillates between symmetric bounds $[-A, A]$ and its angular frequency and corresponding periodic solution are dependent on the amplitude A (see Fig. 1).

We consider Eq. (9), its variational formulation is

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}u'^2 + \frac{1}{2}u^2 - \lambda\sqrt{1+u^2} \right) dt \tag{11}$$

Choosing the trial function $u(t)$=A cos(ωt) into Eq. (11) we obtain

$$J(A) = \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2\omega t + \frac{1}{2}A^2\omega^2\cos^2\omega t - \lambda\sqrt{1+A^2\cos^2\omega t} \right) dt \tag{12}$$

The stationary condition with respect to A reads

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \left(-A\,\omega^2 \sin^2(\omega t) + A\,\cos^2(\omega t) - \frac{\lambda\, A\,\cos^2(\omega t)}{\sqrt{1+A^2\cos^2(\omega t)}} \right) dt = 0 \tag{13}$$

Or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2} \left(-A\,\omega^2 \sin^2 t + A\,\cos^2 t - \frac{\lambda A\,\cos^2 t}{\sqrt{1+A^2\cos^2 t}} \right) dt = 0 \tag{14}$$

Solving Eq. (14), according to ω, we have

$$\omega=\sqrt{\frac{\int_0^{\pi/2}\cos^2 t\,d-\int_0^{\pi/2}\frac{\lambda\cos^2 t}{\sqrt{1+A^2\cos^2 t}}dt}{\int_0^{\pi/2}\sin^2 t\,dt}}$$
$$=\sqrt{1+\frac{4\lambda\,\mathrm{K}(\sqrt{-\mathrm{A}^2})}{\pi A^2}-\frac{4\lambda\,\mathrm{E}(\sqrt{-\mathrm{A}^2})}{\pi A^2}} \tag{15}$$

Where $K(m)$ and $E(m)$ are the complete elliptic integrals of the first and second kind, respectively, defined as follows

$$K(m)=\int_0^{\pi/2}\frac{1}{\sqrt{1+A^2 m\cos^2 t}}dt$$
$$E(m)=\int_0^{\pi/2}\sqrt{1+A^2 m\cos^2 t}\,dt \tag{16}$$

According to $u(t)=A\cos(\omega t)$ and (15), we can obtain the following approximate solution

$$u(t)=A\cos\left(\sqrt{1+\frac{4\lambda\,\mathrm{K}(\sqrt{-\mathrm{A}^2})}{\pi A^2}-\frac{4\lambda\,\mathrm{E}(\sqrt{-\mathrm{A}^2})}{\pi A^2}}\right) \tag{17}$$

The exact frequency is given by (Xu 2010)

$$\omega_{ex}=\frac{2\pi}{4\int_0^{\pi/2}[1-2\lambda/(\sqrt{1+A^2\sin t}+\sqrt{1+A^2})]^{-1/2}dt} \tag{18}$$

3.2 Example 2

We consider the motion equation of the pendulum with harmonic stringer point in Fig. 2.

The solution of Mathieu equation of pendulum with harmonic stringer point according to the Variational approach method is

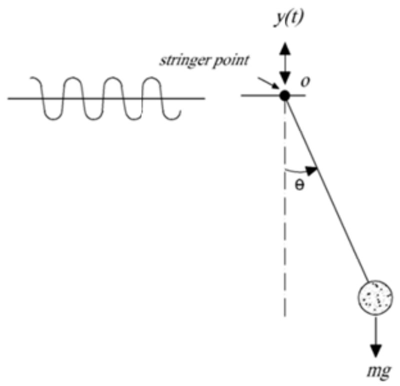


Fig. 2 Pendulum with harmonic stringer point: $y(t)=Y\cos\omega_0 t$

$$\ddot{\theta}+\left(\frac{g}{l}-\frac{\omega_0^2 Y}{l}\cos\omega_0 t\right)\sin\theta=0,\quad \theta(0)=A,\quad \dot{\theta}(0)=0 \tag{19}$$

This equation is as known as Mathieu equation or the system with dependent coefficients to time. In which θ and t are generalized dimensionless displacements and time variables, respectively.

The approximation $\sin(\theta)=\theta-(1/6)\theta^3$ is used.

Its variational formulation can be readily obtained from Eq. (19) as follow

$$J(\theta)=\int_0^t\left(\frac{1}{2}\ddot{\theta}^2+\frac{1}{2}\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\theta^2+\frac{1}{24}\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\theta^4\right)dt \tag{20}$$

Choosing the trial function θ(t)=A cos(ωt) into Eq. (20) we obtain

$$J(A)=\int_0^{T/4}\left(\begin{array}{l}\frac{1}{2}A^2\omega^2 sin^2(\omega t)+\frac{1}{2}A^2\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^2(\omega t)\\ +\frac{1}{24}A^4\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^4(\omega t)\end{array}\right)dt \tag{21}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\left(\begin{array}{l}A\,\omega^2 sin^2(\omega t)+A\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^2(\omega t)\\ +\frac{1}{6}A\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^4(\omega t)\end{array}\right)dt=0 \tag{22}$$

Or

$$\frac{\partial J}{\partial A}=\int_0^{\pi/2}\left(\begin{array}{l}A\,\omega^2 sin^2 t+A\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^2 t\\ +\frac{1}{6}A\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^4 t\end{array}\right)dt=0 \tag{23}$$

Solving Eq. (23), according to ω, we have

$$\omega^2=\frac{\int_0^{\frac{\pi}{2}}\left(\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^2 t+\frac{1}{6}A\left(\frac{g}{l}-\frac{\omega_0^2 Y cos(\omega_0 t)}{l}\right)\cos^4 t\right)dt}{\int_0^{\frac{\pi}{2}}A\ sin^2 t\,dt} \tag{24}$$

Then we have

$$\omega_{VA}=\frac{\sqrt{2}}{4}\sqrt{\frac{128Y\left(A^2+\frac{1}{2}\omega_0^2-8\right)\omega_0 sin\left(\frac{1}{2}\pi\omega_0\right)-g\,\pi\left(\omega_0^2-16\right)\left(\omega_0^2-4\right)\left(A^2-8\right)}{l\left(\omega_0^4-20\omega_0^2+64\right)\pi}} \tag{25}$$

Table1 Comparison of nonlinear frequency of approximate solution (VA) with exact solution corresponding to various parameters of system (example 1)

A	λ	ω_{VA}	ω_{Exact}	Error %
0.1	0.1	0.94888	0.94888	0.00012
0.4	0.1	0.95152	0.95156	0.00348
1	0.1	0.96111	0.96110	0.00081
10	0.1	0.99372	0.99371	0.00024
50	0.1	0.99873	0.99873	0.00000
0.1	0.5	0.70842	0.70842	0.00003
0.4	0.5	0.72588	0.72613	0.03375
1	0.5	0.78652	0.78617	0.04489
10	0.5	0.96817	0.96810	0.00683
50	0.5	0.99361	0.99361	0.00030
0.1	0.75	0.50279	0.50279	0.00018
0.4	0.75	0.53848	0.53921	0.13696
1	0.75	0.65416	0.65277	0.21335
10	0.75	0.95185	0.95170	0.01654
50	0.75	0.99041	0.99040	0.00065
0.1	0.95	0.23139	0.23137	0.00901
0.4	0.95	0.30199	0.31764	4.92676
1	0.95	0.52476	0.52033	0.85137
10	0.95	0.93860	0.93833	0.02820
50	0.95	0.98783	0.98782	0.00104

According to θ(t)=A cos(ωt) and (25), we can obtain the following approximate solution

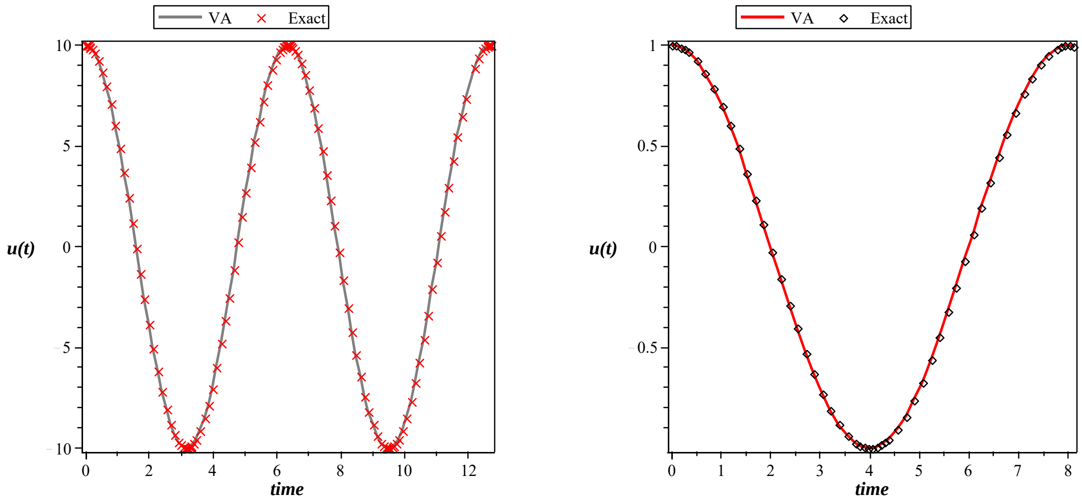
$$\theta(t) = A\cos\left(\frac{\sqrt{2}}{4}\sqrt{\frac{128Y\left(A^2+\frac{1}{2}\omega_0^2-8\right)\omega_0 sin\left(\frac{1}{2}\pi\omega_0\right)-g\,\pi\left(\omega_0^2-16\right)\left(\omega_0^2-4\right)\left(A^2-8\right)}{l\left(\omega_0^4-20\omega_0^2+64\right)\pi}}\right)t \tag{26}$$

4. Results and discussions

In this section, some figures and tables are presnetd to show the accuracy of the presented approach.

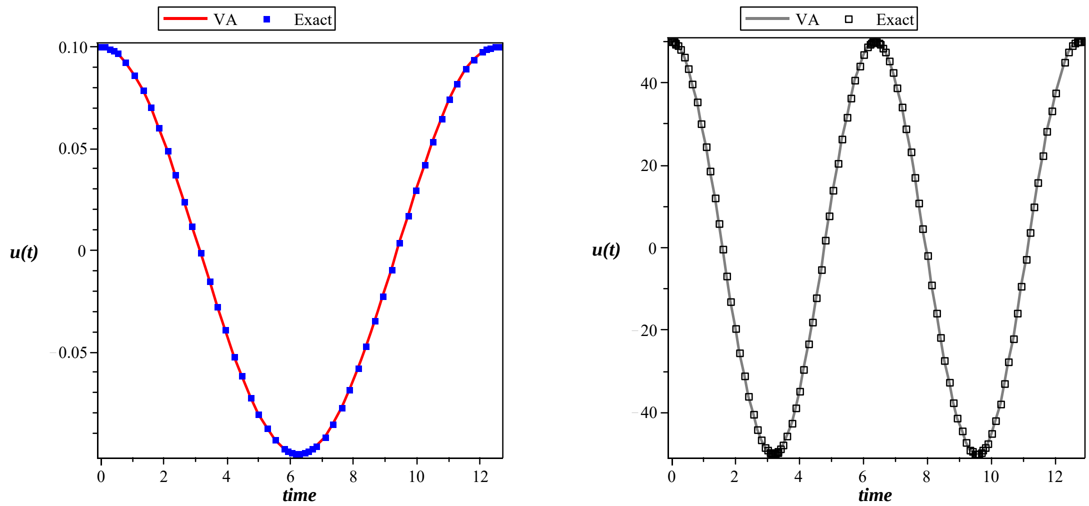
In example 1: Table 1 is the comparison of the variational approach and the exact solution using Runge-Kutta algorithm (Appendix A). Figs. 3 and 4 are the time -displacement comparison of variatioanl approach and exact solution for different parameters values. The maximum error is less than 5 percent. The effects of (λ) and amplitude (A) on nonlinear frequency of the example are also presented in Fig. 5.

In example 2: Table 2 is shown the comparison of the frequency of the system for different parameters with numerical one. In this case the maximum error is less than 2 percent.



(a) (b)

Fig. 3 Comparison of time history response of approximate solution (VA) with the exact solution for (a) λ=0.1, A=10 (b) λ=0.5, A=1



(a) (b)

Fig. 4 Comparison of time history response of approximate solution (VA) with the exact solution for (a) λ=0.75, A=0.1 (b) λ=0.95, A=50

Fig. 6 represents a comparison of analytical solution of $\theta(t)$ based on time with the numerical solution for two different cases. The Fig. 7 shows the effect of length (l) and Y on nonlinear frequency of the system. The Fig. 8 effect of natural frequency (ω_0) and amplitude (A) on nonlinear frequency of the system. In has been indicated the variatioanl approach has an excellent agreement with the numerical solution. It is a simple method and easy to apply to any kind of nonlinear conservative vibration problems.

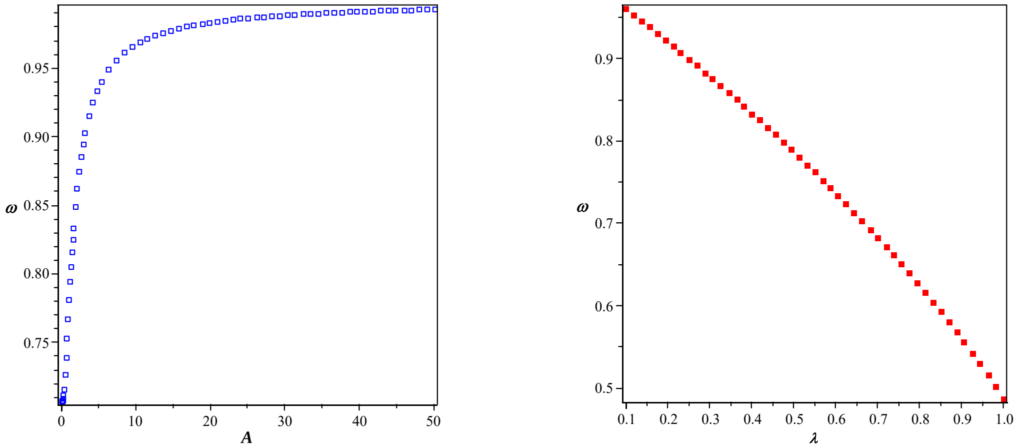
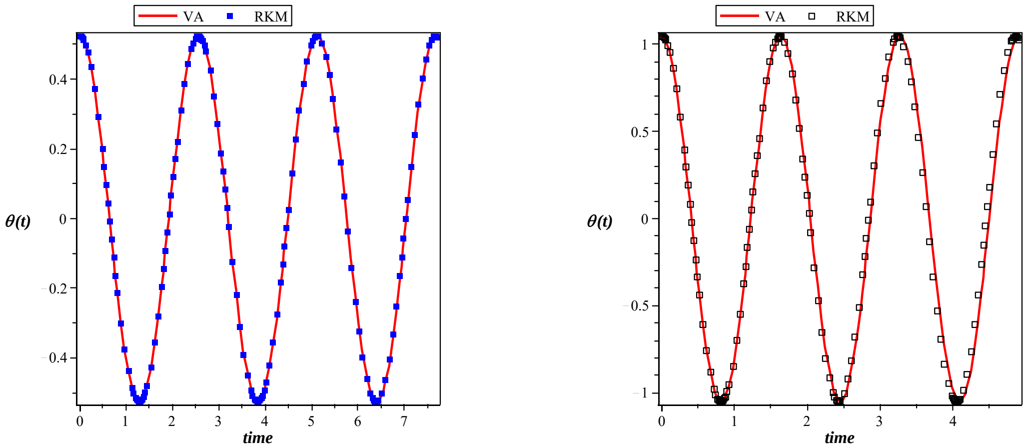


Fig. 5 Effect of parameter (λ) and amplitude (A) on nonlinear frequency



(a) (b)

Fig. 6 Comparison of time history response of approximate solution (VA) with the RKM solution (a) L=1.5 m, ω_0=1.5 rad/sec, Y=0.3 m, g=9.81 m/s^2, A=π/6, (b) L=0.5 m, ω_0=2.5 rad/sec, Y=0.7 m, g=9.81 m/s^2, A=π/3

Table 2 Comparison of nonlinear frequency of approximate solution (VA) with numerical solution (RKM) corresponding to various parameters of system (example 2)

A	g	ω_0	Y	l	ω_{VA}	ω_{RKM}	Error %
$\pi/12$	9.81	3	0.5	0.5	4.2372	4.2460	0.20688
$\pi/12$	9.81	1	0.2	1	3.0916	3.1140	0.72536
$\pi/6$	9.81	1.5	0.3	1.5	2.4535	2.4614	0.32446
$\pi/6$	9.81	2.5	1	1	2.7542	2.7654	0.40602
$\pi/4$	9.81	3	0.2	1.5	2.4224	2.4481	1.06296
$\pi/4$	9.81	1	0.4	1	2.9567	2.9695	0.43509
$\pi/3$	9.81	0.5	0.8	1.5	2.3523	2.3682	0.67797
$\pi/3$	9.81	2.5	0.7	0.5	3.8418	3.8835	1.08406
$\pi/2$	9.81	1.5	0.9	2	1.7167	1.7534	2.14286
$\pi/2$	9.81	3	1	2.5	1.5917	1.6226	1.94062

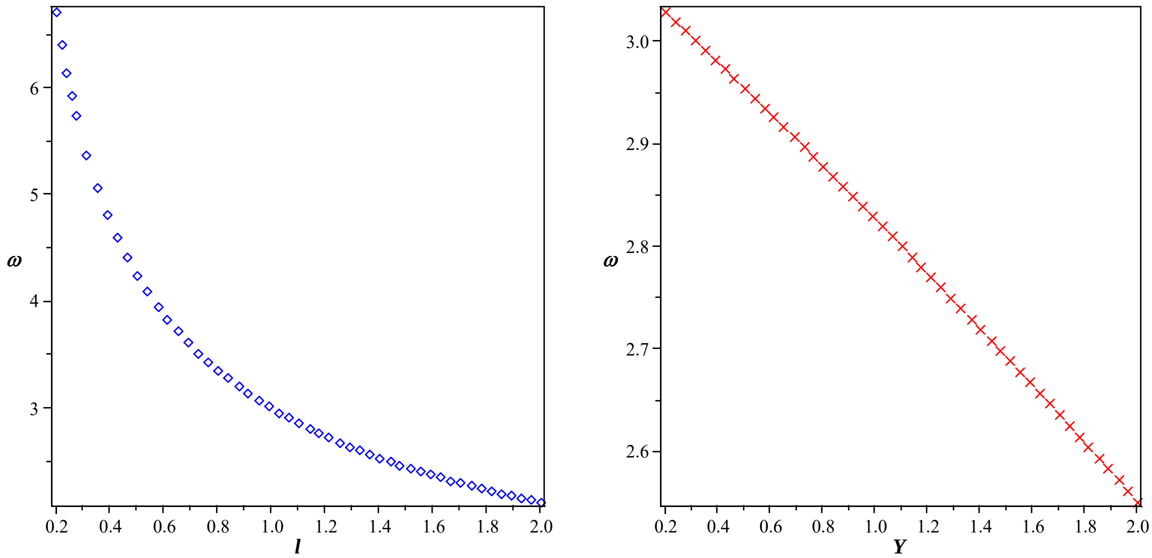


Fig. 7 Effect of length (l) and Y on nonlinear frequency

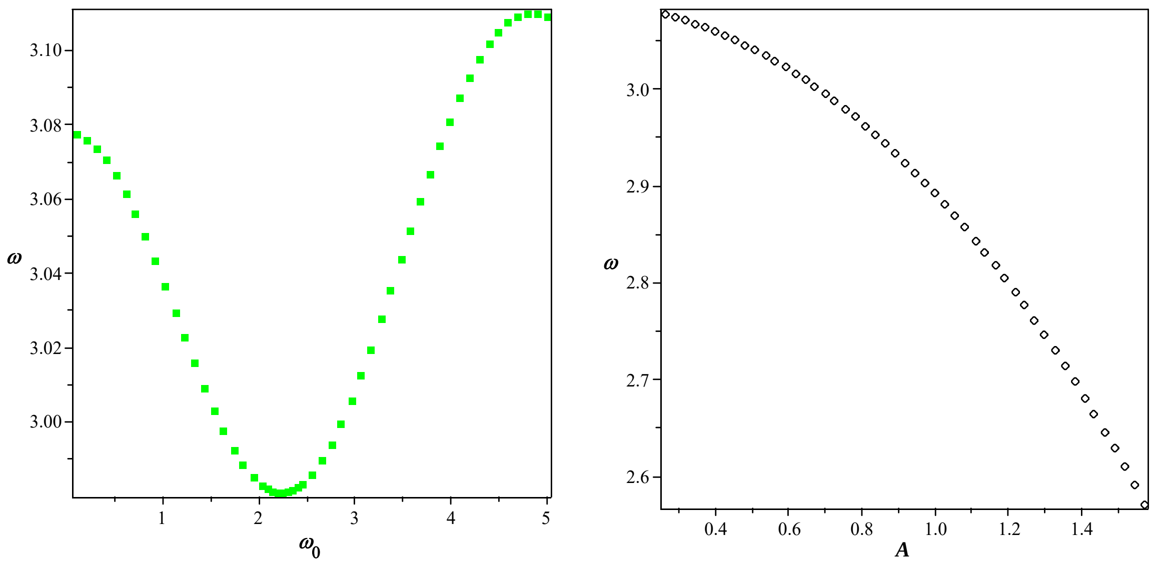


Fig. 8 Effect of natural frequency (ω_0) and amplitude (A) on nonlinear frequency

5. Conclusions

It has been used a quite uncomplicated but productive new methods for non-natural oscillators. The first-order approximate solutions are of a high exactness .Variational Approach was applied successfully for two different nonlinear cases. It has been proved that this approach is an easy to apply method for nonlinear conservative oscillators. The results show good agreement with the numerical solution.

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Appendix A: Basic idea of Runge-Kutta

The Runge-Kutta method is an important iterative method for the approximation solutions of ordinary differential equations. These methods were developed by the German mathematician Runge and Kutta around 1900. For simplicity, we explain one of the important methods of Runge-Kutta methods, called forth-order Runge-Kutta method.

Consider an initial value problem be specified as follows

$$\dot{u} = f\left(t, u\right), \quad u\left(t_0\right) = u_0 \tag{A.1}$$

u is an unknown function of time t which we would like to approximate. Then RK4 method is given for this problem as below

$$\begin{aligned} u_{n+1} &= u_n + \frac{1}{6}h\left(k_1 + 2k_2 + 2k_3 + k_4\right), \\ t_{n+1} &= t_n + h. \end{aligned} \tag{A.2}$$

for n = 0, 1, 2, 3, . . . , using

$$\begin{aligned} k_1 &= f\left(t_n, u_n\right), \\ k_2 &= f\left(t_n + \frac{1}{2}h,\, u_n + \frac{1}{2}hk_1\right), \\ k_3 &= f\left(t_n + \frac{1}{2}h,\, u_n + \frac{1}{2}hk_3\right), \\ k_4 &= f\left(t_n + h,\, u_n + hk_3\right). \end{aligned} \tag{A.3}$$

Where u_{n+1} is the RK4 approximation of $u(t_{n+1})$. The fourth-order Runge-Kutta method requires four evaluations of the right hand side per step h.