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Rotational effect on Rayleigh, Love and Stoneley waves in non-homogeneous fibre-reinforced anisotropic general viscoelastic media of higher order

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Abstract. In this paper, we investigated the propagation of surface waves in a nonhomogeneous rotating fibre-reinforced viscoelastic anisotropic media of higher order of nth order including time rate of strain. The general surface wave speed is derived to study the effect of rotation on surface waves. Particular cases for Stoneley, Love and Rayleigh waves are discussed. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. Also results for homogeneous media can be deduced from this investigation. For order zero our results are well agreed to fibre-reinforced materials. Also by neglecting the reinforced elastic parameters, the results reduce to well known isotropic medium. It is also observed that, surface waves cannot propagate in a fast rotating medium. Comparison was made with the results obtained in the presence and absence of rotation and parameters for fibre-reinforced of the material medium Numerical results are given and illustrated graphically. The results indicate that the effect of rotation and parameters for fibre-reinforced of the material are very pronounced.

Keywords: fibre-reinforced; viscoelastic; surface waves; rotation; anisotropic; nonhomogeneous

1. Introduction

The dynamical problem of propagation of surface waves in a homogeneous and non-homogeneous elastic and thermoplastic media are of considerable importance in earthquake, engineering and seismology on account of the occurrence of non-homogeneities in the earth's crust, as the earth is made up of different layers. Surface waves have been well recognized in the study of earthquake, seismology, geophysics and geodynamics. A good amount of literature for surface waves is available (Bullen 1965, Ewing and Jardetzkyin 1957, Rayleigh 1885, Stoneley 1924). Acharya and Singupta (1978), Pal and Sengupta (1987) studied surface waves under influence of varies parameters. These waves usually have greater amplitudes as compared with body waves and travel more slowly than body waves. There are many types of surface waves but

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we only discussed Stoneley, Love and Rayleigh waves. In earthquake the movement is due to the surface waves. These are also used for detecting cracks and other defects in materials. Lord Rayleigh (1885) was the first to observe such kind of waves in 1885. That's why we called it Rayleigh waves. Sengupta and Nath (2001) investigated surface waves in fibre-reinforced anisotropic elastic media, but their decomposition of displacement vector was not correct that's why some errors are found in their investigations.

The idea of continuous self-reinforcement at every point of an elastic solid was introduced by Belfield *et al*. (1983). The superiority of fibre-reinforced composite materials over other structural materials attracted many authors to study different types of problems in this field. Fibre-reinforced composite structures are used due to their low weight and high strength. Two important components, namely concrete and steel of a reinforced medium are bound together as a single unit so that there can be no relative displacement between them i.e., they act together as a single anisotropic unit. The artificial structures on the surface of the earth are excited during an earthquake, which give rise to violent vibrations in some cases (Acharya and Roy 2009, Acharya and Roy 2009). Engineers and architects are in search of such reinforced elastic materials for the structures that resist the oscillatory vibration. The propagation of waves depends upon the ground vibration and the physical properties of the material structure. Schoenberg and Censor (1973) introduced the idea of elastic waves in rotating media. Surface wave propagation in fiber reinforced media was discussed by various authors (Sapan *et al*. 2011, Singh 2006, Kakar *et al*. 2013, Abd-Alla *et al*. 2013, Chattopadhyay *et al*. 2002, Singh and Singh 2004). Abd-Alla *et al*. (Abd-Alla *et al*. 2013, Abd-Alla *et al*. 2012, Abd-Alla *et al*. 2015, Abo-Dahab *et al*. 2015, Abd-Alla *et al*. 2015) discussed various surface wave propagation in non-homogeneous isotropic media. The extensive literature on the topic is now available and we can only mention a few recent interesting investigations in (Ait Yahia *et al*. 2015, Bourada *et al*. 2015, Belabed *et al*. 2014, Hebali *et al*. 2014, Mahi *et al*. 2015, Bennoun *et al*. 2016, Belabed *et al*. 2014, Hebali *et al*. 2014, Abo-Dahab *et al*. 2016, Abd-Alla *et al*. 2012, Abd-Alla *et al*. 2011, Abd-Alla *et al*. 2010, Abd-Alla *et al*. 1996, Abd-Alla *et al*. 2004).

The aim of this paper is to investigate the propagation of surface waves in a rotating nonhomogeneous fibre-reinforced viscoelastic anisotropic media of higher order. The general surface wave speed is derived to study the effect of rotation on surface waves. Particular cases for Stonely, Love and Rayleigh waves are discussed. The results obtained in this investigation are more general in the sense that some earlier published results are obtained from our result as special cases. For order zero our results are well agreed to fibre-reinforced materials. It is also observed that the corresponding classical results follow from this analysis, in viscoelastic media of order zero, by neglecting reinforced parameters and rotational effects. Results for homogeneous media can be deduced from this investigation. Numerical results are given and illustrated graphically.

2. Formulation of the problem

Medium is consisting of two non-homogeneous anisotropic fibre-reinforced semi-infinite elastic solid media M_1 and M_2 with different elastic and reinforcement parameters. The non-homogeneity of the material is depending on the space variable. It is assumed that non-homogeneity grows or decays slowly. Its rate of growth or decay is proportional to its value at that point, i.e.,

$$\frac{d\lambda}{dx_2} = \alpha\lambda;$$ where λ is an elastic parameter.

This implies

$$\frac{d\lambda}{dx_2} = m\lambda,$$

where m is a constant, which is positive for inhomogeneity growth and negative for decay.

Above equation implies

$$\lambda = \lambda_o e^{mx_2}$$

For m=0, $\lambda = \lambda_0$, Thus for m=0, the medium is homogeneous.

The two media are perfectly welded in contact at a plane interface. Let us take orthogonal Cartesian axes $Ox_1x_2x_3$ with the origin at O. Ox_2 is pointing vertically upwards into the medium $M(x_2>0)$. Each of the media M_1 ($x_2>0$) and M_2 ($x_2<0$) separated at $x_2=0$. Both media are rotating about an axis.

It is assumed that the waves travel in the positive direction of the x_1-axis and at any instant, all particles have equal displacements in any direction parallel to Ox_3. In view of those assumptions, the propagation of waves will be independent of x_3.

The propagation equations of small elastic disturbances are as follows (Schoenberg and Censor 1973).

$\tau_{ij,j} = \rho\{\ddot{u}_i + \Omega_j u_j \Omega_i - \Omega^2 u_i - 2\varepsilon_{ijk}\Omega_j \dot{u}_k\}$, where ε_{ijk} is the Levi-Civita tensor, τ_{ij} are components of stress, ρ is the mass density and u_i is the displacement vector. Upper suffix dot shows the time derivative with respect to time and comma followed by index shows the partial derivative with respect to coordinate. It is assumed that the body is rotating about z-axis with an angular frequency Ω i.e., $\mathbf{\Omega} = \Omega(0,0,1)$

In component form, the equation of motion becomes

$$\left.\begin{aligned} \tau_{11,1} + \tau_{12,2} + \tau_{13,3} &= \rho\{\ddot{u}_1 - \Omega^2 u_1 - 2\Omega\dot{u}_2\}, \\ \tau_{21,1} + \tau_{22,2} + \tau_{23,3} &= \rho\{\ddot{u}_2 - \Omega^2 u_2 + 2\Omega\dot{u}_1\}, \\ \tau_{31,1} + \tau_{32,2} + \tau_{33,3} &= \rho\ddot{u}_3. \end{aligned}\right\} \tag{2.1}$$

The general equation for a fibre-reinforced linearly elastic anisotropic media w.r.t. a direction $\bar{a} = (a_1, a_2, a_3)$ is as under Sengupta and Nath (2001), Acharya and Roy (2009), Kakar *et al*. (2013).

$$\tau_{ij} = D_\lambda \varepsilon_{kk}\delta_{ij} + 2D_{\mu_T}\varepsilon_{ij} + D_\alpha(a_k a_m \varepsilon_{km}\delta_{ij} + \varepsilon_{kk}a_i a_{j)+} 2(D_{\mu_L} - D_{\mu_T})(a_i a_k \varepsilon_{kj} + a_j a_k \varepsilon_{ki}) + D_\beta(a_k a_m \varepsilon_{km} a_i a_j),$$

Strain tensor is $\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$ and D_λ, $D_{\mu T}$ are elastic parameters. D_α, D_β and $(\mathrm{D}_{\mu_L} - D_{\mu_T})$ are reinforced anisotropic viscoelastic parameters of higher order, *s*.

In the present problem we consider exponentially decaying non-homogeneous material. Hence density, elastic module and elastic parameters may be taken in the following form.

$$\rho = \rho_0 e^{-mx_2}$$

$$D_\lambda = \lambda_k \left(\frac{\partial}{\partial t}\right)^k e^{-mx_2} \qquad D_\mu = \mu_k \left(\frac{\partial}{\partial t}\right)^k e^{-mx_2}$$

$$D_\alpha = \alpha_k \left(\frac{\partial}{\partial t}\right)^k e^{-mx_2} \qquad D_{\mu_L} = \mu_{L_k} \left(\frac{\partial}{\partial t}\right)^k e^{-mx_2}$$

$$D_\beta = \beta_k \left(\frac{\partial}{\partial t}\right)^k e^{-mx_2} \qquad D_{\mu_T} = \mu_{T_k} \left(\frac{\partial}{\partial t}\right)^k e^{-mx_2}$$

$$k = 0,1,2...s.$$

An Einstein summation convention for repeated indices is used.
By choosing the fibre direction as $\bar{a} = (1,0,0)$, the components of stress becomes as follows

$$\begin{aligned}
\tau_{11} &= (D_\lambda + 2D_\alpha + 4D_{\mu_L} - 2D_{\mu_T} + D_\beta)\varepsilon_{11} + (D_\lambda + D_\alpha)\varepsilon_{22} + (D_\lambda + D_\alpha)\varepsilon_{33},\\
\tau_{22} &= (D_\lambda + D_\alpha)\varepsilon_{11} + (D_\lambda + 2D_{\mu_T})\varepsilon_{22} + D_\lambda\varepsilon_{33},\\
\tau_{33} &= (D_\lambda + D_\alpha)\varepsilon_{11} + D_\lambda\varepsilon_{22} + (D_\lambda + 2D_{\mu_T})\varepsilon_{33},\\
\tau_{13} &= 2D_{\mu_L}\varepsilon_{13},\\
\tau_{12} &= 2D_{\mu_L}\varepsilon_{12},\\
\tau_{23} &= 2D_{\mu_T}\varepsilon_{23}.
\end{aligned}$$

By using strain tensor, the above equations and taking all derivatives w.r.t. x_3 zero.
The Eq. (2.1) of motion takes the following form

$$\begin{aligned}
(D_\lambda + 2D_\alpha + 4D_{\mu_L} - 2D_{\mu_T} + D_\beta)u_{1,11} + (D_\alpha + D_\lambda + D_{\mu_L})u_{2,21} + D_{\mu_L}u_{1,22}&\\
- mD_{\mu_T}\left(u_{1,2} + u_{2,1}\right) = \rho\{\ddot{u}_1 - \Omega^2 u_1 - 2\Omega\dot{u}_2\},&
\end{aligned}\tag{2.2a}$$

$$\begin{aligned}
(D_\alpha + D_\lambda + D_\mu)u_{1,12} + D_\mu u_{2,11} + (D_\lambda + 2D_{\mu_T})u_{2,22} - m\left(D_\lambda + D_\alpha\right)u_{1,1}&\\
- m\left(D_\lambda + 2D_{\mu_T}\right)u_{2,2} = \rho\{\ddot{u}_2 - \Omega^2 u_2 + 2\Omega\dot{u}_1\},&
\end{aligned}\tag{2.2b}$$

$$D_{\mu_L} u_{3,11} + D_{\mu_T} u_{3,22} - m D_{\mu_T} u_{3,2} = \rho\ddot{u}_3, \tag{2.2c}$$

Similarly, we can get similar relations in medium M_2 with ρ, D_α, D_λ, D_{μ_L}, D_{μ_T} and D_β are replaced by ρ', $D_{\alpha'}$, $D_{\lambda'}$, $D_{\mu'_L}$, $D_{\mu'_T}$ and $D_{\beta'}$.

3. Solution of the problem

We seek harmonic solutions in the form

$$u_1, u_2, u_3 = \hat{u}_1(x_2), \hat{u}_2(x_2), \hat{u}_3(x_2)\exp\{i\omega(x_1 - ct)\},$$

Thus, coupled Eqs. (2.2a), (2.2b) of motion becomes

$$\left[\mu_{Lk}\left(-i\omega c\right)^k D^2 - m\mu_{Tk}\left(-i\omega c\right)^k D + \omega^2\left\{\rho c^2 - (\lambda_k + 2\alpha_k + 4\mu_{Lk} - 2\mu_{Tk} + \beta_k)\left(-i\omega c\right)^k\right\} + \rho\Omega^2\right]\hat{u}_1 +$$
$$i\omega\left[(\alpha_k + \lambda_k + \mu_{Lk})\left(-i\omega c\right)^k D - m\mu_{Tk}\left(-i\omega c\right)^k - 2\rho c\Omega\right]\hat{u}_2 = 0$$

$$\left[(\lambda_k + 2\mu_{Tk})\left(-i\omega c\right)^k D^2 - m(\lambda_k + 2\mu_{Tk})\left(-i\omega c\right)^k D + \omega^2\left\{\rho c^2 - \mu_{Lk}\left(-i\omega c\right)^k\right\} + \rho\Omega^2\right]\hat{u}_2$$
$$+ i\omega\left[(\alpha_k + \lambda_k + \mu_{Lk})\left(-i\omega c\right)^k D - m(\alpha_k + \lambda_k)\left(-i\omega c\right)^k + 2\rho c\Omega\right]\hat{u}_1 = 0.$$

where, $D = \dfrac{d}{dx_2}$

Similarly, we can get similar relations in M_2. Above equations can be written as follows

$$(\hbar_1 D^2 - m\hbar_2 D - \omega^2\hbar_3 + \omega^3\rho c^2 + \rho\Omega^2)\hat{u}_1 + i\omega(\hbar_2 D - m\hbar_2 - 2c\rho\Omega)\hat{u}_2 = 0,$$

$$(\hbar_4 D^2 - m\hbar_4 D - \omega^2\hbar_1 + \omega^2\rho c^2 + \rho\Omega^2)\hat{u}_2 + i\omega(\hbar_2 D - m(\hbar_2 - \hbar_1) + 2c\rho\Omega)\hat{u}_1 = 0,$$

and uncoupled equation becomes

$$\left\{\hbar_5 D^2 - m\hbar_5 D - \omega^2(\hbar_1 - \rho c^2)\right\}\hat{u}_3 = 0$$

where

$$\hbar_1 = \mu_{Lk}(-i\omega c)^k,$$
$$\hbar_2 = (\alpha_k + \lambda_k + \mu_{Lk})(-i\omega c)^k,$$
$$\hbar_3 = (\lambda_k + 2\alpha_k + 4\mu_{Lk} - 2\mu_{Tk} + \beta_k)(-i\omega c)^k\ ,$$
$$\hbar_4 = (\lambda_k + 2\mu_{Tk})(-i\omega c)^k \ \ and\ \ \hbar_5 = \mu_{Tk}(-i\omega c)^k$$

The uncoupled equation has the following solution

$$u_3 = \left(Ee^{-\eta_1\omega x_2} + E_1 e^{-\eta_2\omega x_2}\right)e^{i\omega(x_1 - ct)},$$

where η_1 and η_2 are roots of the equation $\hbar_5\eta^2 - m\hbar_5\eta - \omega^2(\hbar_1 - \rho c^2) = 0$.

$$\eta_{1,2} = \frac{1}{2}\left(m \pm \sqrt{m^2 + \frac{4\left(\hbar_1 - \rho c^2\right)}{\hbar_5}}\right)$$

For positive real root η_1, it is necessary that $0 < 4\rho c^2 < \hbar_5 m^2 + 4\hbar_1$ and in the homogeneous medium $0 < \rho c^2 < \hbar_1$ otherwise transverse component does not exist. For boundedness

$$u_3 = Ee^{-\eta_1\omega x_2}\, exp\left\{i\omega(x_1 - ct)\right\},$$

The above set of coupled equations can be written as

$$\left.\begin{aligned}&(\hbar_1 D^2 - m\hbar_2 D - A_1)\hat{u}_1 + i\omega(\hbar_2 D - m\hbar_2 - 2c\rho\Omega)\hat{u}_2 = 0\\&(\hbar_4 D^2 - m\hbar_4 D - A_2)\hat{u}_2 + i\omega(\hbar_2 D - m(\hbar_2 - \hbar_1) + 2c\rho\Omega)\hat{u}_1 = 0\end{aligned}\right\} \tag{3.1}$$

where

$$A_1 = \omega^2\hbar_3 - \omega^2\rho c^2 - \rho\Omega^2$$
$$A_2 = \omega^2\hbar_1 - \omega^2\rho c^2 - \rho\Omega^2$$

From abvove set of equations, we have

$$\begin{vmatrix}(\hbar_1 D^2 - m\hbar_2 D - A_1) & i\omega(\hbar_2 D - m\hbar_2 - 2c\rho\Omega)\\ i\omega(\hbar_2 D - m(\hbar_2 - \hbar_1) + 2c\rho\Omega) & (\hbar_4 D^2 - m\hbar_4 D - A_2)\end{vmatrix}(\hat{u}_1, \hat{u}_2) = 0$$

This implies

$$(D^4 - C_1 D^3 - C_2 D^2 + C_3 D + C_4(\hat{u}_1, \hat{u}_2) = 0 \tag{3.2}$$

where

$$C_1 = \frac{m\hbar_4}{\hbar_1\hbar_4}(\hbar_2 + \hbar_1)$$

$$C_2 = \frac{1}{\hbar_1\hbar_4}\left(\hbar_4 A_1 + \hbar_1 A_2 - \omega^2\hbar_2^2 - m^2\hbar_2\hbar_4\right)$$

$$C_3 = \frac{m}{\hbar_1\hbar_4}\hbar_2\left\{A_2 + \hbar_1 - 2\omega^2\hbar_2\right\}$$

$$C_4 = \frac{1}{\hbar_1\hbar_4}\left\{A_1 A_2 - \omega^2\left(2\rho c\Omega + m\hbar_2\right)\left(2\rho c\Omega - m\hbar_2 + m\hbar_1\right)\right\}$$

For homogeneous medium, m=0, this implies C_1=C_3=0 and C_2, C_4 must be positive for real positive roots. Here C_2, C_4 must be positive impose a necessary and sufficient condition upon the frequency of rotation of the medium through which a surface wave cannot propagate in a fast rotating medium. i.e.,

$$\frac{\Omega^2}{\omega^2} < \frac{\hbar_3}{\rho},$$

Let α_i, i=1,2,…4 be four positive real roots of Eq. (3.2), then solution by normal mode method has the following form

$$\hat{u}_1 = \sum_{n=1}^{4} M_n\, e^{-\alpha_n x_2},$$

$$\hat{u}_2 = \sum_{n=1}^{4} M_{1n}\, e^{-\alpha_n x_2},$$

where, M_n, M_{1n} are some parameters depending on c and ω.

By using Eqs. (3.1), we get the following relations,

$$M_{1n} = H_{1n} M_n$$

where

$$H_{1n}=\frac{\left\{\left(\hbar_4\alpha_n^2\right)+m(\alpha_n\hbar_4)-A_2\right\}}{i\omega\left\{\hbar_2\alpha_n-2\rho c\Omega+m(\hbar_2-\hbar_1)\right\}},\qquad n=1,2,3,4.$$

Hence we obtain the expressions of the displacement components and stresses as follows

$$u_1=\sum_{n=1}^{4}M_n\ e^{-\alpha_n x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$u_2=\sum_{n=1}^{4}H_{1n}M_n\ e^{-\alpha_n x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$u_3=Ee^{-\eta_1\omega x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$\tau_{12}=\sum_{n=1}^{4}D_{\mu_L}\left(-\alpha_n+i\omega H_{1n}\right)M_n e^{-\alpha_n x_2}\ exp\left\{i\omega(x_1-ct)\right\}$$

$$\tau_{22}=\sum_{n=1}^{4}\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4\alpha_n H_{1n}\right\}M_n e^{-\alpha_n x_2}\ exp\left\{i\omega(x_1-ct)\right\}$$

$$\tau_{23}=-\eta_1\omega E\hbar_5\, e^{-\eta_1\omega x_2}\ exp\left\{i\omega(x_1-ct)\right\}.$$

Similar expressions can be obtained for second mediun and present them with dashes as follows

$$u_1'=\sum_{n=1}^{4}M_n'\ e^{-\alpha_n' x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$u_2'=\sum_{n=1}^{4}H_{1n}'M_n'\ e^{-\alpha_n' x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$u_3'=Fe^{-\eta_1'\omega x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$\tau_{12}'=\sum_{n=1}^{4}\hbar_1'\left(-\alpha_n'+i\omega H_{1n}'\right)M_n' e^{-\alpha_n' x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$\tau_{22}'=\sum_{n=1}^{4}\left\{i\omega(\hbar_2'-\hbar_1')-\hbar_4'\alpha_n' H_{1n}'\right\}M_n' e^{-\alpha_n' x_2}\ exp\left\{i\omega(x_1-ct)\right\},$$

$$\tau_{23}'=-\eta'\omega F\hbar_5' e^{-\eta'\omega x_2}\ exp\left\{i\omega(x_1-ct)\right\}.$$

In order to determine the secular equations, we have the following boundary conditions.

4. Boundary conditions

1) The displacement components and their rate of change w.r.t. x_2, between the mediums is continuous, i.e.,

$u_1=u_1',\ u_2=u_2',\ u_3=u_3',\ u_{1,2}=u_{1,2}',\quad u_{2,2}=u_{2,2}',\ \ u_{3,2}=u_{3,2}'$ on $x_2=0$, for all x_1 and *t*.

2) Stress and their deritive w.r.t x_2 are continuous, i.e.,
$\tau_{12}=\tau'_{12}$, $\tau_{22}=\tau'_{22}$, $\tau_{23}=\tau'_{23}$, $\tau_{12,2}=\tau'_{12,2}$, $\tau_{22,2}=\tau'_{22,2}$, $\tau_{23,2}=\tau'_{23,2}$ on $x_2=0$, for all x_1 and t.
Boundary conditions imply the following equations

$$\left.\begin{aligned}
&M_1+M_2+M_3+M_4=M'_1+M'_2+M'_3+M'_4\\
&H_{11}M_1+H_{12}M_2+H_{13}M_3+H_{14}M_4=H'_{11}M'_1+H'_{12}M'_2+H'_{13}M'_3+H'_{14}M'_4\\
&E=F\\
&\alpha_1M_1+\alpha_2M_2+\alpha_3M_3+\alpha_4M_4=\alpha'_1M'_1+\alpha'_2M'_2+\alpha'_3M'+\alpha'_4M'_4\\
&\alpha_1H_{11}M_1+\alpha_2H_{12}M_2+\alpha_3H_{13}M_3+\alpha_4H_{14}M_4=\alpha'_1H'_{11}M'_1+\alpha'_2H'_{12}M'_2+\alpha'_3H'_{13}M'+\alpha'_4H'_{14}M'_4\\
&\eta_1E=\eta'_1F
\end{aligned}\right\} \tag{4.1}$$

$$\begin{aligned}
&\sum_{n=1}^{4}\hbar_1\left(-\alpha_n+i\omega H_{1n}\right)M_n=\sum_{n=1}^{4}\hbar'_1\left(-\alpha'_n+i\omega H'_{1n}\right)M'_n\ ,\\
&\sum_{n=1}^{4}\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4\alpha_nH_{1n}\right\}M_n=\sum_{n=1}^{4}\left\{i\omega(\hbar'_2-\hbar'_1)-\hbar'_4\alpha'_nH'_{1n}\right\}M'_n\ ,\\
&\hbar_5\eta_1E=\hbar'_5\eta'_1F,\\
&\sum_{n=1}^{4}\hbar_1\alpha_n\left(-\alpha_n+i\omega H_{1n}\right)M_n=\sum_{n=1}^{4}\hbar'_1\alpha'_n\left(-\alpha'_n+i\omega H'_{1n}\right)M'_n\ ,\\
&\sum_{n=1}^{4}\alpha_n\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4\alpha_nH_{1n}\right\}M_n=\sum_{n=1}^{4}\alpha'_n\left\{i\omega(\hbar'_2-\hbar'_1)-\hbar'_4\alpha'_nH'_{1n}\right\}M'_n\ ,\\
&\hbar_5\eta_1^2E=\hbar'_5\eta_1'^2F,
\end{aligned}$$

From the above set of equations, the four equations containing E and F implies that $E=F=0$. From remaining eight equations, for non-trivial solution we have

$$\det\left(a_{ij}\right)=0,\quad i=j=1,2,...,8. \tag{4.2}$$

where

$$\begin{aligned}
&a_{11}=1,\ a_{12}=1,\ a_{13}=1,\ \ a_{14}=1,\ a_{15}=-1,\ a_{16}=-1,\ a_{17}=-1,\ \ a_{18}=-1,\\
&a_{21}=H_{11},\ a_{22}=H_{12},\ a_{23}=H_{13},\ \ a_{24}=H_{14},\ a_{25}=-H'_{11},\ a_{26}=-H'_{12},\ a_{27}=-H'_{13},\ \ a_{28}=-H'_{14},\\
&a_{31}=\alpha_1,\ a_{32}=\alpha_2,\ a_{33}=\alpha_3,\ \ a_{34}=\alpha_4,\ a_{35}=-\alpha'_1,\ a_{36}=-\alpha'_2,\ a_{37}=-\alpha'_3,\ \ a_{38}=-\alpha'_4,\\
&a_{41}=\alpha_1H_{11},\ a_{42}=\alpha_2H_{12},\ a_{43}=\alpha_3H_{13},\ \ a_{44}=\alpha_4H_{14},\\
&a_{45}=-\alpha'_1H'_{11},\ a_{46}=-\alpha'_2H'_{12},\ a_{47}=-\alpha'_3H'_{13},\ \ a_{48}=-\alpha'_4H'_{14},
\end{aligned}$$

$$a_{5p}=\hbar_1\left(-\alpha_p+i\omega H_{1p}\right),\ p=1,2,...,4;\qquad a_{5q}=-\hbar'_1\left(-\alpha'_q+i\omega H'_{1q}\right),\ q=5,6,...,8.$$

$$a_{6p}=\left\{i\omega(\hbar_2-\hbar_1)-\hbar_4\alpha_pH_{1p}\right\},\ p=1,2,...,4;\qquad a_{6q}=-\left\{i\omega(\hbar'_2-\hbar'_1)-\hbar'_4\alpha'_qH'_{1q}\right\},\ q=5,6,...,8.$$

$$a_{7p} = \hbar_1 \alpha_p \left(-\alpha_p + i\omega H_{1p}\right),\ p = 1,2,...,4; \qquad a_{7q} = -\hbar_1' \alpha_q \left(-\alpha_q' + i\omega H_{1q}'\right),\ q = 5,6,...,8.$$

$$a_{6p} = \alpha_p \left\{ i\omega(\hbar_2 - \hbar_1) - \hbar_4 \alpha_p H_{1p} \right\},\ p = 1,2,...,4; \qquad a_{6q} = -\alpha_q' \left\{ i\omega(\hbar_2' - \hbar_1') - \hbar_4' \alpha_q' H_{1q}' \right\},\ q = 5,6,...,8.$$

5 Particular cases

5.1 Stoneley waves

It is also a surface wave and may be considered as the generalized form of Rayleigh waves propagating at the common boundary of M and M_1. Hence, the wave velocity Eq. (34.2) for general surface waves may also be considered for the Stoneley waves in a fibre-reinforced elastic media along the common boundary. Since the wave velocity Eq. (34.2) for Stoneley waves under the present circumstances does not contain explicitely, such types of waves are not dispersive like the classical one. Eq. (34.2) is the secular equation for Stonely waves. If rotational effects and fiber-reinforced parameters are ignored, then for k=0, in homogeneous media, the results are same as Stoneley (1924).

5.2 Love waves

To investigate the rotational effects on Love waves in a fibre reinforced viscoelastic media of higher order, we replace the medium M_1 by an infinitely extended horizontal plate of finite thickness d and bounded by two horizontal plane surfaces x_2=0 and x_2=d. Medium M is semi infinite as in the general case.

The boundary conditions of Love wave are as follows

The displacement component u_3 and τ_{12} between the mediums is continuous, i.e.,

$$u_3 = u_3' \quad \text{and} \quad \tau_{23} = \tau_{23}' \quad \text{on} \quad x_2 = 0$$

$$\tau_{23}' = 0 \quad \text{on} \quad x_2 = d, \qquad \text{for all } x_1 \text{ and } t,$$

where

$$u_3 = E e^{-\eta_1 \omega x_2} e^{i\omega(x_1 - ct)},$$

$$u_3' = E' e^{\eta_1' \omega x_2} e^{i\omega(x_1 - ct)} + F' e^{-\eta_1' \omega x_2} e^{i\omega(x_1 - ct)},$$

This implies

$$\begin{aligned} & E - E' - F' = 0, \\ & \mu_{Tk} \left(-i\omega c\right)^k \eta_1 E + \mu_{Tk}' \left(-i\omega c\right)^k \eta_1' E' - \mu_{Tk}' \left(-i\omega c\right)^k \eta_1' F' = 0, \\ & e^{\omega \eta_1 d} E' - e^{-\omega \eta_1 d} F' = 0. \end{aligned}$$

For non trivial solution implies

$$\begin{vmatrix} 1 & -1 & -1 \\ \mu_{Tk} \left(-i\omega c\right)^k \eta_1 & \mu_{Tk}' \left(-i\omega c\right)^k \eta_1' & -\mu_{Tk}' \left(-i\omega c\right)^k \eta_1' \\ 0 & e^{\omega \eta_1' d} & -e^{-\omega \eta_1' d} \end{vmatrix} = 0,$$

This gives the wave velocity of Love waves propagating in a fiber-reinforced viscoelastic medium of higher order. For k=0, the results are exactly same as in literature. It is interesting to note that rotation and non-homogeneity does not affect the velocity of Love waves.

5.3 Rayleigh waves

Rayleigh wave is a special case of the above general surface wave. In this case we consider a model where the medium M_2 is replaced by vacuum. Since the boundary x_2=0 is adjacent to vacuum. It is free from surface traction. So the stress boundary condition in this case may be expressed as

$\tau_{12} = 0$, $\tau_{22} = 0$, $\tau_{12,2} = 0$, $\tau_{22,2} = 0$, $\tau_{23} = 0$ and $\tau_{23,2} = 0$ on $x_2 = 0$, for all x_1 and t

Thus above set of equations reduces to

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ H_{11} & H_{12} & H_{13} & H_{14} \\ \alpha_1 & \alpha_2 & \alpha_3 & \alpha_4 \\ \alpha_1 H_{11} & \alpha_2 H_{12} & \alpha_3 H_{13} & \alpha_4 H_{14} \end{vmatrix} = 0 .$$

Which is secular equation for Rayleigh wave.

6. Numerical results and discussion

The following values of elastic constants are considered Chattopadhyay *et al*. (2002), Singh (2006), for mediums M and M_1 respectively.

$$\rho = 2660 kG/m^3, \quad \lambda = 5.65x10^{10}\,Nm^{-2}, \quad \mu_T = 2.46x10^9\,Nm^{-2}, \quad \mu_L = 5.66x10^9\,Nm^{-2}$$
$$\alpha = -1.28x10^9\,Nm^{-2}, \quad \beta = 220.90x10^9\,Nm^{-2},$$

$$\rho = 7800 kG/m^3, \quad \lambda = 5.65x10^{10}\,Nm^{-2}, \quad \mu_T = 2.46x10^{10}\,Nm^{-2}, \quad \mu_L = 5.66x10^{10}\,Nm^{-2}$$
$$\alpha = -1.28x10^{10}\,Nm^{-2}, \quad \beta = 220.90x10^{10}\,Nm^{-2}$$

$$T_0 = 293K, \quad \tau_0 = 0.1, \quad \nu_0 = 0.2$$

Taking into consideration, the numerical technique outlined above was used to obtain secular equation, surface wave velocity and attenuation coefficients under the effects of rotation and inhomogeneity in two models considering the real part indicates to the surface wave velocity, but the imaginary part indicates to the attenuation coefficient. For the sake of brevity, some computational results are being presented here. The variations are shown in Figs. 1-3 respectively.

Figs. (1a-1i) Show that the variation of the magnitude of the frequency equation $|\Delta|$, Stoneley wave velocity Re($|\Delta|$) and attenuation coefficient Im($|\Delta|$) with respect to frequency ω for different values of inhomogeneity m, rotation Ω and higher order k of nth order including time rate of strain. The magnitude of the frequency equation decreases and increases with increasing of frequency and it vanish at ω=0.4 when effect of inhomogeneity, while it increases in the interval [0,0.4] with increasing of inhomogeneity and it decreases in the interval [0.4,1] with increasing of inhomogeneity, as well the magnitude of the frequency equation increases with increasing of frequency and rotation, while it decreases with increasing of higher order k of nth order including time rate of strain, the Stoneley wave velocity decreases and increases with increasing of frequency and it vanish at ω=0.4,0.6 when effect of inhomogeneity, while it increases in the interval [0,0.4] and in the interval [0.6,1] with increasing of inhomogeneity, as well the Stoneley wave velocity increases and decreases with increasing of frequency, while it increases with increasing of rotation, as well it increases with increasing of higher order k of nth order including time rate of strain and the attenuation coefficient decreases and increases with increasing of frequency and it vanish at ω=0.3, 0.4, 0.7 when effect of inhomogeneity, while it increases in the interval [0,0.2] and in the interval [0.7,1] with increasing of inhomogeneity, as well the attenuation coefficient decreases with increasing of frequency and rotation, as well it increases with increasing of higher order k of nth order including time rate of strain. However from the Stoneley wave velocity the combined effectof rotation and fibre reinforcing has a significant role to play on the Stoneley wave velocity

Figs. (2a-2i) Show that the variation of the magnitude of the frequency equation $|\Delta|$, Love wave velocity $\mathrm{Re}(|\Delta|)$ and attenuation coefficient $\mathrm{Im}(|\Delta|)$ with respect to frequency ω for different values of inhomogeneity m, thickness d and higher order k of nth order including time rate of strain. The magnitude of the frequency equation decreases with increasing of frequency, while it increases and decreases with increasing of frequency when effect of higher order, as well the magnitude of the frequency equation increases with increasing of inhomogeneity, while it decreases with increasing of higher order k of nth order including time rate of strain and thickness, the Love wave velocity increases with increasing of frequency, while it decreases and increases with increasing of frequency when effect of higher order, as well the Love wave velocity decreases with increasing of inhomogeneity, while it increases with increasing of thickness and higher order and the attenuation coefficient decreases with increasing of frequency, while it decreases and increases with increasing of frequency when effect of higher order, as well the attenuation coefficient increases with increasing of inhomogeneity and higher order, while it decreases with increasing of thickness. The physical fact which emerges out of the above analysis is that fibre-reinforcement plays a vital role in the Love wave velocity where as the presence of rotation can not influence the same. Moreover the thickness of the fibre-reinforced layer has a pronounced effect on the Love wave velocity.

Figs. (3a-3l) Show that the variation of the magnitude of the frequency equation $|\Delta|$, Rayleigh wave velocity $\mathrm{Re}(|\Delta|)$ and attenuation coefficient $\mathrm{Im}(|\Delta|)$ with respect to frequency ω for different values of inhomogeneity m, rotation Ω and higher order k of nth order including time rate of strain. The magnitude of the frequency equation decreases and increases with increasing of frequency when effect of inhomogeneity, while it increases in the interval [0,0.4] with increasing of inhomogeneity and it decreases in the interval [0.4,1] with increasing of inhomogeneity, as well the magnitude of the frequency equation increases with increasing of frequency and rotation, while it increases with increasing of frequency and there is no effect of higher order k of nth order including time rate of strain, the Rayleigh wave velocity decreases and increases with increasing of frequency, while it increases with increasing of inhomogeneity, as well the Rayleigh wave velocity increases and decreases with increasing of frequency, while it increases with increasing of rotation, as well it decreases with increasing of higher order k of nth order including time rate of strain and the attenuation coefficient decreases with increasing of frequency, while it increases with increasing of inhomogeneity, as well the attenuation coefficient decreases with increasing of frequency and rotation, as well it decreases with increasing of frequency and there is no effect of higher order k of nth order including time rate of strain. The above discussion expresses the physical fact, in general, that the rotation as well as fibre-reinforcement which are generally

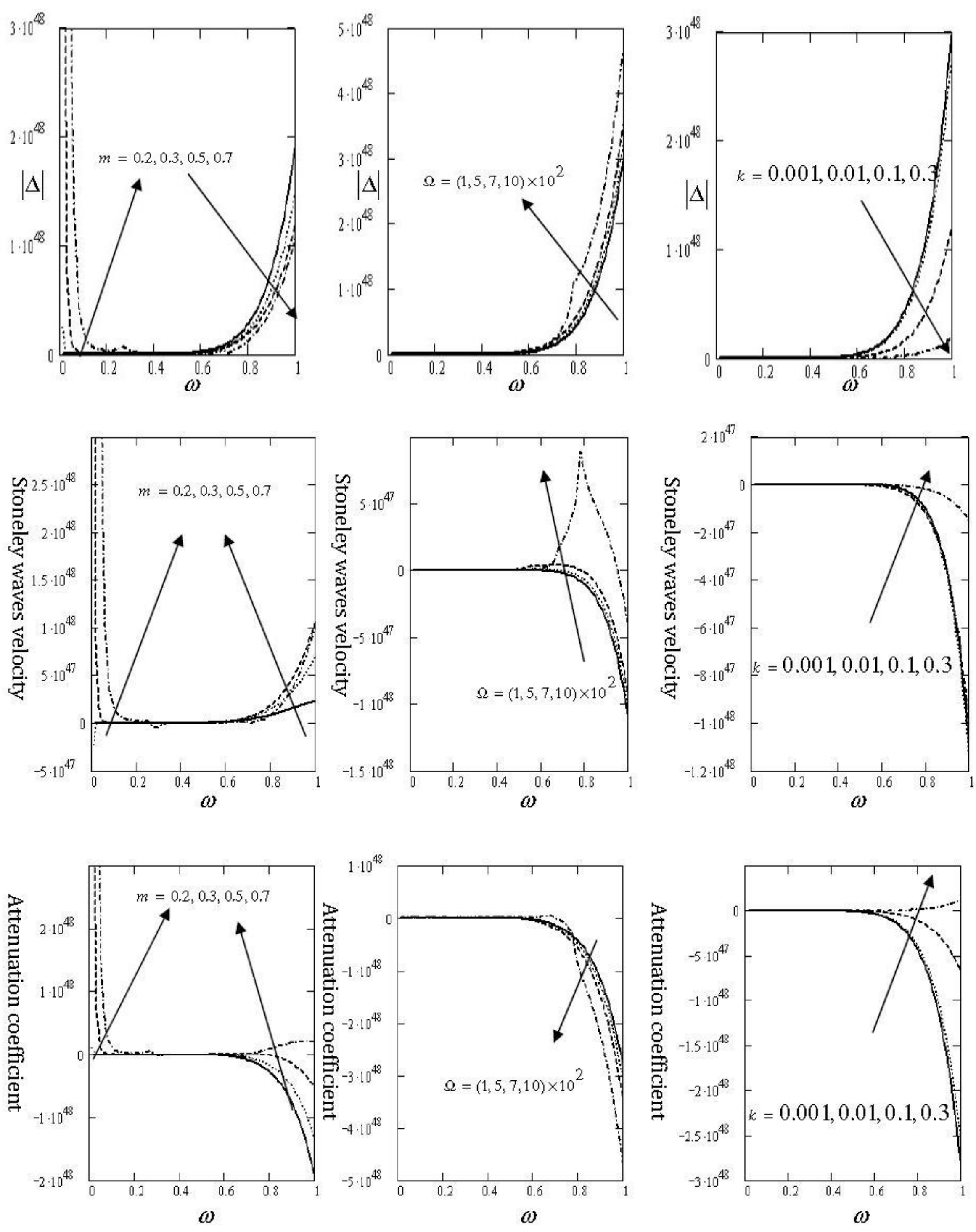

Fig. 1 Variation of $|\Delta|$, velocity (Re($|\Delta|$)) and attenuation coefficient (Im($|\Delta|$)) for Stoneley waves with respect to c with the variation of Ω and k

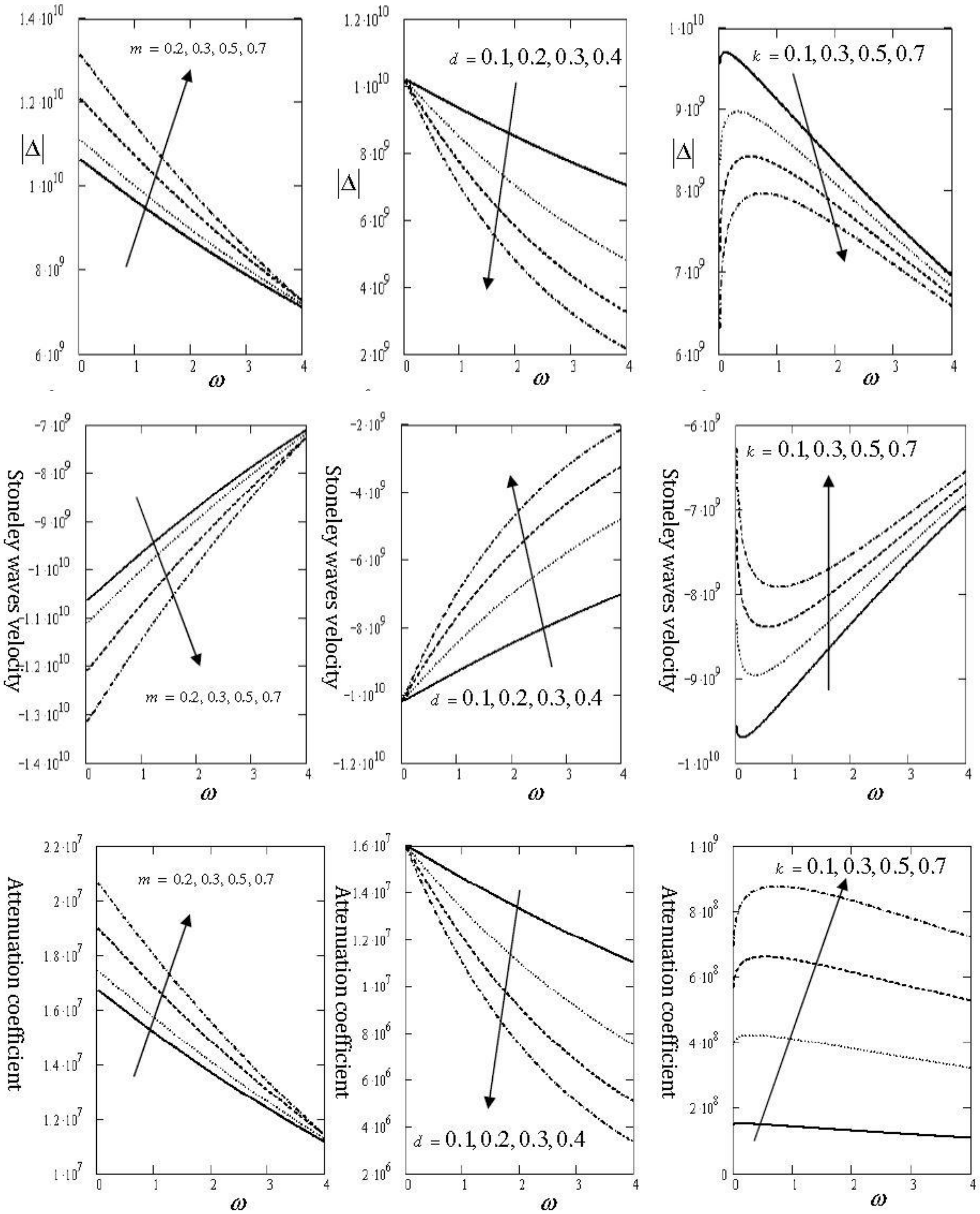

Fig. 2 Variation of |Δ|, velocity (Re(|Δ|)) and attenuation coefficient (Im(|Δ|)) for Love waves with respect to *ω* with the variation of *m*, *d* and *k*

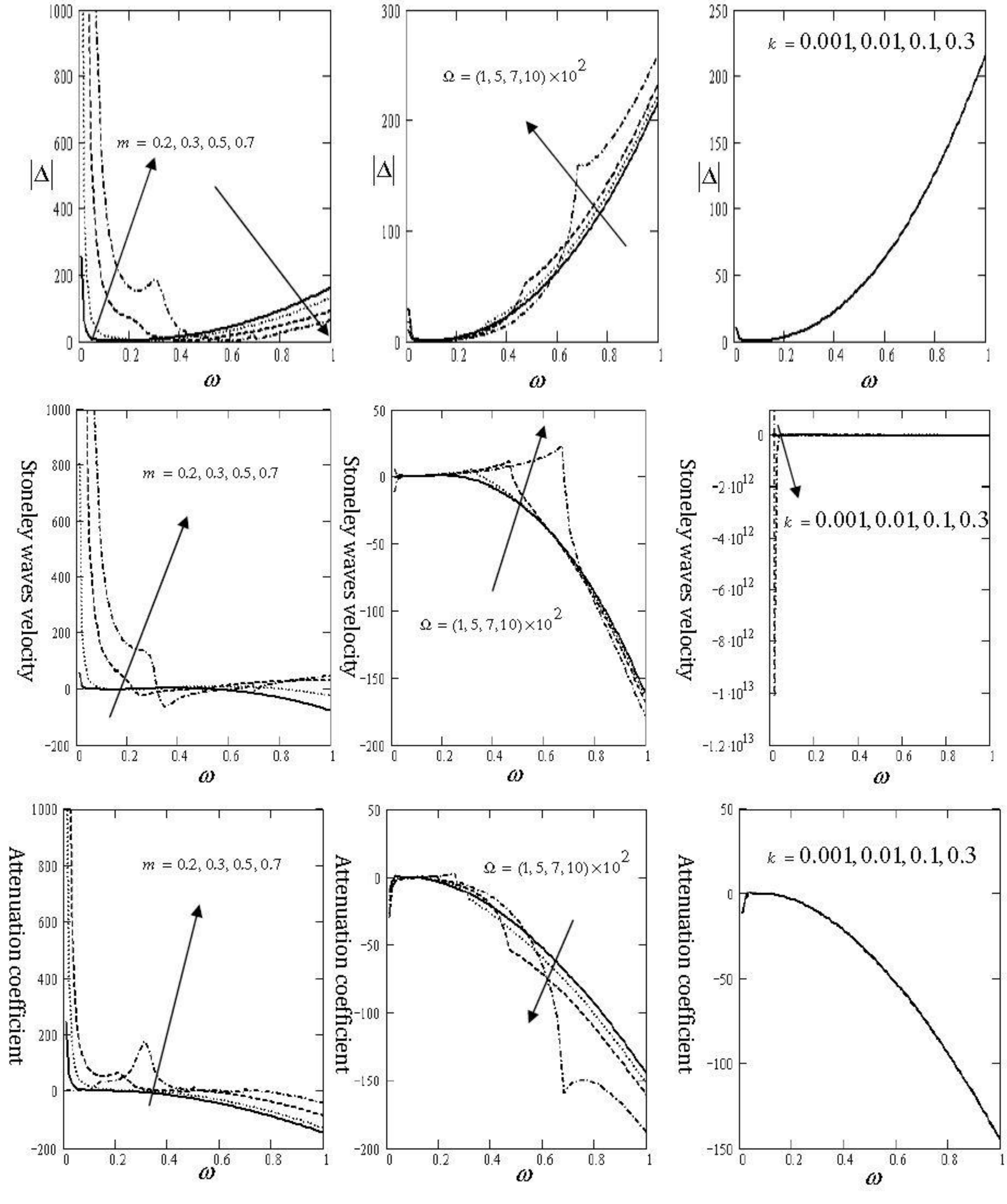

Fig. 3 Variation of |Δ|, velocity (Re(|Δ|)) and attenuation coefficient (Im(|Δ|)) for Rayleigh waves with respect to ω with variation of m, Ω and k

neglected in corresponding classical problems, influences the Rayleigh wave velocity to a considerable extent.

7. Conclusions

1. The surface waves in non-homogeneous, anisotropic, fibre-reinforced viscoelastic solid media under the rotation and higher order k of nth order including time rate of strain are investigated. It is observed that viscoelastic surface waves are affected by rotation, inhomogeneity, frequency and the time rate of strain parameters. These parameters influence on the wave velocity to an extent depending on the corresponding constants characterizing and viscoelasticity of the material. So the results of this analysis become useful in circumstances where these effects cannot be neglected. These velocities depend upon the fibre-reinforced parameters 'a' confirming that these waves are affected by the rotation of the media.
2. Love waves in non-homogeneous media; these are only affected by viscosity, rotation, frequency, higher order k of net order, including time rate of strain, frequency and thickness of the medium. In the absence of all fields, the dispersion equation is in complete agreement with the corresponding classical result.
3. Rayleigh waves in non-homogeneous, general viscoelastic solid medium of higher order, including time rate of change of strain, we find that the wave velocity equation, proves that there is a dispersion of waves due to the presence of rotation, frequency, inhomogeneity and viscosity. The results are in complete agreement with the corresponding classical results in the absence of all fields.
4. The wave velocity equation of Stoneley waves is very similar to the corresponding problem in the classical theory of elasticity. The dispersion of waves is due to the presence of rotation, phase velocity, frequency and viscosity of the solid. Also, wave velocity equation of this generalized type of surface waves is in complete agreement with the corresponding classical result in the absence of all fields.
5. The result provides a motivation to investigate fibre-reinforced viscoelastic anisotropic media of higher order as a new class of applicable fibre-reinforced viscoelastic media . The results presented in this paper should prove useful for researchers in material science, designers of new materials, physicists as well as for those working on the development of fibre-reinforced elasticity and in practical situations as in geophysics, optics, acoustics, geomagnetic and oil prospecting etc . The used methods in the present article is applicable to a wide range of problems in thermodynamics and thermoelasticity.
6. The rotation plays a significant role in the distribution of all the physical quantities. The parameters of all the physical quantities vary (increase or decrease) as rotation increases..
7. All the physical quantities satisfy the boundary conditions.
8. If we neglect the effect of rotation then our results coincide with Abd-Alla *et al*. (2015), Finally, if the rotation and fibre-reinforced are neglected , the relevant results obtained are deduced to the results obtained by Sengupta and Nat (2001). It is observed that in a fast rotating medium the surface wave cannot propagate.

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