Buckling of plates including effect of shear deformations: a hyperelastic formulation

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Abstract. Consistent finite strain Plate constitutive relations are derived based on a hyperelastic formulation for an isotropic material. Plate equilibrium equations under finite strain are derived following a static kinematic approach. Three Euler angles and four shear angles, based on Timoshenko beam theory, represent the kinematics of the deformations in the plate cross section. The Green deformation tensor has been expressed in term of a deformation tensor associated with the deformation and stretches of an embedded plate element. Buckling formulation includes the in-plane axial deformation prior to buckling and transverse as well as in-plane shear deformations. Numerical results for a simply supported thick plate under uni-axial compression force are presented.

Keywords: plate; buckling; shear deformations; hyperelasticity

1. Introduction

The Classical thin plate theory has been widely employed in predicting stresses and strains and for estimating the buckling load in problems involving thin plates with reasonable results. Yet, due to several shortcomings specifically the neglect of shear deformations, it is not suitable for the investigation of buckling of thick plate where stress can vary through the thickness of the plate. Effort has been made to develop refined theories which could overcome the limitations of the classical plate theory and led to the development of shear deformation theories which include the effect of shear deformations.

Reissner (1945) developed his two dimensional plate theory based on an assumed stress field. It was the first plate theory to include the effect of transverse shear and for the first time the correct number of boundary conditions was satisfied. Mindlin (1951) developed a two dimensional plate theory, which included the effect of transverse shear, based on an assumed displacement field.

Since the development of Mindlin's plate theory, many shear deformation theories were developed which usually, assume a displacement field in the form of power series in the thickness coordinate. Reddy and Chao (1981) introduced a non-linear first order shear deformation theory. They adopted the displacement field in Mindlin's plate theory using non-linear strain-displacement relationships (Von Karman strain displacement equations). Ziegler (1983) pioneered the

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investigation of plate buckling employing a Mindlin type plate theory, which included the effect of transverse and in-plane shear deformations, based on linear elastic analysis. Reddy (1984) improved his first order plate theory by upgrading it into a third order theory. Di Sciuva (1986) assumed a displacement model which accounts for linear distribution of the in-plane displacements across the plate thickness and allows the surface conditions of the transverse shear stresses to be satisfied. Based on such a displacement field, he developed a linear theory for statics of thick multilayered anisotropic plates. Stein (1986) formulated a two-dimensional non-linear plate theory including shear deformations effect. The assumed displacement field is a trigonometric series representation of the displacements through the thickness. Murty and Vellaichamy (1988) formulated a two dimensional higher order shear deformation theory based on a displacement field where cubic variation of in-plane displacements and parabolic variation of the transverse displacement across the thickness is assumed. Reddy (1990) revised his earlier work in Reddy (1984) and derived a third order deformation theory with taking account to the stretching of transverse normal in the displacement field. Matsunaga (1992) assumed a displacement field in the form of power series in the thickness coordinate in deriving a two dimensional plate theory. Shariat and Eslami (2007) employed a third order theory in buckling analysis of rectangular thick functionally graded plates. Vijayakumar (2011) modified the classical plate theory by solving the boundary condition paradox. He derived a sixth-order partial differential equation which satisfies all the plate boundary conditions. He considered the in plane displacement as linearly distributed across plate thickness and modified them through additional gradients of an auxiliary harmonic function. Mantari *et al.* (2012) derived a two-dimensional higher order shear deformation theory for sandwich and composite laminated plates. In their theory the displacements of the middle surface are expanded as a combination of exponential and trigonometric functions of the thickness coordinate. Thai and Kim (2012) assumed a displacement field, which yields parabolic variation of transverse shear stress through the thickness, in the formulation of a two dimensional higher order shear deformation theory. Using a linear strain-displacement relations (small deformations) and through the principle of minimum total potential energy they derived the governing equations.

Many nonlinear stability analyses of plates are available in the literature, based on higher order shear deformation theories, yet they did not use the appropriate constitutive relations that account for large deformations. Since buckling problem in general is nonlinear, the plate theories which employ linear strain-displacement relations will give neither accurate buckling load nor the correct mode shape into which a structural element buckles.

A plate theory which accounts for large deformations and predicts accurate stresses, deformations and buckling loads should be derived from a consistent finite strain hyperelastic formulation, and the correct buckling formula which includes shear and axial deformation should be closely linked with the correct finite strain constitutive relationship and hence the correct expression for the strain energy density.

Linear elastic analysis assumes small deformations and linear relationship between stress and strain (Hook's law). Hyperelastic material modelling assumes that material behaviour can be described by mean of a strain energy density function, from which the constitutive equations can be derived. In other words, a material is hyperelastic when there exists a potential function called the strain energy density function, whose derivative with respect to a strain component, determines the corresponding stress component. Reddy (2007).

In this paper, a strain energy density for isotropic hyperelastic materials under finite strain proposed by Attard and Hunt (2004) is used to derive the constitutive relationships for plates which include effect of transverse and in-plane shear deformations and a detailed hyperelastic
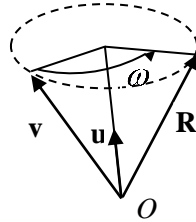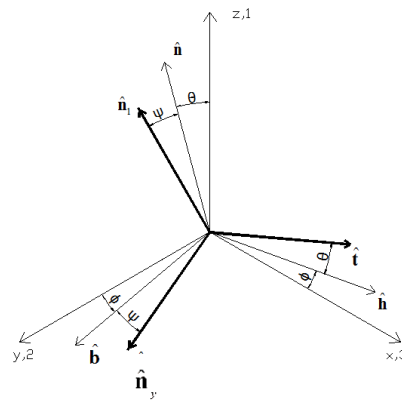
Fig. 1 Rotation tensor *R*

Fig. 2 Rotation of the tangent base vectors

formulation for thick plate buckling. The equations derived are then compared with equations in the literature.The symbolic manipulator Maple is used for the detailed mathematical derivations.

2. Derivation of constitutive relationship

Consider a rectangular plate having thickness *t* and side lengths *a* and *b*. The plate is assumed to be initially unstressed i.e., the initial curvature and stretches are assumed to be zero. It is further assumed that the plate is made of homogeneous isotropic perfectly elastic material. The plate is referred to rectangular Cartesian coordinates *x*, *y*, *z* where *x* and *y* lie in the middle plane of the plate.

In the following derivation, a bold lower case symbol such as **u** represents a vector.

2.1 Rotation of tangent base vectors

A rotation tensor **R** (Fig. 1), rotates a given vector **v** about an axis parallel to a normalized axis vector **u** (**u**×**u**=1) through an angle $0 \leq \omega \leq 2\pi$ (see Attard and Kim 2010). The angle is defined as positive using the right hand screw rule where the thumb of the right hand is extended in the direction of the axis vector **u** and the closing fingers define a positive rotation. The rotation vector **R** is given by

$$\mathbf{R} = \cos\omega\mathbf{I} + (1 - \cos\omega)\mathbf{u} \otimes \mathbf{u} + \sin\omega\mathbf{u} \times \tag{1}$$

Where, $\mathbf{I}$ is the identity tensor.

Consider a differential plate element embedded in the plate and located at the mid-surface of the plate. The plate element can be resolved into its component with respect to the Cartesian reference frame. The component can be considered as tangent base vectors $\mathbf{i}_1$, (z), $\mathbf{i}_2$, (y) and $\mathbf{i}_3$, (x) with its origin at the mid-plane of a thick plate. Assume that during the deformation, the tangent base vectors are rotated by different angles and assume that the rotation order is as follow (Fig. 2)

First consider a rotation of ϕ about the unit vector $\mathbf{i}_1$ in the 1,z axis, such that:

$$\mathbf{R}_\phi = \cos\phi\mathbf{I} + (1 - \cos\phi)\mathbf{i}_1 \otimes \mathbf{i}_1 + \sin\phi\mathbf{i}_1 \times \tag{2}$$

Applying this rotation to $\mathbf{i}_2$ and $\mathbf{i}_3$ about $\mathbf{i}_1$, $\mathbf{i}_2$ becomes $\hat{\mathbf{b}}$ and $\mathbf{i}_3$ becomes $\hat{\mathbf{h}}$

$$\hat{\mathbf{b}} = \mathbf{R}_\phi \mathbf{i}_2 = \cos\phi\mathbf{i}_2 + \sin\phi\mathbf{i}_3 \tag{3}$$

$$\hat{\mathbf{h}} = \mathbf{R}_\phi \mathbf{i}_3 = -\sin\phi\mathbf{i}_2 + \cos\phi\mathbf{i}_3 \tag{4}$$

Then consider a rotation of θ about $\hat{\mathbf{b}}$

$$\mathbf{R}_\theta = \cos\theta\mathbf{I} + (1 - \cos\theta)\hat{\mathbf{b}} \otimes \hat{\mathbf{b}} + \sin\theta\hat{\mathbf{b}} \tag{5}$$

Applying this rotation to $\hat{\mathbf{h}}$ and $\mathbf{i}_1$, $\hat{\mathbf{h}}$ becomes $\hat{\mathbf{t}}$ and $\mathbf{i}_1$ becomes $\hat{\mathbf{n}}$

$$\hat{\mathbf{t}} = \mathbf{R}_\theta\hat{\mathbf{h}} = \cos\theta\hat{\mathbf{h}} + \sin\theta\mathbf{i}_1 \tag{6}$$

$$\hat{\mathbf{n}} = \mathbf{R}_\theta\mathbf{i}_1 = -\sin\theta\hat{\mathbf{h}} + \cos\theta\mathbf{i}_1 \tag{7}$$

Finally, rotate $\hat{\mathbf{n}}$ and $\hat{\mathbf{b}}$ about $\hat{\mathbf{t}}$ with an angel of ψ, so that $\hat{\mathbf{n}}$ becomes $\hat{\mathbf{n}}_1$ and $\hat{\mathbf{b}}$ becomes $\hat{\mathbf{n}}_y$

$$\mathbf{R}_\psi = \cos\psi\mathbf{I} + (1 - \cos\psi)\hat{\mathbf{t}} \otimes \hat{\mathbf{t}} + \sin\psi\hat{\mathbf{t}} \times \tag{8}$$

$$\hat{\mathbf{n}}_1 = \mathbf{R}_\psi\hat{\mathbf{n}} = \cos\psi\hat{\mathbf{n}} + \sin\psi\hat{\mathbf{b}} \tag{9}$$

$$\hat{\mathbf{n}}_y = \mathbf{R}_\psi\hat{\mathbf{b}} = -\sin\psi\hat{\mathbf{n}} + \cos\psi\hat{\mathbf{b}} \tag{10}$$

The three vectors $\hat{\mathbf{n}}_1$, $\hat{\mathbf{t}}$ and $\mathbf{n}_y$ form a moving orthonormal triad. To add the effect of shear deformations, Timoshenko's beam theory concept will be adopted. Four shear angles α_o, φ_o, ω_2 and ω_3 will be added to account for two transverse and two in-plane shear deformations.

To represent the transverse shear deformation in yz plane (perpendicular to x axis), rotate $\hat{\mathbf{t}}$ about $\hat{\mathbf{n}}_y$ with an angle φ_o, then to represent the in-plane shear deformation in the direction of y-axis, rotate $\hat{\mathbf{t}}$ additionally about $\hat{\mathbf{n}}_1$ with an angle $-\omega_3$ so that $\hat{\mathbf{t}}$ becomes $\hat{\mathbf{n}}_3$

$$\mathbf{R}_{\varphi_o} = \cos\varphi_o\mathbf{I} + (1 - \cos\varphi_o)\hat{\mathbf{n}}_y \otimes \hat{\mathbf{n}}_y + \sin\varphi_o\hat{\mathbf{n}}_y \times \tag{11}$$

$$\mathbf{R}_{-\omega_3} = \cos(\omega_3)\mathbf{I} + (1 - \cos\omega_3)\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{n}}_1 - \sin\omega_3\hat{\mathbf{n}}_1 \times \tag{12}$$

Applying the above rotations, $\hat{\mathbf{t}}$ becomes $\hat{\mathbf{n}}_3$

$$\hat{\mathbf{n}}_3 = \lambda_3 \mathbf{R}_{-\omega_3} \mathbf{R}_{\varphi_\circ} \hat{\mathbf{t}} \tag{13}$$

$$\hat{\mathbf{n}}_3 = \lambda_3 (\cos\varphi_o \cos\omega_3 \hat{\mathbf{t}} + \sin\varphi_o \hat{\mathbf{n}}_1 + \sin\omega_3 \cos\varphi_o \hat{\mathbf{n}}_y)$$

$$= \gamma_{33} \hat{\mathbf{t}} + \gamma_{31} \hat{\mathbf{n}}_1 + \gamma_{32} \hat{\mathbf{n}}_y \tag{14}$$

Similarly, to represent the transverse shear deformation in *xz* plane (perpendicular to *y* axis), rotate $\hat{\mathbf{n}}_y$ about $\hat{\mathbf{t}}$ with an angle α_o, then to represent the in-plane shear deformation in the direction of *x*-axis, rotate $\hat{\mathbf{n}}_y$ additionally about $\hat{\mathbf{n}}_1$ with an angle ω_2 so that $\hat{\mathbf{n}}_y$ becomes $\hat{\mathbf{n}}_2$

$$\mathbf{R}_{\alpha_o} = \cos\alpha_o \mathbf{I} + (1-\cos\alpha_o)\hat{\mathbf{t}} \otimes \hat{\mathbf{t}} + \sin\alpha_o \hat{\mathbf{t}} \times \tag{15}$$

$$\mathbf{R}_{\omega_o} = \cos\omega_2 \mathbf{I} + (1-\cos\omega_2)\hat{\mathbf{n}}_1 \otimes \hat{\mathbf{t}} + \sin\alpha_o \hat{\mathbf{t}} \times \tag{16}$$

$$\hat{\mathbf{n}}_2 = \lambda_2 \mathbf{R}_{\omega_2} \mathbf{R}_{\alpha_o} \hat{\mathbf{n}}_y \tag{17}$$

$$\hat{\mathbf{n}}_2 = \lambda_2 (\cos\alpha_o \cos\omega_2 \hat{\mathbf{n}}_y + \sin\alpha_o \hat{\mathbf{n}}_1 + \sin\omega_2 \cos\alpha_o \hat{\mathbf{t}}$$

$$= \gamma_{22} \hat{\mathbf{n}}_y + \gamma_{21} \hat{\mathbf{n}}_1 + \gamma_{23} \hat{\mathbf{t}} \tag{18}$$

Where, λ_2 and λ_3 represent the stretches in the *y* and *x* axis, respectively.

The terms γ_{22} and γ_{33} represent the axial stretch component in the normal directions, while γ_{31}, γ_{32}, γ_{21} and γ_{23} are measures of the transverse and in-plane shear deformation components taken in the tangent base vector directions, in the deformed state, in *yz* and *xz* planes, respectively.
Where

$$\gamma_{33} = \lambda_3 \cos\varphi_\circ \cos\omega_3 \qquad \gamma_{32} = \lambda_3 \sin\omega_3 \cos\varphi_\circ \qquad \gamma_{31} = \lambda_3 \sin\varphi_\circ \tag{19}$$

$$\gamma_{22} = \lambda_2 \cos\alpha_\circ \cos\omega_2 \qquad \gamma_{23} = \lambda_2 \sin\omega_2 \cos\alpha_\circ \qquad \gamma_{21} = \lambda_2 \sin\alpha_\circ \tag{20}$$

We can write the triad $\hat{\mathbf{n}}_1$, $\hat{\mathbf{t}}$ and $\mathbf{n}_y$ in terms of $\mathbf{i}_1$, $\mathbf{i}_2$, and $\mathbf{i}_3$

$$\hat{\mathbf{n}}_1 = \cos\psi\cos\theta\mathbf{i}_1 + (\cos\psi\sin\theta\sin\phi + \sin\psi\cos\phi)\mathbf{i}_2 + (\sin\psi\sin\phi - \cos\psi\cos\phi\sin\theta)\mathbf{i}_3 \tag{21}$$

$$\hat{\mathbf{n}}_y = (-\sin\psi\cos\theta)\mathbf{i}_1 + (\cos\psi\cos\phi - \sin\psi\sin\theta\sin\phi)\mathbf{i}_2 + (\sin\psi\sin\theta\cos\phi + \cos\psi\sin\phi)\mathbf{i}_3 \tag{22}$$

$$\hat{\mathbf{t}} = (\sin\theta)\mathbf{i}_1 - (\cos\theta\sin\phi)\mathbf{i}_2 + (\cos\theta\cos\phi)\mathbf{i}_3 \tag{23}$$

2.2 Curvature and torsion

The deformation curvatures and torsions about the unit normal of the deformed cross section can be defined as

$$\hat{\mathbf{n}}_{1,2} \bullet \hat{\mathbf{n}}_y = \kappa_2 \tag{24}$$

$$\hat{\mathbf{n}}_{1,2} \cdot \hat{\mathbf{t}} = \tau_2 \tag{25}$$

$$\hat{\mathbf{n}}_{1,3} \cdot \hat{\mathbf{t}} = \kappa_3 \tag{26}$$

$$\hat{\mathbf{n}}_{1,3} \bullet \hat{\mathbf{n}}_y = \tau_3 \tag{27}$$

$$\hat{\mathbf{n}}_{y,3} \cdot \hat{\mathbf{t}} = \kappa_{32} \tag{28}$$

$$\hat{\mathbf{t}}_{,2} \cdot \hat{\mathbf{n}}_y = \kappa_{23} \tag{29}$$

Where, κ_2 is the curvature along y-axis, κ_3 is the curvature along x-axis, τ_2 is the torsion along y-axis, τ_3 is the torsion along x-axis, κ_{23} and κ_{32} are the in-plane curvatures. $\hat{\mathbf{n}}_{1,2}$ represents the derivative of $\hat{\mathbf{n}}_1$ with respect to axis 2 (y axis) and so forth.

Now consider two curvature vectors $\boldsymbol{K}_a$, $\boldsymbol{K}_b$ as follow

$$\boldsymbol{K}_a = \kappa_2 \hat{\mathbf{n}}_1 - \tau_2 \hat{\mathbf{n}}_y + \kappa_2 \hat{\mathbf{t}} \tag{30}$$

$$\boldsymbol{K}_b = \kappa_3 \hat{\mathbf{n}}_1 - \kappa_3 \hat{\mathbf{n}}_y + \tau_3 \hat{\mathbf{t}} \tag{31}$$

Therefore we have

$$\boldsymbol{K}_a \times \hat{\mathbf{n}}_1 = \tau_2 \hat{\mathbf{t}} + \kappa_2 \hat{\mathbf{n}}_y \tag{32}$$

$$\boldsymbol{K}_b \times \hat{\mathbf{n}}_1 = \kappa_3 \hat{\mathbf{t}} + \tau_3 \hat{\mathbf{n}}_y \tag{33}$$

$$(\boldsymbol{K}_a \times \hat{\mathbf{n}}_1) \bullet \hat{\mathbf{n}}_y = \kappa_2 \tag{34}$$

$$(\boldsymbol{K}_a \times \hat{\mathbf{n}}_1) \bullet \hat{\mathbf{t}} = \tau_2 \tag{35}$$

$$(\boldsymbol{K}_b \times \hat{\mathbf{n}}_1) \bullet \hat{\mathbf{t}} = \kappa_3 \tag{36}$$

$$(\boldsymbol{K}_b \times \hat{\mathbf{n}}_1) \bullet \hat{\mathbf{n}}_y = \tau_3 \tag{37}$$

Therefore

$$\hat{\mathbf{n}}_{1,2} = \boldsymbol{K}_a \times \hat{\mathbf{n}}_1 = \tau_2 \hat{\mathbf{t}} + \kappa_2 \hat{\mathbf{n}}_y \tag{38}$$

$$\hat{\mathbf{n}}_{1,3} = \boldsymbol{K}_b \times \hat{\mathbf{n}}_1 = \kappa_3 \hat{\mathbf{t}} + \tau_3 \hat{\mathbf{n}}_y \tag{39}$$

Let's now denote the stretched form of the vectors $\hat{\mathbf{n}}_1, \hat{\mathbf{n}}_2$ and $\hat{\mathbf{n}}_3$ by $\hat{\mathbf{g}}_1, \hat{\mathbf{g}}_2$ and $\hat{\mathbf{g}}_3$ Where

$$\hat{\mathbf{g}}_1 = \lambda_1 \hat{\mathbf{n}}_1 \tag{40}$$

$$\hat{\mathbf{g}}_2 = \lambda_2 \cos\alpha_o \cos\omega_2 \hat{\mathbf{n}}_y + z(\tau_2 \hat{\mathbf{t}} + \kappa_2 \hat{\mathbf{n}}_y) + \lambda_2 \sin\omega_2 \cos\alpha_o \hat{\mathbf{t}} + \lambda_2 \sin\alpha_o \hat{\mathbf{n}}_1$$

$$= \gamma_{22} \hat{\mathbf{n}}_y + z(\tau_2 \hat{\mathbf{t}} + \kappa_2 \hat{\mathbf{n}}_y) + \gamma_{23} \hat{\mathbf{t}} + \gamma_{21} \hat{\mathbf{n}}_1 \tag{41}$$

$$\hat{\mathbf{g}}_3 = \lambda_3 \cos\varphi_o \cos\omega_3 \hat{\mathbf{t}} + z(\kappa_3 \hat{\mathbf{t}} + \tau_3 \hat{\mathbf{n}}_y) + \lambda_3 \sin\omega_3 \cos\varphi_o \hat{\mathbf{n}}_y + \lambda_3 \sin\varphi_o \hat{\mathbf{n}}_1$$

$$= \gamma_{33}\hat{\mathbf{t}} + z(\kappa_3\hat{\mathbf{t}} + \tau_3\hat{\mathbf{n}}_y) + \gamma_{32}\hat{\mathbf{n}}_y + \gamma_{31}\hat{\mathbf{n}}_1 \tag{42}$$

From Eqs. (21)-(23) and Eqs. (24)-(29), following relationships can be obtained for the curvatures and torsions:

$$\kappa_2 = \frac{\partial\psi}{\partial y} + \sin\theta\frac{\partial\phi}{\partial y} \tag{43}$$

$$\tau_2 = -\frac{\partial\theta}{\partial y}\cos\psi + \cos\theta\sin\psi\frac{\partial\phi}{\partial y} \tag{44}$$

$$\kappa_3 = -\frac{\partial\theta}{\partial x}\cos\psi + \cos\theta\sin\psi\frac{\partial\phi}{\partial x} \tag{45}$$

$$\tau_3 = \frac{\partial\psi}{\partial x} + \sin\theta\frac{\partial\phi}{\partial x} \tag{46}$$

$$\kappa_{32} = \sin(\psi)\frac{\partial\theta}{\partial x} + \cos(\psi)\cos(\theta)\frac{\partial\phi}{\partial x} \tag{47}$$

$$\kappa_{23} = \sin(\psi)\frac{\partial\theta}{\partial y} + \cos(\psi)\cos(\theta)\frac{\partial\phi}{\partial y} \tag{48}$$

2.3 Deformation tensor F*

Define a deformation tensor F^* associated with the deformation and stretches of the tangent base vectors. From Eqs. (40)-(42), we can obtain a matrix form of F^* as follow

$$\mathbf{F}^* = \begin{pmatrix} 1 & \gamma_{21} & \gamma_{31} \\ 0 & \gamma_{22} + z\kappa_2 & z\tau_3 + \gamma_{32} \\ 0 & z\tau_2 + \gamma_{23} & \gamma_{33} + z\kappa_3 \end{pmatrix} \tag{49}$$

2.4 Stresses and forces

A physical Lagrangian stress system is defined with respect to the directions of a moving orthonormal triad frame. The transformation between the second Piola Kirchhoff stress tensor **II** components and the Lagrangian physical stresses **S** can be established using vector transformation. The transformation between the second Piola Kirchhoff stress tensor and Lagrangian stresses is

$$\mathbf{S} = \mathbf{II}\mathbf{F}^{*\mathbf{T}} = \begin{bmatrix} S^{nn} & S^{nt} & S^{nb} \\ S^{tn} & S^{tt} & S^{tb} \\ S^{bn} & S^{bt} & S^{bb} \end{bmatrix} \tag{50}$$

The normal stresses S^{nn} are taken normal to the cross-sectional plane while the tangential shear stresses S^{nt} & S^{nb} are taken within the cross-sectional plane. The stress tensor $\mathbf{S}$ is not symmetric but needs to satisfy the following symmetry condition

$$\mathbf{S^T} = \mathbf{F^*}\mathbf{S}\mathbf{F^{*-T}} \tag{51}$$

The Green deformation tensor $\mathbf{C}$ and the Jacobian or volume invariant J can then be expressed in terms of the deformation tensor, that is

$$\mathbf{C} = \mathbf{F^{*T}}\mathbf{F^*}$$

$$J = \det\left(\mathbf{F^*}\right) = \gamma_{22}\gamma_{33} - \gamma_{32}\gamma_{23} - z\tau_3\gamma_{23} - z\tau_2\gamma_{32} + \gamma_{22}z\kappa_3 + \gamma_{33}z\kappa_2 + z^2\kappa_2\kappa_3 - z^2\tau_2\tau_3 \tag{52}$$

The strain energy density function U for a compressible isotropic neo-Hookean material (see Attard and Hunt 2004) is given by

$$U = \tfrac{1}{2}G\left(tr\left(\mathbf{C}-\mathbf{I}\right) - 2\ln J\right) + \tfrac{1}{2}\Lambda\left(\ln J\right)^2 \tag{53}$$

Here, tr symbolizes the trace of a tensor and $\mathbf{C}$ is the right Cauchy-Green deformation tensor. The constitutive relationship for a hyperelastic material can be established for the second Piola Kirchhoff stress tensor $\mathbf{\Pi}$ by (see Attard and Hunt 2004)

$$\mathbf{\Pi} = 2\frac{\partial U}{\partial \mathbf{C}} = G\mathbf{I} - p_h\mathbf{C^{-1}} \qquad p_h = G - \Lambda\ln J \tag{54}$$

Where, $G = \dfrac{E}{2\left(1+\upsilon\right)}$ is the shear modulus, $\Lambda = \dfrac{2G\upsilon}{\left(1-2\upsilon\right)}$ is the Lamé constant, E is the elastic modulus and υ is the Poisson's ratio. In the above equation, p_h represents a hydrostatic stress. Incorporating Eq. (54) and Eq. (50), the constitutive relationship for the physical Lagrangian stress is then

$$\mathbf{S} = \mathbf{\Pi}\mathbf{F^{*T}} = G\mathbf{F^{*T}} - p_h\mathbf{C^{-1}}\mathbf{F^{*T}} = G\mathbf{F^{*T}} - p_h\mathbf{F^{*-1}}$$
$$\cong G\left(\mathbf{F^{*T}} - \mathbf{F^{*-1}}\right) + \Lambda\left(J-1\right)\mathbf{F^{*-1}} \tag{55}$$

Incorporating the constitutive relations given in Eq. (55), using Eq. (52), expanding to second order in terms of the deformations, we can write the constitutive relations as

$$S^{11} = \Lambda(\gamma_{22} + \gamma_{33} - 2) + z\Lambda(\kappa_2 + \kappa_3) \tag{56}$$

$$S^{22} = \bar{E}(\gamma_{22} - 1 + z\kappa_2) + \Lambda(z\kappa_3 + \gamma_{33} - 1) \tag{57}$$

$$S^{33} = \bar{E}(\gamma_{33} - 1 + z\kappa_3) + \Lambda(z\kappa_2 + \gamma_{22} - 1) \tag{58}$$

$$S^{23} = S^{32} = G.(z\tau_2 + z\tau_3 + \gamma_{32} + \gamma_{23}) \tag{59}$$

$$S^{21} = G\gamma_{21} \tag{60}$$

$$S^{31} = G\gamma_{31} \tag{61}$$

$$S^{12} = \Lambda(2\gamma_{21} - z\kappa_3\gamma_{21} - z\kappa_2\gamma_{21} - \gamma_{22}\gamma_{21} - \gamma_{33}\gamma_{21}) + G(2\gamma_{21} - z\kappa_2\gamma_{21} - \gamma_{22}\gamma_{21} - z\tau_2\gamma_{31} - \gamma_{31}\gamma_{23}) \quad (62)$$

$$S^{13} = G.(\gamma_{31}.\gamma_{22} - \gamma_{21}\gamma_{32} - z\gamma_{21}\tau_3 - z\gamma_{31}\kappa_2) \quad (63)$$

Where

$$\bar{E} = 2G + \Lambda = \frac{E(1-\upsilon)}{(1-2\upsilon)(1+\upsilon)} \quad (64)$$

By integrating the stresses we obtain the normal forces per meter, and in turn we can obtain the bending moments and shears per meter of the plate

$$N^{22} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{22} dz = \frac{Et}{(1+\upsilon)(1-2\upsilon)}[(1-\upsilon)(\gamma_{22} - 1) + \upsilon(\gamma_{33} - 1)] \quad (65)$$

$$M^{22} = \int_{\frac{-t}{2}}^{\frac{t}{2}} zS^{22} dz = \frac{Et^3}{12(1+\upsilon)(1-2\upsilon)}[\kappa_2(1-\upsilon) + \kappa_3\upsilon] \quad (66)$$

$$N^{33} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{33} dz = \frac{Et}{(1+\upsilon)(1-2\upsilon)}[(1-\upsilon)(\gamma_{33} - 1) + \upsilon(\gamma_{22} - 1)] \quad (67)$$

$$M^{33} = \int_{\frac{-t}{2}}^{\frac{t}{2}} zS^{33} dz = \frac{Et^3}{12(1+\upsilon)(1-2\upsilon)}[\kappa_3(1-\upsilon) + \kappa_2\upsilon] \quad (68)$$

$$N^{11} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{11} dz = \frac{E\upsilon t}{(1+\upsilon)(1-2\upsilon)}(\gamma_{22} + \gamma_{33} - 2) \quad (69)$$

$$M^{11} = \int_{\frac{-t}{2}}^{\frac{t}{2}} zS^{11} dz = \frac{E\upsilon t^3}{12(1+\upsilon)(1-2\upsilon)}(\kappa_2 + \kappa_3) \quad (70)$$

$$N^{32} = N^{23} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{23} dz = \frac{Et}{2(1+\upsilon)}(\gamma_{23} + \gamma_{32}) \quad (71)$$

$$M^{23} = M^{32} = \int_{\frac{-t}{2}}^{\frac{t}{2}} zS^{23} dz = \frac{Et^3}{24(1+\upsilon)}(\tau_2 + \tau_3) \quad (72)$$

$$Q^{21} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{21} dz = \frac{Et}{2(1+\upsilon)} \gamma_{21} \tag{73}$$

$$M^{21} = \int_{\frac{-t}{2}}^{\frac{t}{2}} z S^{21} dz = 0 \tag{74}$$

$$Q^{31} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{31} dz = \frac{Et}{2(1+\upsilon)} \gamma_{31} \tag{75}$$

$$M^{31} = \int_{\frac{-t}{2}}^{\frac{t}{2}} z S^{31} dz = 0 \tag{76}$$

$$Q^{12} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{12} dz = \frac{Et}{2(1+\upsilon)(-1+2\upsilon)} (\gamma_{22}\gamma_{21} + 2\upsilon\gamma_{33}\gamma_{21} - 2\gamma_{21} - 2\upsilon\gamma_{31}\gamma_{23} + \gamma_{31}\gamma_{23}) \tag{77}$$

$$M^{12} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{12} z dz = \frac{Et^3}{24(1+\upsilon)(-1+2\upsilon)} [\gamma_{21}(\kappa_2 + 2\upsilon\kappa_3) + \gamma_{31}\tau_2(1-2\upsilon)] \tag{78}$$

$$Q^{13} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{13} dz = \frac{Et}{2(1+\upsilon)} (\gamma_{31}\gamma_{22} - \gamma_{21}\gamma_{32}) \tag{79}$$

$$M^{13} = \int_{\frac{-t}{2}}^{\frac{t}{2}} S^{13} z dz = \frac{Et^3}{24(1+\upsilon)} (\gamma_{31}\kappa_2 - \gamma_{21}\tau_3) \tag{80}$$

3. Equilibrium equations under finite strain

An element of the plate, in the deformed state, under the effect of internal forces is shown in Fig. 3. The internal forces and moments are shown in vector form. The forces $\mathbf{N}_i$ and moments $\mathbf{M}_i$ are forces and moments per unit of undeformed length respectively.

The equilibrium equations in vectorial form are

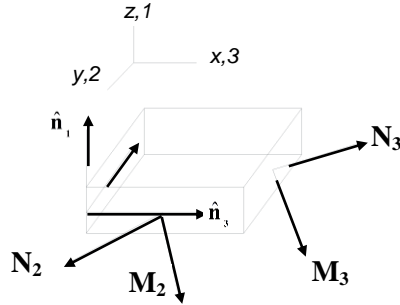$$\mathbf{N}_{3,3} + \mathbf{N}_{2,2} = \mathbf{0} \tag{81}$$

Fig. 3 Plate element in the deformed state

$$\mathbf{M}_{3,3} + \mathbf{M}_{2,2} + \hat{\mathbf{n}}_3 \times \mathbf{N}_3 + \hat{\mathbf{n}}_2 \times \mathbf{N}_2 = 0 \tag{82}$$

Each of the two vector equilibrium equations is equivalent to three scalar component equations. The component representation for the forces and moments along the orthogonal triad tangent base vectors in the deformed state, $\hat{\mathbf{n}}_1$, $\hat{\mathbf{n}}_y$ and $\hat{\mathbf{t}}$, can be obtained using the following relationships

$$\mathbf{N_i} = N_{ij}\mathbf{t_j} + Q_i\mathbf{n} \tag{83}$$

$$\mathbf{M_i} = M_{ij}(\mathbf{n} \times \mathbf{t_j}) \tag{84}$$

In Eqs. (83)-(84), the subscripts *i* and *j* can be 2 and 3 only and the summation convention is invoked. Where, N_{ij} represents an axial force, Q_{ij} represents a shear force and M_{ij} represents a moment.

$\mathbf{t_j}$ represents the tangential base vectors $\hat{\mathbf{n}}_y$ and $\hat{\mathbf{t}}$ while $\mathbf{n}$ represents the normal base vector $\hat{\mathbf{n}}_1$.

From Eq. (83) we can get

$$\mathbf{N}_2 = N^{22}\hat{\mathbf{n}}_y + N^{23}\hat{\mathbf{t}} + Q^{21}\hat{\mathbf{n}}_1 \tag{85}$$

$$\mathbf{N}_3 = N^{32}\hat{\mathbf{n}}_y + N^{33}\hat{\mathbf{t}} + Q^{31}\hat{\mathbf{n}}_1 \tag{86}$$

Taking the derivative of Eq. (85) with respect to *y*-axis (axis 2) and the derivative of Eq. (86) with respect to *x*-axis (axis 3) we get the following two equations

$$\mathbf{N_{2,2}} = N^{22}_{,2}\hat{\mathbf{n}}_y + N^{22}\hat{\mathbf{n}}_{y,2} + N^{23}_{,2}\hat{\mathbf{t}} + N^{23}\hat{\mathbf{t}}_{,2} + Q^{21}_{,2}\hat{\mathbf{n}}_1 + Q^{21}\hat{\mathbf{n}}_{1,2} \tag{87}$$

$$\mathbf{N_{3,3}} = N^{32}_{,3}\hat{\mathbf{n}}_y + N^{32}\hat{\mathbf{n}}_{y,3} + N^{33}_{,3}\hat{\mathbf{t}} + N^{33}\hat{\mathbf{t}}_{,3} + Q^{31}_{,3}\hat{\mathbf{n}}_1 + Q^{31}\hat{\mathbf{n}}_{1,3} \tag{88}$$

Substituting Eqs. (84)-(88) in to Eq. (81) to get

$$N^{32}_{,3}\hat{\mathbf{n}}_y + N^{32}\hat{\mathbf{n}}_{y,3} + N^{33}_{,3}\hat{\mathbf{t}} + N^{33}\hat{\mathbf{t}}_{,3} + Q^{31}_{,3}\hat{\mathbf{n}}_1 + Q^{31}\hat{\mathbf{n}}_{1,3} + N^{22}_{,2}\hat{\mathbf{n}}_y + N^{22}\hat{\mathbf{n}}_{y,2} + N^{23}_{,2}\hat{\mathbf{t}} + N^{23}\hat{\mathbf{t}}_{,2} + Q^{21}_{,2}\hat{\mathbf{n}}_1 + Q^{21}\hat{\mathbf{n}}_{1,2} = 0 \tag{89}$$

Previously, we derived the following relationships (Eqs. (38)-(39))

$$\hat{\mathbf{n}}_{1,2} = \tau_2\hat{\mathbf{t}} + \kappa_2\hat{\mathbf{n}}_y \tag{90}$$

$$\hat{\mathbf{n}}_{1,3} = \kappa_3\hat{\mathbf{t}} + \tau_3\hat{\mathbf{n}}_y \tag{91}$$

In a similar way the following relationships can be obtained

$$\hat{\mathbf{t}}_{,3} = -\kappa_{32}\hat{\mathbf{n}}_y - \kappa_3\hat{\mathbf{n}}_1 \tag{92}$$

$$\hat{\mathbf{n}}_{y,3} = \kappa_{32}\hat{\mathbf{t}} - \tau_3\hat{\mathbf{n}}_1 \tag{93}$$

$$\hat{\mathbf{t}}_{,2} = -\kappa_{23}\hat{\mathbf{n}}_y - \kappa_2\hat{\mathbf{n}}_1 \tag{94}$$

$$\hat{\mathbf{n}}_{y,2} = \kappa_{32}\hat{\mathbf{t}} - \tau_2\hat{\mathbf{n}}_1 \tag{95}$$

Substituting Eqs. (90)-(95) in to Eq. (89) to get

$$N_{,3}^{32}\hat{\mathbf{n}}_y + N^{32}\kappa_{32}\hat{\mathbf{t}} - N^{32}\tau_3\hat{\mathbf{n}}_1 + N_{,3}^{33}\hat{\mathbf{t}} - N^{33}\kappa_{32}\hat{\mathbf{n}}_y - N^{33}\kappa_3\hat{\mathbf{n}}_1 + Q_{,3}^{31}\hat{\mathbf{n}}_1 + Q^{31}\kappa_3\hat{\mathbf{t}} + Q^{31}\tau_3\hat{\mathbf{n}}_y$$

$$+ N_{,2}^{22}\hat{\mathbf{n}}_y + N^{22}\kappa_{23}\hat{\mathbf{t}} - N^{22}\kappa_2\hat{\mathbf{n}}_1 + N_{,2}^{23}\hat{\mathbf{t}} - N^{23}\kappa_{23}\hat{\mathbf{n}}_y - N^{23}\kappa_{23}\hat{\mathbf{n}}_y - N^{23}\tau_2\hat{\mathbf{n}}_1 + Q_{,2}^{21}\hat{\mathbf{n}}_1 + Q^{21}\tau_2\hat{\mathbf{t}} + Q^{21}\kappa_2\hat{\mathbf{n}}_y = 0 \tag{96}$$

Putting all terms, in $\hat{\mathbf{t}}$ direction, equal to zero the first force equilibrium equation will be obtained

$$N_{,3}^{33} + N_{,2}^{23} + N^{32}\kappa_{32} + N^{22}\kappa_{23} + Q^{31}\kappa_3 + Q^{21}\tau_2 = 0 \tag{97}$$

Similarly the terms in the $\hat{\mathbf{n}}_y$ and $\hat{\mathbf{n}}_1$ directions will yield the following two equilibrium equations

$$N_{,3}^{32} + N_{,2}^{22} - N^{33}\kappa_{32} - N^{23}\kappa_{23} + Q^{31}\tau_3 + Q^{21}\kappa_2 = 0 \tag{98}$$

$$Q_{,3}^{31} + Q_{,2}^{21} - N^{33}\kappa_3 - N^{22}\kappa_2 - N^{32}\tau_3 - N^{23}\tau_2 = 0 \tag{99}$$

To derive the moment equilibrium equations, Eq. (84) will be put in component form

$$\mathbf{M}_2 = M^{22}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) \tag{100}$$

$$\mathbf{M}_3 = M^{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{33}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) \tag{101}$$

Taking the derivative of Eq. (100) with respect to *y*-axis (axis 2) and the derivative of Eq. (101) with respect to *x*-axis (axis 3) we get the following two equations

$$\mathbf{M}_{2,2} = M_{,2}^{22}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{22}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y)_{,2} + M_{,2}^{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}})_{,2}$$

$$= M_{,2}^{22}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{22}(\hat{\mathbf{n}}_{1,2} \times \hat{\mathbf{n}}_y) + M^{22}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_{y,2}) + M_{,2}^{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{23}(\hat{\mathbf{n}}_{1,2} \times \hat{\mathbf{t}}) + M^{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}_{,2}) \tag{102}$$

$$\mathbf{M}_{3,3} = M_{,3}^{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y)_{,3} + M_{,3}^{33}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{33}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}})_{,3}$$

$$= M_{,3}^{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{32}(\hat{\mathbf{n}}_{1,3} \times \hat{\mathbf{n}}_y) + M^{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_{y,3}) + M_{,3}^{33}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{33}(\hat{\mathbf{n}}_{1,3} \times \hat{\mathbf{t}}) + M^{33}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}_{,3}) \quad (103)$$

Substituting Eqs. (90)-(95) into Eqs. (102)-(103) we get

$$\mathbf{M}_{2,2} = M_{,2}^{22}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{22}\tau_2(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_y) + M^{23}\kappa_{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M_{,2}^{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{23}\kappa_2(\hat{\mathbf{n}}_y \times \hat{\mathbf{t}}) - M^{23}\kappa_{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) \quad (104)$$

$$\mathbf{M}_{3,3} = M_{,3}^{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{32}\kappa_3(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_y) + M^{32}\kappa_{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M_{,3}^{33}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{33}\tau_3(\hat{\mathbf{n}}_y \times \hat{\mathbf{t}}) - M^{33}\kappa_{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) \quad (105)$$

The component of $\hat{\mathbf{n}}_2$ and $\hat{\mathbf{n}}_3$ were previously obtained to be

$$\hat{\mathbf{n}}_2 = \gamma_{22}\hat{\mathbf{n}}_y + \gamma_{21}\hat{\mathbf{n}}_1 + \gamma_{23}\hat{\mathbf{t}} \quad (106)$$

$$\hat{\mathbf{n}}_3 = \gamma_{33}\hat{\mathbf{n}}_y + \gamma_{31}\hat{\mathbf{n}}_1 + \gamma_{32}\hat{\mathbf{n}}_y \quad (107)$$

Substituting Eqs. (85)-(86) and Eqs. (104)-(107) into Eq. (82) to get and simplifying to get

$$M_{,3}^{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{32}\kappa_3(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_y) + M^{32}\kappa_{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M_{,3}^{33}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{33}\tau_3(\hat{\mathbf{n}}_y \times \hat{\mathbf{t}}) - M^{33}\kappa_{32}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y)$$

$$M_{,2}^{22}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + M^{22}\tau_2(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_y) + M^{22}\kappa_{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M_{,2}^{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + M^{23}\kappa_2(\hat{\mathbf{n}}_y \times \hat{\mathbf{t}}) - M^{23}\kappa_{23}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y)$$

$$+ N^{32}\gamma_{33}(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_y) + Q^{31}\gamma_{33}(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_1) + N^{32}\gamma_{31}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + N^{33}\gamma_{31}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) + N^{32}\gamma_{32}(\hat{\mathbf{n}}_y \times \hat{\mathbf{t}}) + Q^{31}\gamma_{32}(\hat{\mathbf{n}}_y \times \hat{\mathbf{n}}_1)$$

$$+ N^{23}\gamma_{22}(\hat{\mathbf{n}}_y \times \hat{\mathbf{t}}) + Q^{21}\gamma_{22}(\hat{\mathbf{n}}_y \times \hat{\mathbf{n}}_1) + Q^{21}\gamma_{23}(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_1) + N^{22}\gamma_{21}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y) + N^{32}\gamma_{23}(\hat{\mathbf{t}} \times \hat{\mathbf{n}}_y) + N^{23}\gamma_{21}(\hat{\mathbf{n}}_1 \times \hat{\mathbf{t}}) \quad (108)$$

To satisfy equilibrium condition, all the moments in the direction of vectors $\hat{\mathbf{n}}_1 \times \hat{\mathbf{n}}_y$, $\hat{\mathbf{n}}_1 \times \mathbf{t}$ and $\hat{\mathbf{n}}_y \times \hat{\mathbf{t}}$ must vanish and so yielding the following three equilibrium equations

$$M_{,3}^{32} + M_{,2}^{22} - M^{33}\kappa_{32} - M^{23}\kappa_{23} - Q^{21}\gamma_{22} - Q^{31}\gamma_{32} + N^{32}\gamma_{31} + N^{22}\gamma_{21} = 0 \quad (109)$$

$$M_{,2}^{23} + M_{,3}^{33} + M^{32}\kappa_{32} + M^{22}\kappa_{23} - Q^{31}\gamma_{33} - Q^{21}\gamma_{23} + N^{33}\gamma_{31} + N^{23}\gamma_{21} = 0 \quad (110)$$

$$M^{32}\kappa_3 - M^{33}\tau_3 + M^{22}\tau_2 - M^{23}\kappa_2 + N^{33}\gamma_{32} - N^{22}\gamma_{23} - N^{32}\gamma_{33} + N^{23}\gamma_{22} = 0 \quad (111)$$

Eqs. (97)-(99) and (109)-(111) are the six plate equilibrium equations in finite strain.
The above equilibrium equations are consistent with those derived by Reissner (see Fung (1974) P. 41) and by Taber (1988), individually, in their two-dimensional shell theories, after converting them into plate equations.

4. Solved example

Consider a rectangular plate with two simply supported edges parallel to y-axis and free edges on the two edges parallel to x-axis with length a, width b and thickness t subjected to a uniformly distributed axial force along x axis (Fig. 4).
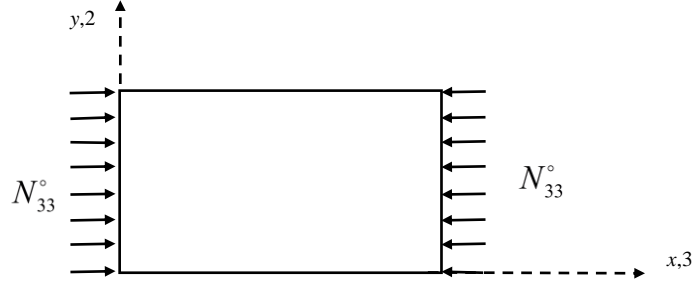
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Fig. 4 Plate under uniaxial compression force

It is assumed that the plate has no out-of-plane deformations prior to buckling, i.e., the plate is flat. The buckling load can be established by looking at the equilibrium equations under small perturbations or variations about the initial loaded state. The variation symbol δ will be used to indicate small perturbations.

A term with the superscript $\circ$ represents a quantity in the initial conditions.

The boundary conditions are:

$$\text{At } x=0 \text{ , } x=a$$

$$w=0, M^{33}=0, M^{32}=0 \quad \text{or} \quad w=0 \text{ , } \kappa_3=0 \text{ , } \tau_3=0$$

where w is the transverse deflection. Since we have axial force along x-axis direction only, it can be considered as a plane stress condition, hence we have the following conditions (see Ziegler 1983)

$$N^{22}=0 \text{ , } N^{23}=0 \text{ , } Q^{21}=0 \text{ , } M^{22}=0 \text{ , } M^{23}=0, M^{32}=0$$

$$\gamma_{21}=0 \text{ , } \gamma_{23}=0 \text{ , } \tau_2=0, \quad \kappa_2=0$$

And because the applied load is uniformly distributed, the in-plane curvatures will vanish:

$$\kappa_{23}=0 \text{ , } \kappa_{32}=0$$

Additionally, we have the following initial values equal to zero:

$$Q^{\circ}_{31}=0, \gamma^{\circ}_{31}=0, \kappa^{\circ}_3=0, \kappa^{\circ}_2=0, \tau^{\circ}_3=0 \text{ , } N^{\circ}_{32}=0$$

Incorporating the above conditions the six buckling equations will be reduced to the following two equations

$$\frac{\partial \delta Q_{31}}{\partial x} - N^{\circ}_{33} \delta \kappa_3 = 0 \tag{112}$$

$$\frac{\partial \delta M^{33}}{\partial x} + N^{\circ}_{33} \delta \gamma_{31} - \delta Q_{31} \gamma^{\circ}_{33} = 0 \tag{113}$$

Substituting the force-displacement relationships in the above two equations and further simplifying to get

$$\beta C \frac{\partial \delta \gamma_{31}}{\partial x} - N_{33}^{\circ} \delta \kappa_3 = 0 \tag{114}$$

$$D(\frac{2\upsilon-1}{\upsilon-1}) \frac{\partial^2 (\delta \kappa_3)}{\partial x^2} + N_{33}^{\circ} \frac{\partial \delta \gamma_{31}}{\partial x} - N_{33}^{\circ} \delta \kappa_3 \gamma_{33}^{\circ} = 0 \tag{115}$$

Where, β is the shear correction factor to account for the non-linear distribution of transverse shear deformation.

The above two equations can be merged into one equation. Substituting $\gamma_{33}^{\circ} = 1 - \frac{N_{33}^{\circ}}{EA}$ to get

$$D(\frac{2\upsilon-1}{\upsilon-1}) \frac{\partial^2 (\delta \kappa_3)}{\partial x^2} + (\frac{(N_{33}^{\circ})^2}{\beta C} - N_{33}^{\circ} + \frac{(N_{33}^{\circ})^2}{EA}) \delta \kappa_3 = 0 \tag{116}$$

Solving the above quadratic equation for the buckling load (N_{33}°) will yield two values, the one which satisfy is

$$N_{33}^{\circ} = \frac{1}{2} \frac{\beta CEAa(\upsilon-1) + \sqrt{\beta CEA(\upsilon-1)(8\upsilon\beta CDm^2\pi^2 + 8AE\upsilon Dm^2\pi^2 + Aa^2\upsilon\beta CE - 4\beta CDm^2\pi^2 - 4EADm^2\pi^2 - Aa^2\beta CE}}{a(\upsilon-1)(EA+\beta C)} \tag{117}$$

Where, m is the number of half waves in the x-axis direction, A is the cross-sectional area of the plate ($A=bt$).

Substituting $C=Gt$, and $D = \frac{Gt^3}{6(1-2\upsilon)}$ where $G = \frac{E}{2(1+\upsilon)}$ is the shear moduli, and $m=1$ into the above equation to get:

$$N_{33}^{\circ} = \frac{Gt}{2} \left(\frac{\beta EA}{EA+\beta Gt} - \frac{\sqrt{AE\beta(1-\upsilon)[AE\beta a^2(1-\upsilon) + \frac{2}{3}\pi^2 t^2(EA+\beta Gt)]}}{a(1-\upsilon)(EA+\beta Gt)} \right)$$

The above equation gives the buckling load per unit width of the plate.

Ziegler (1983) in his two dimensional plate theory, for a similar case, obtained the following formula for the buckling load: $P_{Ziegler} = \left(1 + \frac{(1-\upsilon^2)P_{eul}}{Et} - \frac{P_{eul}}{\beta Gt} \right) P_{eul}$

Where P_{eul} is the well-known Euler column buckling load.

Attard and Hunt (2008) obtained the following column buckling formula:

$$P_{Attard} = \frac{EA}{2(1-\frac{E}{G})} \left(1 \pm \sqrt{1 - \frac{4n^2 P_{eul}}{EA}(1-\frac{E}{G})} \right)$$

Fig. 5 shows a comparison of the results, for a steel plate having thickness $=0.1$ m, (such a plate can be a sheet pile or can be found in ship hulls) obtained using the buckling formula derived in the current theory and the above formulas. It reveals very limited difference which is due to
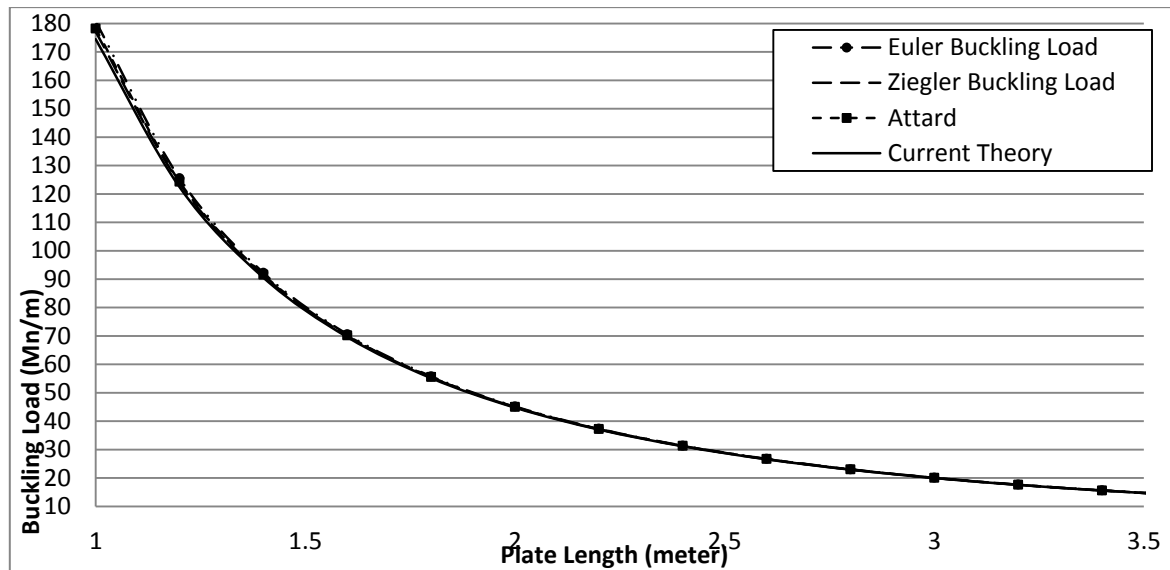
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Fig. 5 Buckling Load for a simply supported plate under uni-axial compression with various plate lengths *E*=200 Gpa, Thickness=0.1 m, Width=1 m, Poisson ratio=0.3, *β*=0.83
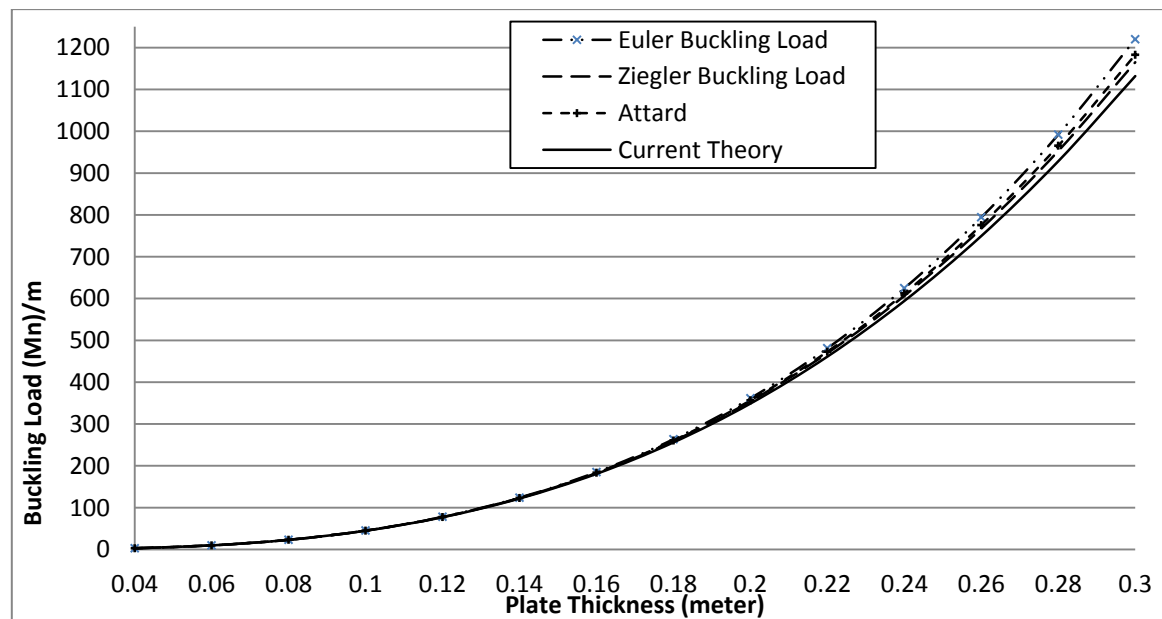

Fig. 6 Buckling Load for a simply supported plate under uni-axial compression with various plate thickness. *E*=200 Gpa, Length=2 m, Width=1 m, Poisson ratio=0.3, *β*=0.83

limited effect of transverse shear in a column like plate under uni-axial load. It is obvious that as the length is increased the difference disappear which is due to superiority of bending deformations on shear deformations.

Fig. 6 shows that as the thickness is increased, the effect of transverse shear on the buckling
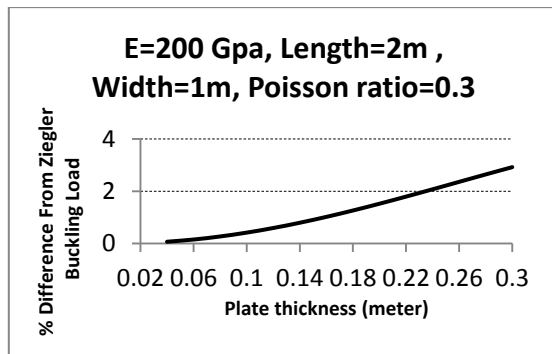
Fig. 7 Percentage difference between Ziegler buckling load and buckling load from the current theory with varying plate thickness
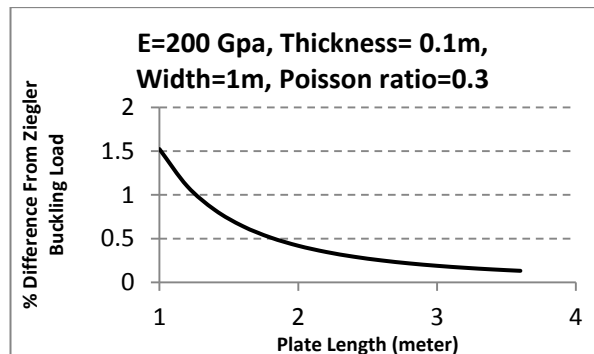
Fig. 8 Percentage difference between Ziegler buckling load and buckling load from the current theory with varying plate length
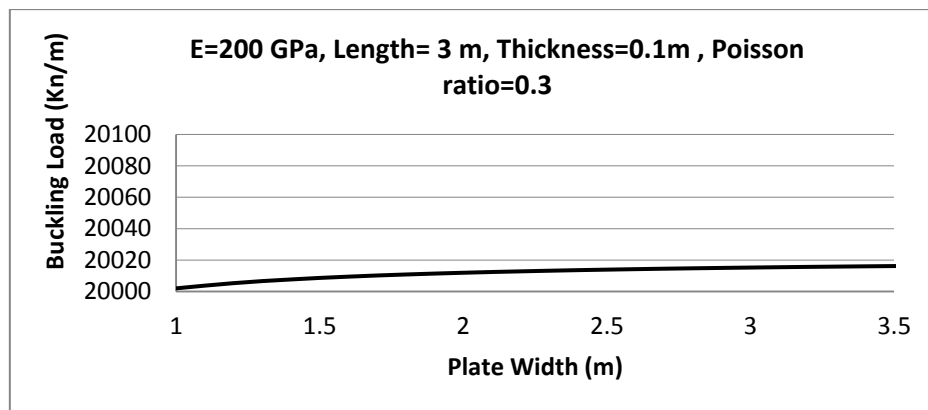

Fig. 9 Effect of plate width on the buckling load

load is quite obvious.

Figs. 7-8 show the difference between the results obtained from Ziegler's plate theory, where linear elastic analysis was employed including effect of transverse and in-plane shear deformations, and results predicted from the current theory. It shows that as the thickness increases, the erroneous results predicted by the linear elastic analysis.

Fig. 9 shows that if a plate strip is buckled individually or as a part of a very wide plate, the buckling load per unit width is slightly different.

5. Conclusions

A two-dimensional nonlinear plate theory has been formulated. The deformation across the thickness coordinate is neglected hence no plate thickness changes during deformation will occur. A deformation tensor which accounts for rotations and stretches has been derived. A hyperelastic strain energy density function for isotropic compressible neo-Hooken material has been employed to derive the constitutive equations which include the effect of transverse and in-plane shear deformations. The equilibrium equations have been derived following the equilibrium approach.

Buckling formulation for a simply supported rectangular plate under uni-axial compression force has been introduced. The results reveal that effect of shear is distinct as the thickness is increased. Hence, the current theory will give much better results when estimating the buckling load in thick plates.

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