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A novel approximate solution for nonlinear problems of vibratory systems

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Abstract. In this research, an approximate analytical solution has been presented for nonlinear problems of vibratory systems in mechanical engineering. The new method is called Variational Approach (VA) which is applied in two different high nonlinear cases. It has been shown that the presented approach leads us to an accurate approximate analytical solution. The results of variational approach are compared with numerical solutions. The full procedure of the numerical solution is also presented. The results are shown that the variatioanl approach can be an efficient and practical mathematical tool in field of nonlinear vibration.

Keywords: variational approach method; nonlinear vibration; numerical method

1. Introduction

It is very difficult to find an exact solution for nonlinear differential problems, therefore it is some new semi-analytical methods have been proposed recently and the results are verified with numerical solutions. Sadighi and Ganji (2007) studied the application of the Adomian decomposition method (ADM) to prepare an analytical solution for solutions of linear and nonlinear Schrödinger equations. The presented method does not need any small parameters and avoid linearization and physically unrealistic assumptions. Chen *et al.* (2009) considered differential transformation method to solve free vibration of the fifth-order nonlinear problems. Runge-kutta algorithm was used to obtain the numerical solution of the problem. The DTM solves the problems with a process of inverse Transformation. Barania *et al.* (2010) studied the heat diffusion and heat transfer equation with a powerful analytical method called homotopy analysis method (HAM). HAM has an auxiliary parameter h, which controls the convergence region of solution series. Cai and Liu (2011) solved nonlinear equations via He's frequency formulation. Ganji *et al.* (2007) applied the variational iteration method (VIM) for the equations of Generalized Hirota-Satsuma coupled KdV equation, Kawahara equation and FKdV equations. Other related paper are available in which they have applied standard methods for solving nonlinear partial differential equations suh as: Homotopy perturbation method (Bayat *et al.* 2013a, 2014a),

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Hamiltonian approach (He 2010, Xu 2010, Bayat *et al.* 2014b, c, d, e, f, 2013b, Bayat and Pakar 2013c), Energy balance method (Jamshidi *et al.* 2010, Bayat *et al.* 2014g, Mehdipour 2010), Variational iteration method (Dehghan 2010), Amplitude frequency formulation (He 2008), Max-Min approach (Shen *et al.* 2009, Zeng *et al.* 2009), Variational approach (He 2007, Bayat and Pakar 2012a, Bayat *et al.* 2012b, Bayat and Pakar 2013a, Bayat *et al.* 2013b, Shahidi *et al.* 2011, Pakar and Bayat 2013), and the other analytical and numerical (Bayat and Abdollahzade 2011, Pakar *et al.* 2014a, b, 2011, Xu 2009, Alicia *et al.* 2010, Bor-Lih *et al.* 2009, Wu 2011, Odibat *et al.* 2008, Zhifeng *et al.* 2013, Rajasekaran 2013, Akgoz 2013, Akgoz and Civalek 2011, Atmane *et al.* 2011, Cunedioglu and Beylergil 2014, Radomirovic *et al.* 2015, Filobello-Nino *et al.* 2015, Xu *et al.* 2015).

Among of the mentioned papers and approaches, the Variational Approach (VA) is considered to solve the nonlinear vibration equations in this paper.

The paper has been collocated as follows:

In section 2, the basic idea of the He's Variational approach is introduced in detail. Section 3, contains the full procedure of the Runge- Kutta algorithm. Section 4 is the application of the variational approach in two different cases for high nonlinear vibratory systems. The validation of the approach and also the discussion on the nonlinear parameters of the systems and the comparison with the numerical results are studied in section 5. Finally, it has been demonstrated that the variational approach can be a precise cyclic solution for nonlinear systems.

2. Basic concept of variational approach

He suggested a variational approach which is different from the known variational methods in open literature (He 2007). Hereby we give a brief introduction of the method

$$\ddot{u} + f(u) = 0 \tag{1}$$

Its variational principle can be easily established utilizing the semi-inverse method

$$J(u) = \int_0^{T/4} \left(-\frac{1}{2}\dot{u}^2 + F(u) \right) dt \tag{2}$$

Where T is period of the nonlinear oscillator, $\partial F / \partial u = f$. Assume that its solution can be expressed as

$$u(t) = A\cos(\omega t) \tag{3}$$

Where A and ω are the amplitude and frequency of the oscillator, respectively. Substituting Eq. (3) into Eq. (2) results in

$$\begin{aligned} J(A,\omega) &= \int_0^{T/4} \left(-\frac{1}{2}A^2\omega^2 \sin^2 \omega t + F(A\cos\omega t) \right) dt \\ &= \frac{1}{\omega}\int_0^{\pi/2} \left(-\frac{1}{2}A^2\omega^2 \sin^2 t + F(A\cos t) \right) dt \\ &= -\frac{1}{2}A^2\omega \int_0^{\pi/2} \sin^2 t\, dt + \frac{1}{\omega}\int_0^{\pi/2} F(A\cos t)\, dt \end{aligned} \tag{4}$$

Applying the Ritz method, He require

$$\frac{\partial J}{\partial A} = 0 \tag{5}$$

$$\frac{\partial J}{\partial \omega} = 0 \tag{6}$$

But with a careful inspection, for most cases He fined that

$$\frac{\partial J}{\partial \omega} = -\frac{1}{2} A^2 \int_0^{\pi/2} \sin^2 t\, dt - \frac{1}{\omega^2} \int_0^{\pi/2} F(A \cos t)\, dt < 0 \tag{7}$$

Thus, He modify conditions Eq. (5) and Eq. (6) into a simpler form

$$\frac{\partial J}{\partial \omega} = 0 \tag{8}$$

From which the relationship between the amplitude and frequency of the oscillator can be obtained.

3. Basic concept of Runge-Kutta

For the numerical approach to verify the analytic solution, the fourth RK (Runge-Kutta) method has been used. This iterative algorithm is written in the form of the following formulae for the second-order differential equation

$$\begin{aligned} \dot{u}_{i+1} &= \dot{u}_i + \frac{\Delta t}{6}\left(h_1 + 2h_2 + 2h_3 + h_4\right) \\ u_{i+1} &= u_i + \Delta t \left(\dot{u}_i + \frac{\Delta t}{6}\left(h_1 + h_2 + h_3\right)\right) \end{aligned} \tag{9}$$

Where, Δt is the increment of the time and h_1, h_2, h_3, and h_4 are determined from the following formulae

$$\begin{aligned} h_1 &= f\left(\dot{u}, u_i, \dot{u}_i\right), \\ h_2 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i,\ \dot{u}_i + \frac{\Delta t}{2} h_1\right), \\ h_3 &= f\left(t_i + \frac{\Delta t}{2},\ u_i + \frac{\Delta t}{2}\dot{u}_i + \frac{1}{4}\Delta t^2 h_1,\ \dot{u}_i + \frac{\Delta t}{2} h_2\right), \\ h_4 &= f\left(t_i + \Delta t,\ u_i + \Delta t\, \dot{u}_i + \frac{1}{2}\Delta t^2 h_2,\ \dot{u}_i + \Delta t\, h_3\right). \end{aligned} \tag{10}$$

The numerical solution starts from the boundary at the initial time, where the first value of the displacement function and its first-order derivative are determined from initial condition. Then, with a small time increment Δt, the displacement function and its first-order derivative at the new position can be obtained using Eq. (9). This process continues to the end of the time limit.

4. Application

In this section, two well-known problems are studied to show the accuracy of the variational approach method. The application of the method are presented in detail for these problems.

4.1 Example 1

First example is equation Helmholtz-Duffing oscillator. This is the euqation of the systems include shallow arches, ship roll dynamics, some electrical circuits, microperforated panel absorber and heavy symmetric gyroscope. The conservative governing equation of the systm is presented as follow

$$\ddot{u} + u + (1-\sigma)u^2 + \sigma u^3 = 0, \tag{11}$$

with initial conditions

$$u(0) = A, \quad \dot{u}(0) = 0 \tag{12}$$

where σ is an asymmetric parameter representing the extend of asymmetry and an over dot denotes differentiation with respect to t. When σ=1, Eq. (11) is a classical Duffing oscillator. Eq. (11) becomes a Helmholtz oscillator with a single-well potential when σ=0.

Its variational formulation can be readily obtained Eq. (11) as follows

$$J(u) = \int_0^t \left(\frac{1}{2}\dot{u}^2 + \frac{1}{2}u^2 + \frac{1}{3}u^3 - \frac{1}{3}\sigma u^3 + \frac{1}{4}\sigma u^4 \right) dt \tag{13}$$

Choosing the trial function $u(t)$=A cos(ωt) into Eq.(13) we obtain

$$J(A) = \int_0^{T/4} \begin{pmatrix} \frac{1}{2}\omega^2 A^2 \sin^2(\omega t) + \frac{1}{2}A^2\cos^2(\omega t) + \frac{1}{3}A^3\cos^3(\omega t) \\ -\frac{1}{3}\sigma A^3\cos^3(\omega t) + \frac{1}{4}\sigma A^4 \cos^4(\omega t) \end{pmatrix} dt \tag{14}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A} = \int_0^{T/4} \begin{pmatrix} \omega^2 A \sin^2(\omega t) + A\cos^2(\omega t) + A^2\cos^3(\omega t) \\ -\sigma A^2 \cos^3(\omega t) + \sigma A^3\cos^4(\omega t) \end{pmatrix} dt = 0 \tag{15}$$

Or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2} \left(\omega^2 A \sin^2 t + A\cos^2 t + A^2 \cos^3 t - \sigma A^2\cos^3 t + \sigma A^3 \cos^4 t \right) dt = 0 \tag{16}$$

Solving Eq. (16), according to ω, we have

$$\omega^2 = \frac{\int_0^{\frac{\pi}{2}} \left(A\cos^2 t + A^2\cos^3 t - \sigma A^2 \cos^3 t + \sigma A^3\cos^4 t \right) dt}{\int_0^{\frac{\pi}{2}} A\, sin^2 t\, dt} \tag{17}$$

Then we have

$$\omega_{VA}=\sqrt{1+\frac{3}{4}\sigma A^2+\frac{8}{3}\frac{1}{\pi}A-\frac{8}{3}\frac{\sigma}{\pi}A} \tag{18}$$

According to $u(t)=A\cos(\omega t)$ and (18), we can obtain the following approximate solution

$$u(t)=A\cos\left(\sqrt{1+\frac{3}{4}\sigma A^2+\frac{8}{3}\frac{1}{\pi}A-\frac{8}{3}\frac{\sigma}{\pi}A}\,t\right) \tag{19}$$

4.2 Example 2

Second example is the governing equation of a mass connected with a spring. The motion of the system is nonlinear. Fig. 1 is the scheme of the motion of the systems and the governing equation of it is as follow

$$J\ddot{\theta}+k_1\theta+k_2\theta^2+k_3\theta^3=0 \tag{20}$$

with initial conditions

$$\theta(0)=A,\quad \dot{\theta}(0)=0 \tag{21}$$

Its variational formulation can be readily obtained Eq. (20) as follows

$$J(\theta)=\int_0^t\left(\frac{1}{2}J\dot{\theta}^2+\frac{1}{2}k_1\theta^2+\frac{1}{3}k_2\theta^3+\frac{1}{4}k_3\theta^4\right)dt \tag{22}$$

Choosing the trial function $u(t)=A\cos(\omega t)$ into Eq. (22) we obtain

$$J(A)=\int_0^{T/4}\left(\begin{array}{l}\frac{1}{2}J\omega^2A^2\sin^2(\omega t)+\frac{1}{2}k_1A^2\cos^2(\omega t)+\frac{1}{3}k_2A^3\cos^3(\omega t)\\+\frac{1}{4}k_3A^4\cos^4(\omega t)\end{array}\right)dt \tag{23}$$

The stationary condition with respect to A leads to

$$\frac{\partial J}{\partial A}=\int_0^{T/4}\left(\begin{array}{l}J\omega^2A\sin^2(\omega t)+k_1A\cos^2(\omega t)+k_2A^2\cos^3(\omega t)\\+k_3A^3\cos^4(\omega t)\end{array}\right)dt=0 \tag{24}$$

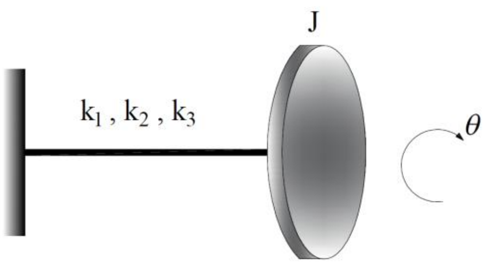


Fig. 1 The mass-nonlinear spring systems

Or

$$\frac{\partial J}{\partial A} = \int_0^{\pi/2} \left(J\,\omega^2 A \sin^2 t + k_1 A \cos^2 t + k_2 A^2 \cos^3 t + k_3 A^3 \cos^4 t \right) dt = 0 \tag{25}$$

Solving Eq. (25), according to ω, we have

$$\omega^2 = \frac{\int_0^{\frac{\pi}{2}} \left(k_1 A \cos^2 t + k_2 A^2 \cos^3 t + k_3 A^3 \cos^4 t \right) dt}{\int_0^{\frac{\pi}{2}} JA\, sin^2 t\, dt} \tag{26}$$

Then we have

$$\omega_{VA} = \sqrt{\frac{k_1}{J} + \frac{3}{4}\frac{k_3}{J} A^2 + \frac{8}{3}\frac{k_2}{\pi J} A} \tag{27}$$

According to $\theta(t)$=A cos(ωt) and (27), we can obtain the following approximate solution

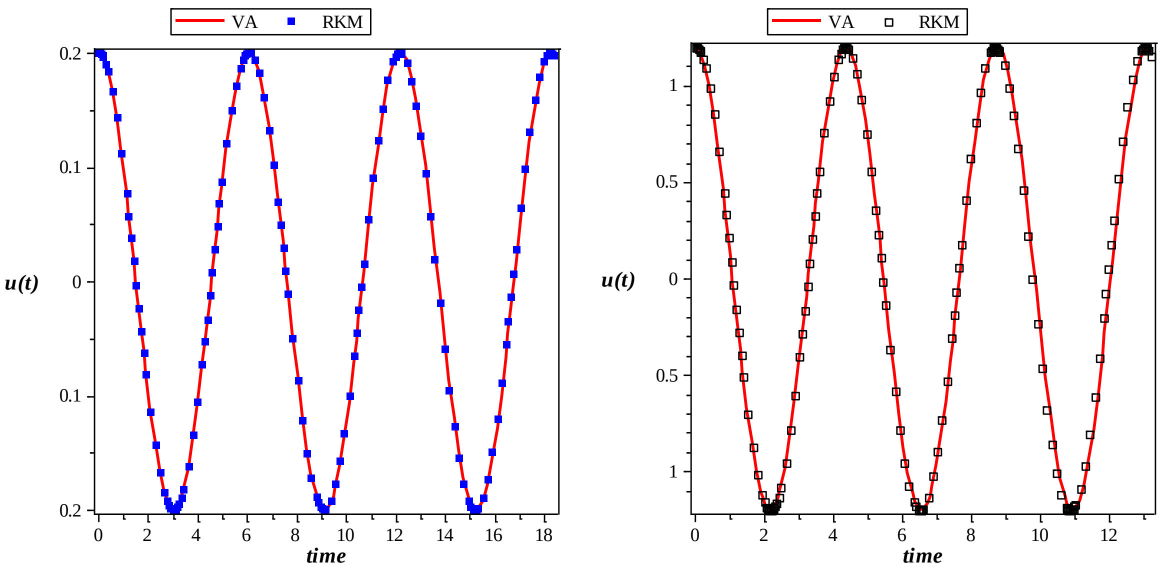
$$\theta(t) = A \cos\left(\sqrt{\frac{k_1}{J} + \frac{3}{4}\frac{k_3}{J} A^2 + \frac{8}{3}\frac{k_2}{\pi J} A}\; t \right) \tag{28}$$

5. Results and discussions

In this section, a detailed comparsion have been done on the results of variational approach and Runge-Kutta algorithm for two examples. Table 1 shows the effects of the two importanat parameters in the Helhomz equation (A and σ). Differant valeus are studeid and it has been shown the maximum relative error is less than two percrnt in high values of amplitudes. The motion of the syetem is shown in Fig. 2. It has a periodic motion for differet paramerets. Fig. 3 is the effect of asymmetric parameter (σ) and amplitude (A) on nonlinear frequency of the system. By

Table 1 Comparison of nonlinear frequency of approximate solution (VA) with numerical solution (RKM) corresponding to various parameters of system (example 1)

A	σ	VA	RKM	Error %
0.1	0.9	1.0076	1.0136	0.59045
0.2	0.8	1.0286	1.032157	0.34887
0.5	0.5	1.1428	1.152238	0.82721
0.8	0.1	1.2881	1.300277	0.94677
0.9	0.3	1.3103	1.326653	1.24435
1	0.7	1.3340	1.342864	0.6619
1.2	0.6	1.4337	1.448624	1.04238
1.5	0.8	1.6139	1.624878	0.68068
1.8	1	1.8520	1.867256	0.82237
2	0.9	1.9672	1.99263	1.29413



(a) (b)

Fig. 2 (Ex1) Comparison of analytical solution of $u(t)$ based on time with the RKM solution (a) A=0.2, σ=0.8 (b) A=0.2, σ=0.6

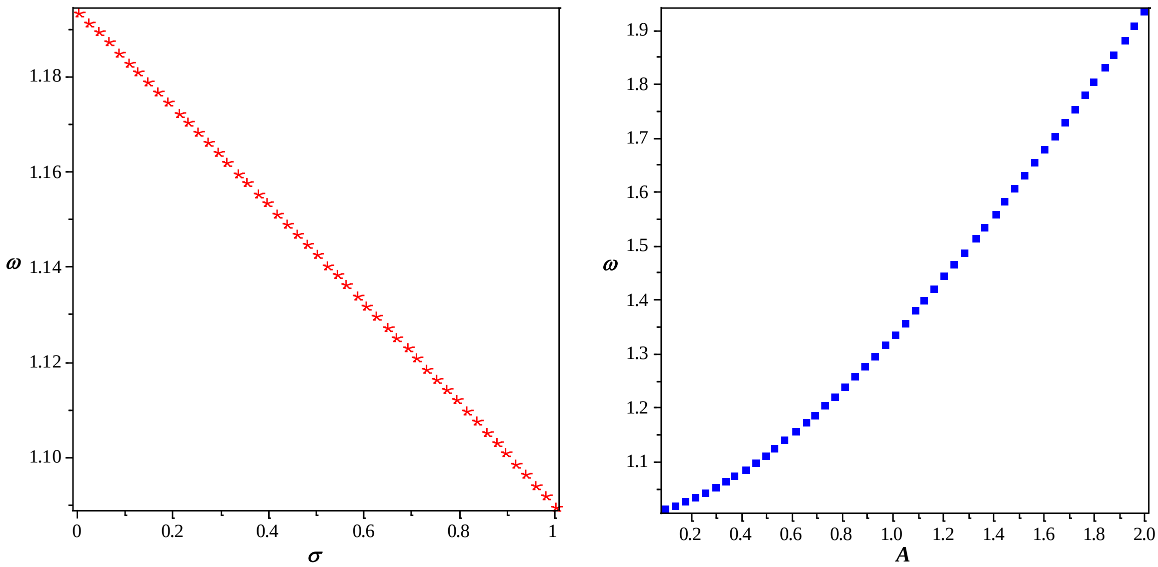
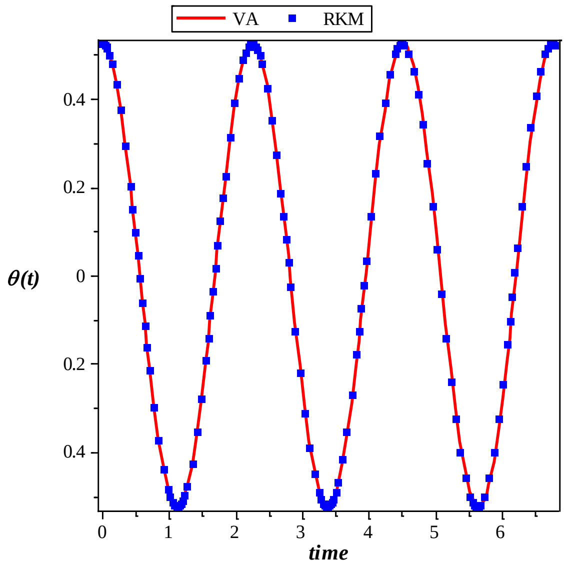


Fig. 3 (Ex1) Effect of asymmetric parameter (σ) and amplitude (A) on nonlinear frequency

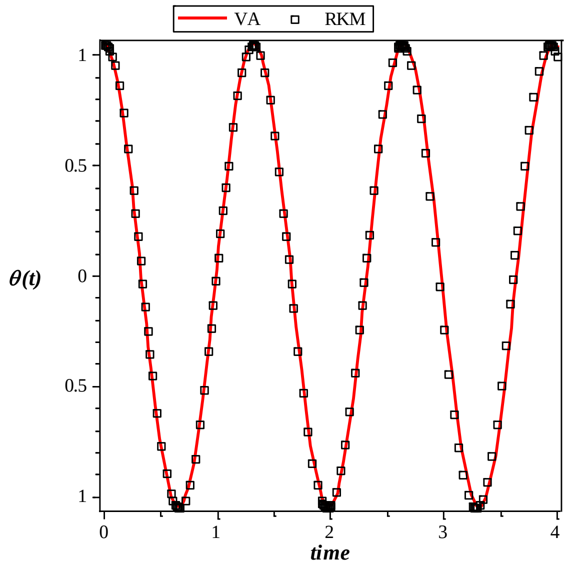
increasing the amplitude the frequency of the system will increase and by increasing the parameter (σ) the frequency will decrease. In Table 2, it shows the comparison of nonlinear frequency of approximate solution (VA) with numerical solution (RKM) corresponding to various parameters

Table 2 Comparison of nonlinear frequency of approximate solution (VA) with numerical solution (RKM) corresponding to various parameters of system (example 2)

A	J	k_1	k_2	k_3	VA	RKM	Error %
$\pi/12$	2	10	20	10	2.7348	2.7422	0.26889
$\pi/12$	1	20	10	10	4.7683	4.7751	0.14426
$\pi/6$	5	30	10	20	2.7769	2.7891	0.43864
$\pi/6$	2.5	50	30	20	5.1941	5.2210	0.51841
$\pi/4$	1	10	50	5	6.7562	6.7952	0.57714
$\pi/4$	3	20	30	50	4.5874	4.6325	0.98318
$\pi/3$	4	10	20	30	3.6212	3.6922	1.96229
$\pi/3$	2.5	30	20	10	4.7330	4.7936	1.28195
$\pi/2$	1.5	30	30	20	8.4463	8.5736	1.50723
$\pi/2$	1	20	20	5	7.4779	7.5561	1.04541



(a)



(b)

Fig. 4 (Ex2) Comparison of analytical solution of $\theta(t)$ based on time with the RKM solution (a) $A=\pi/6$, $J=5$, $k_1=30$, $k_2=10$, $k_3=20$ (b) $A=\pi/3$, $J=2.5$, $k_1=30$, $k_2=20$, $k_3=10$

of system (example 2). The maximum relative error for different valeus of stiffness is about two percent. It has been shown a good agrrement with numerical solutions. The motion of the syetm are shown for two different caes in Fig. 4. The direct effects of the k_1, k_2 and k_3 and the amplitude on the frequcny of the system is shown in Figs. 5 and 6. The results of the variational approach are in high accuaret .

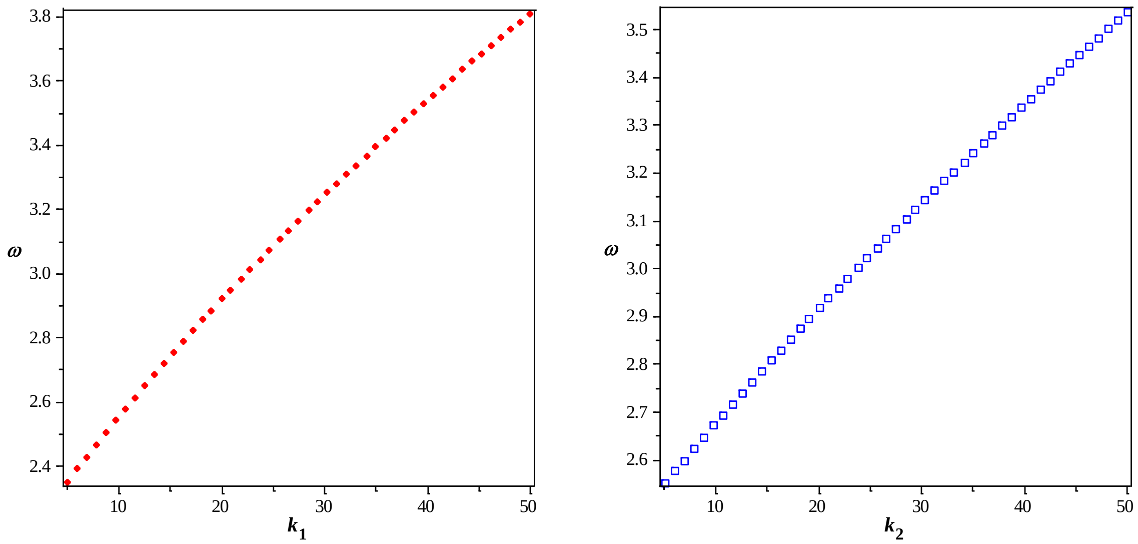


Fig. 5 (Ex2) Effect of spring (k_1) and (k_2) on nonlinear frequency

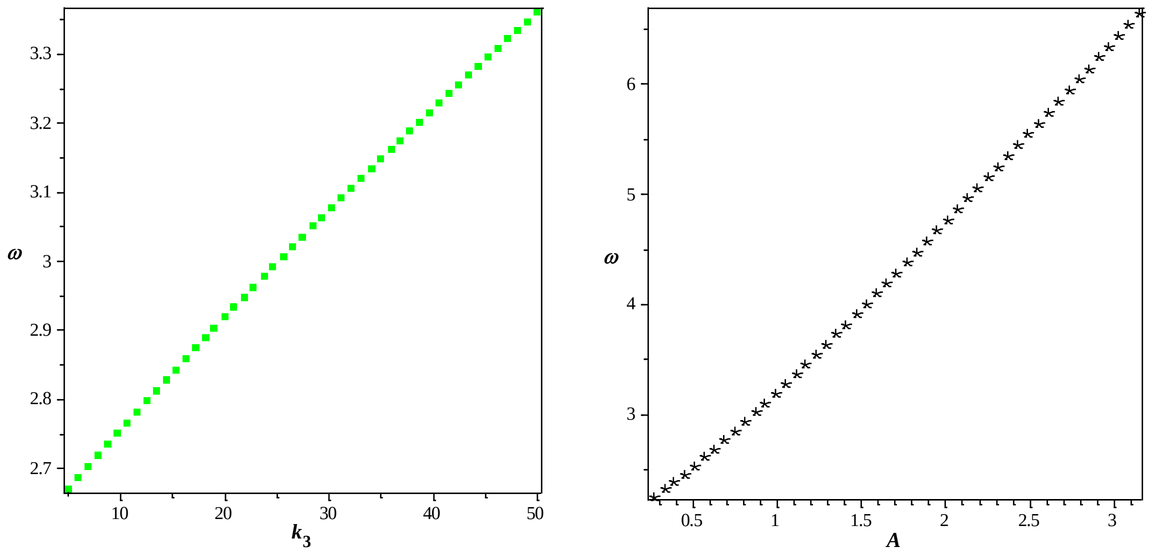


Fig. 6 (Ex2) Effect of spring (k_1, k_2 and k_3) and amplitude (A) on nonlinear frequency

6. Conclusions

In this paper, a new application of the variational approach was preseted. Two different high nonlinear problems were studied and solved via variational approach. The results of the presented approach were compared to numerical solution using Runge-Kutta algorithm. For the both cases the maximum error is less than two percent. Variational approach is an easy method to apply and can be easily extend to conservative nonlinear equations. The first itreation of the approach prepares a high accurate approximation.

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