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Examination of non-homogeneity and lamination scheme effects on deflections and stresses of laminated composite plates

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Abstract. In this study, a convenient formulation for the bending of laminated composite plates that hold non-homogeneous properties is examined. The constitutive equations of first order shear deformation plate theory are obtained using Hamilton Principle. The effect of non-homogeneity, lamination schemes and aspect ratio on the deflections and stresses is analysed. It is understood from the study that economical and optimum designs for laminated composite plates can be achieved by changing lamination scheme and by considering non-homogeneity response of composite plate.

Keywords: First Order Shear Deformation Theory (FSDT); laminated composite plate; non-homogeneous plates; non-homogeneity effect

1. Introduction

When historical development of the usage of material is investigated, usually three main stages stand out. First one is that the materials obtained from nature were used without changing their properties according to peoples usage aims. Second one, the materials were improved by utilizing some methods to facilitate human life. The last one, people have begun to produce composites using some different materials obtained from nature depending on the specific scientific and technological developments. This stage corresponds to the industrial revolution. After that time, usage of machinery has facilitated composite material production (Reddy 2004).

Composite materials have been used commonly after developments of material technology for last century. While the usage of composite materials was limited and only in some specific fields in the past, nowadays they are used in huge number of diverse industries such as air craft industry, defense industry and especially structural strengthening applications. The usage of composite materials have been expanded significantly due to their light-weight, high stiffness and high strength compared to classical structural materials (Patel 2014, Sadoune *et al*. 2014).

In literature there are numerous number of theories related to laminated composite analysis.

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Equivalent Single-Layer (ESL) Theory is one of the most important of those and it is divided into two subgroups as Classical Laminated Plate Theory (CLPT) and Shear Deformation Theory (SDT). The classical laminated plate theory is an extension of the classical plate theory based on Kirchhoff plate theory which neglects the shear deformation through the thickness of laminate.

Kinematic approaches for FSDT are an extension of the CLPT by including linear transverse shear deformation occured through plate thickness. However, the classical elasticity theory represents that transverse shear stress is distributed parabolically through the plate thickness. Because of that, FSDT requires a shear correction factor (K) to compensate the difference between this parabolic shear stress distribution and assumed constant stress distribution. The shear correction factor depends on geometrical parameters, boundary and loading conditions.

Materials are generally considered as homogeneous and isotropic in classical elasticity theory because of simplicity in calculation. On the contrary, material anisotropic properties should be included to be able to obtain more accurate and sensitive analysis results.

As is known, there are two material constants in an isotropic body, modulus of elasticity and poisson's ratio. However, number of elastic constants increase in an anisotropic body. In such a body should be analysed by utilizing anisotropic elasticity theory in order to determine stress and strain (Khoroshun *et al*. 1988, Kolpakov 1999, Lal 2007, Leknitskii and Fern 1963, Lomakin 1976).

The linear elasticity theory of non-homogeneous materials is based on Hooke Law, and material elastic properties differ functionally through the thickness of plate. This is more realistic in terms of mathematical and physical modeling. In this case, the physical characteristic of the material changes point to point continually and it becomes the continuous function of the point coordinates (Beena and Parvathy 2014, Delale and Erdogan 1983, Erdogan *et al*. 1991, Fares and Zenkour 1999, Hashin and Shtrikman 1962, He *et al*. 2013, He *et al*. 2012, Kant and Swaminathan 2002, Khoroshun *et al*. 1988, Kolpakov 1999, Lal 2007, Leknitskii and Fern 1963, Lomakin 1976, Schmitz and Horst 2014, Sofiyev and Kuruoglu 2014, Sofiyev *et al*. 2008, Stürzenbecher and Hofstetter 2011, Zenkour and Fares 1999).

Due to the fact that the analyses of non-homogeneous anisotropic structural element includes both in-plane and out-plane deformation effects, so the solution of non-homogeneous laminates becomes more complicated.

A new simple first order shear deformation theory almost the same as CLPT was derived in terms of parameters such as equation of motion and boudary conditions (Thai and Choi 2013a). Lots of theories acceptable for homogeneous laminated plates were modified into the behaviours of buckling and free vibration of non-homogeneous rectangle plates. The effects of non-homogeneity and thickness ratio on natural vibration and critical buckling load were determined. In this study, it is expressed that CLPT is not convenient method to investigate the structural behaviours of non-homogeneous plates (Fares and Zenkour 1999). Higher order shear deforrmation theory has been modified into a simple C^0 finite element formulation, a method which shear correction factor is unnecessary anymore has been developed and assumes that transverse shear deformation was changed parabolically through the laminate thickness (Goswami 2006). The non-homogeneity effects on free vibration of non-homogeneous isotropic circular plates of non-linear thickness were analysed. The non-homogeneity was related to variation of Young's modulus and density of plate material (Gupta *et al*. 2006). The non-homogeneity behaviours of non-homogeneous rectangle plates were pointed out by means of small parameter method, and the effects of non-homogeneity and material anisotropy on deflection and stress values were evaluated (Zenkour and Fares 1999). The free vibration analysis of variable thickness non-homogeneous orthotropic rectangle plates and Winkler type elastic foundation was considered according to CLPT and it was assumed that elasticity modulus and density of non-homogeneous plate varied as exponential function (Lal 2007). It was supposed that elasticity modulus differed through the thickness as a power, sigmoid and exponential functions, and poisson ratio was constant through the thickness. The plate was analysed under different loading conditions using CLPT (Beena and Parvathy 2014).

2. Mathematical model

Consider a fiber-reinforced rectangular laminated plate of aspect ratio *a/b* and total thickness *h* consisted of *N* orthotropic non-homogeneous layers with orientation angle θ_1, θ_2,..., θ_N. The coordinate system is assumed that the middle plane of the plate overlaps xy plane, and z axis is perpendicular to the middle plane. The top surface of the plate ($z=-h/2$) is subjected to a transverse distribution load $q(x,y)$.

In FSDT, transverse normals do not remain perpendicular to the midplane after deformation, the third principle of Kirchhoff Hypothesis, is neglected and this results in including additional transverse shear strains in the theory. A point in an element on *x*, *y* and *z* coordinates in undeformed condition is transferred to the $x+u$, $y+v$ and $z+w$ coordinates after deformation occurs. Displacement components for FSDT can be expressed as (Pagano 1970, Pagano and Hatfield 1972, Phan and Reddy 1985, Reddy 1984, Reddy 2004, Reissner 1975, Thai and Choi 2013a, 2013b, Yin *et al*. 2014, Zenkour and Fares 1999)

$$
\begin{aligned}
u(x,y,z) &= u_0(x,y) + z\phi_x \\
v(x,y,z) &= v_0(x,y) + z\phi_y \\
w(x,y,z) &= w(x,y)
\end{aligned}
\tag{1}
$$

where (u_0, v_0, w) are the displacement functions of the plate’s midplane, and ϕ_x and ϕ_y are the slopes in the *xz* and *yz* planes by reason of bending only.

The strains related to the displacements (1) can be presented as (Thai and Choi 2013a, 2013b, Yin *et al*. 2014, Zenkour and Fares 1999)

$$
\begin{aligned}
&\varepsilon_{xx} = \varepsilon_{xx}^{(0)} + z\varepsilon_{xx}^{(1)},\ \varepsilon_{yy} = \varepsilon_{yy}^{(0)} + z\varepsilon_{yy}^{(1)},\ \varepsilon_{xy} = \varepsilon_{xy}^{(0)} + z\varepsilon_{xy}^{(1)} \\
&\varepsilon_{yz} = \varepsilon_{yz}^{(0)},\ \varepsilon_{xz} = \varepsilon_{xz}^{(0)}
\end{aligned}
\tag{2}
$$

where

$$
\begin{aligned}
&\varepsilon_{xx}^{(0)} = \frac{\partial u_0}{\partial x},\ \varepsilon_{yy}^{(0)} = \frac{\partial v_0}{\partial y},\ \varepsilon_{xy}^{(0)} = \frac{\partial u_0}{\partial y} + \frac{\partial v_0}{\partial x},\ \varepsilon_{yz}^{(0)} = \phi_y + \frac{\partial w}{\partial y},\ \varepsilon_{xz}^{(0)} = \phi_x + \frac{\partial w}{\partial x},\ \varepsilon_{xx}^{(1)} = \frac{\partial \phi_x}{\partial x} \\
&\varepsilon_{yy}^{(1)} = \frac{\partial \phi_y}{\partial y},\ \varepsilon_{xy}^{(1)} = \frac{\partial \phi_x}{\partial y} + \frac{\partial \phi_y}{\partial x}
\end{aligned}
\tag{3}
$$

The material elastic properties of the non-homogeneous laminates can be expressed as

$$\begin{aligned}
&E_{11}^{(k)}(z)=E_{01}^{(k)}[1+\mu f^{(k)}(z)] \qquad E_{22}^{(k)}(z)=E_{02}^{(k)}[1+\mu f^{(k)}(z)]\\
&G_{12}^{(k)}(z)=G_{012}^{(k)}[1+\mu f^{(k)}(z)] \qquad G_{13}^{(k)}(z)=G_{013}^{(k)}[1+\mu f^{(k)}(z)]\\
&G_{23}^{(k)}(z)=G_{023}^{(k)}[1+\mu f^{(k)}(z)] \qquad \max\left|\mu f^{(k)}(z)\right|<1\\
&(k=1,2,\ldots,N)
\end{aligned} \tag{4}$$

where $E_{01}^{(k)}$, $E_{02}^{(k)}$, $G_{012}^{(k)}$, $G_{013}^{(k)}$ and $G_{023}^{(k)}$ are the material elastic properties of homogeneous orthotropic laminates. N is total laminate number, μ is a parameter that represents the variation of elasticity modulus through the plate thickness (non-homogeneous coefficient) and $f^{(k)}(z)$ is the continuous functions which express the variation of the elastic properties (Lal 2007, Schmitz and Horst 2014, Sofiyev and Kuruoglu 2014, Sofiyev *et al*. 2008).

In the first order shear deformation theory (FSDT), stress-strain expressions of kth non-homogeneous laminate can be given as (Aydogdu 2009, Phan and Reddy 1985, Reddy 1984, Reddy 2004, Reissner 1975)

$$\begin{Bmatrix}\sigma_{xx}\\ \sigma_{yy}\\ \sigma_{yz}\\ \sigma_{xz}\\ \sigma_{xy}\end{Bmatrix}^{(k)}=\begin{bmatrix}\overline{Q}_{11} & \overline{Q}_{12} & 0 & 0 & \overline{Q}_{16}\\ \overline{Q}_{12} & \overline{Q}_{22} & 0 & 0 & \overline{Q}_{26}\\ 0 & 0 & \overline{Q}_{44} & \overline{Q}_{45} & 0\\ 0 & 0 & \overline{Q}_{45} & \overline{Q}_{55} & 0\\ \overline{Q}_{16} & \overline{Q}_{26} & 0 & 0 & \overline{Q}_{66}\end{bmatrix}^{(k)}\begin{Bmatrix}\varepsilon_{xx}\\ \varepsilon_{yy}\\ \varepsilon_{yz}\\ \varepsilon_{xz}\\ \varepsilon_{xy}\end{Bmatrix}^{(k)} \tag{5}$$

where $\overline{Q}_{ij}$ are the transformed material properties expressed as (Fares 1999, Reddy 2004, Zenkour and Fares 1999)

$$\begin{aligned}
&\overline{Q}_{11}=Q_{11}\cos^4\theta+Q_{22}\sin^4\theta+2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta \quad ; \quad \overline{Q}_{44}=Q_{44}\cos^2\theta+Q_{55}\sin^2\theta\\
&\overline{Q}_{12}=(Q_{11}+Q_{22}-4Q_{66})\sin^2\theta\cos^2\theta+Q_{12}(\sin^4\theta+\cos^4\theta) \quad ; \quad \overline{Q}_{45}=(Q_{55}-Q_{44})\cos\theta\sin\theta\\
&\overline{Q}_{22}=Q_{11}\sin^4\theta+Q_{22}\cos^4\theta+2(Q_{12}+2Q_{66})\sin^2\theta\cos^2\theta \quad ; \quad \overline{Q}_{55}=Q_{44}\sin^2\theta+Q_{55}\cos^2\theta\\
&\overline{Q}_{16}=(Q_{11}-Q_{12}-2Q_{66})\sin\theta\cos^3\theta+(Q_{12}-Q_{22}+2Q_{66})\sin^3\theta\cos\theta\\
&\overline{Q}_{26}=(Q_{11}-Q_{12}-2Q_{66})\cos\theta\sin^3\theta+(Q_{12}-Q_{22}+2Q_{66})\cos^3\theta\sin\theta\\
&\overline{Q}_{66}=(Q_{11}+Q_{22}-2Q_{12}-2Q_{66})\sin^2\theta\cos^2\theta+Q_{66}(\sin^4\theta+\cos^4\theta)
\end{aligned} \tag{6}$$

in which θ is the angle between global x-axis and local x-axis of each laminate. The material properties of the laminate $Q_{ij}^{(k)}$ are given by

$$\begin{aligned}
&Q_{11}^{(k)}=\frac{E_{01}^{(k)}[1+\mu f^{(k)}(z)]}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}} \quad ; \quad Q_{66}^{(k)}=G_{012}^{(k)}[1+\mu f^{(k)}(z)]\\
&Q_{12}^{(k)}=\frac{\nu_{12}^{(k)}E_{02}^{(k)}[1+\mu f^{(k)}(z)]}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}} \quad ; \quad Q_{44}^{(k)}=G_{023}^{(k)}[1+\mu f^{(k)}(z)]\\
&Q_{22}^{(k)}=\frac{E_{02}^{(k)}[1+\mu f^{(k)}(z)]}{1-\nu_{12}^{(k)}\nu_{21}^{(k)}} \quad ; \quad Q_{55}^{(k)}=G_{013}^{(k)}[1+\mu f^{(k)}(z)]
\end{aligned} \tag{7}$$

where $E_{01}^{(k)}$ and $E_{02}^{(k)}$ are modulus of elasticity of homogeneous case in 1 and 2 material-principal directions, respectively; $G_{012}^{(k)}$, $G_{013}^{(k)}$ and $G_{023}^{(k)}$ are shear modulus of homogeneous case in the 1-2, 1-3 and 2-3 surfaces, respectively and v_{ij} are Poisson's ratio.

3. Equations of motion

Hamilton principle is utilized to obtain equations of motion for FSDT

$$0 = \int_T (\delta U + \delta V - \delta K) dt \tag{8}$$

where δU, δV, δK refer to virtual displacament energy, virtual work made by external load and virtual kinetic energy, respectively (Fares 1999, Pagano 1970, Phan and Reddy 1985, Reddy 2004, Zenkour and Fares 1999).

$$\begin{aligned}
\delta U &= \int_A \left\{ \int_{-h/2}^{h/2} \left[\sigma_{xx}^{(k)} (\delta\varepsilon_{xx}^{(0)} + z\delta\varepsilon_{xx}^{(1)}) + \sigma_{yy}^{(k)} (\delta\varepsilon_{yy}^{(0)} + z\delta\varepsilon_{yy}^{(1)}) + \sigma_{xy}^{(k)} (\delta\varepsilon_{xy}^{(0)} + z\delta\varepsilon_{xy}^{(1)}) \right.\right. \\
&\quad \left.\left. + \sigma_{xz}^{(k)} \delta\varepsilon_{xz}^{(0)} + \sigma_{yz}^{(k)} \delta\varepsilon_{yz}^{(0)} \right] dz \right\} dxdy \\
\delta V &= -\int_A q\delta w dA \\
\delta K &= \int_A \left\{ \int_{-h/2}^{h/2} \rho_0 \left[(\dot{u}_0 + z\dot{\phi}_x)(\delta\dot{u}_0 + z\delta\dot{\phi}_x) + (\dot{v}_0 + z\dot{\phi}_y)(\delta\dot{v}_0 + z\delta\dot{\phi}_y) \right.\right. \\
&\quad \left.\left. + \dot{w}\delta\dot{w} \right] dz \right\} dxdy
\end{aligned} \tag{9}$$

When the Eq. (9) are substituted in Eq. (8)

$$\begin{aligned}
0 = \int_0^T & \left\{ \int_A \left[N_{xx}\delta\varepsilon_{xx}^{(0)} + M_{xx}\delta\varepsilon_{xx}^{(1)} + N_{yy}\delta\varepsilon_{yy}^{(0)} + M_{yy}\delta\varepsilon_{yy}^{(1)} + N_{xy}\delta\varepsilon_{xy}^{(0)} + M_{xy}\delta\varepsilon_{xy}^{(1)} \right.\right. \\
& - q\delta w + Q_x\delta\varepsilon_{xz}^{(0)} + Q_y\delta\varepsilon_{yz}^{(0)} - I_0(\dot{u}_0\delta\dot{u}_0 + \dot{v}_0\delta\dot{v}_0 + \dot{w}\delta\dot{w}) \\
& \left.\left. - I_1(\dot{\phi}_x\delta\dot{u}_0 + \delta\dot{\phi}_x\dot{u}_0 + \dot{\phi}_y\delta\dot{v}_0 + \delta\dot{\phi}_y\dot{v}_0) - I_2(\dot{\phi}_x\delta\dot{\phi}_x + \dot{\phi}_y\delta\dot{\phi}_y) \right] dxdy \right\} dt
\end{aligned} \tag{10}$$

where N, M and Q are the stress resultants defined by

$$\begin{aligned}
(N_{xx}, N_{yy}, N_{xy}) &= \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_{xx}^{(k)}, \sigma_{yy}^{(k)}, \sigma_{xy}^{(k)}) dz \\
(M_{xx}, M_{yy}, M_{xy}) &= \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_{xx}^{(k)}, \sigma_{yy}^{(k)}, \sigma_{xy}^{(k)}) z dz \\
(Q_x, Q_y) &= \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} (\sigma_{xz}^{(k)}, \sigma_{yz}^{(k)}) dz
\end{aligned} \tag{11}$$

Substituting Eq. (5) into Eq. (9) and subsequent results into Eq. (11), the stress resultants are obtained as (Phan and Reddy 1985, Reddy 1984, Reddy 2004, Reissner 1975, Thai and Choi 2013a, 2013b, Yin *et al*. 2014)

$$\begin{Bmatrix} N_{xx} \\ N_{yy} \\ N_{xy} \end{Bmatrix} = \begin{bmatrix} A_{11} & A_{12} & A_{16} \\ A_{12} & A_{22} & A_{26} \\ A_{16} & A_{26} & A_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{xy}^{(1)} \end{Bmatrix}; \begin{Bmatrix} Q_y \\ Q_x \end{Bmatrix} = K \begin{bmatrix} A_{44} & A_{45} \\ A_{45} & A_{55} \end{bmatrix} \begin{Bmatrix} \phi_y + \dfrac{\partial w}{\partial y} \\ \phi_x + \dfrac{\partial w}{\partial x} \end{Bmatrix}$$
$$\begin{Bmatrix} M_{xx} \\ M_{yy} \\ M_{xy} \end{Bmatrix} = \begin{bmatrix} B_{11} & B_{12} & B_{16} \\ B_{12} & B_{22} & B_{26} \\ B_{16} & B_{26} & B_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(0)} \\ \varepsilon_{yy}^{(0)} \\ \varepsilon_{xy}^{(0)} \end{Bmatrix} + \begin{bmatrix} D_{11} & D_{12} & D_{16} \\ D_{12} & D_{22} & D_{26} \\ D_{16} & D_{26} & D_{66} \end{bmatrix} \begin{Bmatrix} \varepsilon_{xx}^{(1)} \\ \varepsilon_{yy}^{(1)} \\ \varepsilon_{xy}^{(1)} \end{Bmatrix} \tag{12a}$$

where K is the shear correction factor and A_{ij}, B_{ij}, D_{ij} are the extension (in-plane) stiffness, bending-extension coupling stiffness and bending stiffness of composite plate, respectively, defined by

$$(A_{ij}, B_{ij}, D_{ij}) = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \overline{Q}_{ij}^{(k)} (1, z, z^2) dz \quad ; \quad (I_0, I_1, I_2) = \sum_{k=1}^{N} \int_{z_{k-1}}^{z_k} \rho_0^{(k)} (1, z, z^2) dz \tag{12b}$$

4. Analitical solution

The determination of deflections and stresses are the fundamental process in the design of many constructional components. Non-homogeneous function and non-homogeneous coefficients are used to analyse the non-homogeneous laminated plate.

Boundary conditions of a simply supported rectangular plate are

$$\begin{aligned} &\mathrm{x} = 0, \mathrm{a} \quad u = w = \phi_x = 0\ , \\ &\mathrm{y} = 0, \mathrm{b} \quad v = w = \phi_y = 0 \end{aligned} \tag{13}$$

The considered transverse distribution load can be expanded in a double Fourier series

$$q(x, y) = \sum_{m=1,3,..}^{\infty} \sum_{n=1,3,..}^{\infty} Q_{mn} \sin(\frac{m\pi x}{a}) \sin(\frac{n\pi y}{b}) \tag{14}$$

and

$$Q_{mn} = \begin{cases} q_0 & \text{for sinusoidal load}, m = n = 1 \\ \dfrac{16 q_0}{mn\pi^2} & \text{for uniform load}, m, n = 1, 3, 5, \ldots \end{cases} \tag{15}$$

where q_0 represents the load at the center of the plate.

Navier approach is considered for the analitical solution of the problems. So, it can be assumed that

$$\begin{Bmatrix} w \\ \phi_x \\ \phi_y \end{Bmatrix} = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \begin{Bmatrix} W_{mn}\sin(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \\ X_{mn}\cos(\frac{m\pi x}{a})\sin(\frac{n\pi y}{b}) \\ Y_{mn}\sin(\frac{m\pi x}{a})\cos(\frac{n\pi y}{b}) \end{Bmatrix} \tag{16}$$

where W_{mn}, X_{mn} and Y_{mn} are the arbitrary coefficients. By substituting Eqs. (3), (12a) and (16) into the Eq. (9), the analytical solutions can be obtained from

$$[S]\{\Gamma_{mn}\} = \{F\}\ ;\quad \{\Gamma_{mn}\} = \begin{Bmatrix} W_{mn} \\ X_{mn} \\ Y_{mn} \end{Bmatrix};\quad \{F\} = \begin{Bmatrix} Q_{mn} \\ 0 \\ 0 \end{Bmatrix};\ [S] = \begin{bmatrix} S_{11} & S_{12} & S_{13} \\ S_{12} & S_{22} & S_{23} \\ S_{13} & S_{23} & S_{33} \end{bmatrix} \tag{17}$$

where

$$\begin{gathered} S_{11} = K(A_{55}\alpha^2 + A_{44}\beta^2),\ S_{12} = K\alpha A_{55},\ S_{13} = K\beta A_{44},\ S_{22} = D_{11}\alpha^2 + D_{66}\beta^2 + KA_{55} \\ S_{23} = (D_{12} + D_{66})\alpha\beta,\ S_{33} = D_{66}\alpha^2 + D_{22}\beta^2 + KA_{44}\ ,\ \alpha = \frac{m\pi x}{a}\ ,\ \beta = \frac{n\pi y}{b} \end{gathered} \tag{18}$$

where K is shear correction factor and it is determined as 5/6.

5. Numerical results and discussion

In this section, various numerical examples are analyzed and discussed to confirm the accuracy of the present study. For verification process, the obtained results are compared with the elasticity solutions of (Pagano 1970, Pagano and Hatfield 1972). For the sake of illustration, we discuss the improvement in the prediction of the present transverse displacements and stresses.

The numerical results of stresses and deflections are achieved for symmetric and antisymmetric cross-ply non-homogeneous rectangular laminated composite plates that all four edges simply supported. It is assumed that the thickness and the material properties are the same for all laminates

$$E_{01} = 25E_{02}\ ,\ G_{012} = G_{013} = 0.5E_{02}\ ,\ G_{023} = 0.2E_{02}\ ,\ \nu_{12} = 0.25 \tag{19}$$

The following nondimensionalizations are used to present results

$$\begin{gathered} \bar{w} = \frac{100h^3E_{02}}{q_0a^4} w(\frac{a}{2},\frac{b}{2})\quad ,\quad \bar{\sigma}_x = \frac{h^2}{q_0a^2}\sigma_x(\frac{a}{2},\frac{b}{2},\frac{h}{2}) \\ \bar{\sigma}_y = \frac{h^2}{q_0a^2}\sigma_y(\frac{a}{2},\frac{b}{2},\frac{h}{4})\quad ,\quad \bar{\sigma}_{xy} = \frac{h^2}{q_0a^2}\sigma_{xy}(0,0,-\frac{h}{2}) \\ \bar{\sigma}_{yz} = \frac{h}{q_0a}\sigma_{yz}(\frac{a}{2},0,0)\quad ,\quad \bar{\sigma}_{xz} = \frac{h}{q_0a}\sigma_{xz}(0,\frac{b}{2},0) \end{gathered} \tag{20}$$

It is understood from the results that, the deflections and stresses diminish with increasing the

Table 1 Non-dimensionalized deflections and stresses in four-layer cross-ply (0/90/90/0) square laminates under sinusoidal transverse loads (*Homogeneous)

a/h	Source	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$	$\bar{\sigma}_{xy}$
4	Pagano-Hatfield	1.9540	0.7200	0.6630	0.2920	0.2910	0.0467
	Zenkour-Fares	1.8937	0.6651	0.6322	0.2389	0.2064	0.0440
	Hom*. (present)	1.7101	0.4064	0.5410	0.3495	0.0785	0.0308
	μ=0.01 (present)	1.6917	0.4020	0.5361	0.3493	0.0784	0.0311
10	Pagano-Hatfield	0.7430	0.5590	0.4010	0.1960	0.3010	0.0275
	Zenkour-Fares	0.7147	0.5456	0.3888	0.1531	0.2640	0.0268
	Hom*. (present)	0.6632	0.4994	0.3647	0.4165	0.0517	0.0242
	μ=0.01 (present)	0.6560	0.4941	0.3614	0.4162	0.0516	0.0244
20	Pagano-Hatfield	0.5170	0.5430	0.3080	0.1560	0.3280	0.0230
	Zenkour-Fares	0.5060	0.5393	0.3043	0.1234	0.2825	0.0228
	Hom*. (present)	0.4916	0.5279	0.3108	0.4370	0.0435	0.0221
	μ=0.01 (present)	0.4863	0.5222	0.3079	0.4366	0.0434	0.0223
100	Pagano-Hatfield	0.4385	0.5390	0.2760	0.1410	0.3370	0.0216
	Zenkour-Fares	0.4343	0.5387	0.2708	0.1117	0.2897	0.0213
	Hom*. (present)	0.4341	0.5388	0.2901	0.4448	0.0403	0.0213
	μ=0.01 (present)	0.4295	0.5330	0.2875	0.4445	0.0403	0.0215

Table 2 Non-dimensionalized deflections and stresses in rectangular (a=3b), three-layer cross-ply (0/90/0) laminates under sinusoidal transverse loads (* Homogeneous)

a/h	Source	$\bar{w}$	$\bar{\sigma}_x$	$\bar{\sigma}_y$	$\bar{\sigma}_{xz}$	$\bar{\sigma}_{yz}$	$\bar{\sigma}_{xy}$
4	Pagano	2.8200	1.1000	0.1190	0.3870	0.0334	0.0281
	Zenkour-Fares	2.6411	1.0356	0.1028	0.0348	0.2724	0.0263
	Hom*. (present)	2.3631	0.6095	0.0054	0.4698	0.0123	0.0205
	μ=0.01 (present)	2.3378	0.6030	0.0054	0.4694	0.0123	0.0207
10	Pagano	0.9190	0.7250	0.0435	0.4200	0.0152	0.0123
	Zenkour-Fares	0.8622	0.6924	0.0398	0.0170	0.2859	0.0115
	Hom*. (present)	0.8035	0.6204	0.0354	0.4735	0.0064	0.0105
	μ=0.01 (present)	0.7949	0.6138	0.0351	0.4731	0.0064	0.0106
20	Pagano	0.6100	0.6500	0.0299	0.4340	0.0119	0.0093
	Zenkour-Fares	0.5937	0.6407	0.0289	0.0139	0.2880	0.0091
	Hom*. (present)	0.5789	0.6222	0.0403	0.4741	0.0054	0.0088
	μ=0.01 (present)	0.5727	0.6156	0.0400	0.4737	0.0054	0.0089
100	Pagano	0.5080	0.6240	0.0253	0.4390	0.0108	0.0083
	Zenkour-Fares	0.5077	0.6240	0.0253	0.0129	0.2886	0.0083
	Hom*. (present)	0.5069	0.6228	0.0419	0.4743	0.0051	0.0083
	μ=0.01 (present)	0.5015	0.6162	0.0416	0.4739	0.0051	0.0084

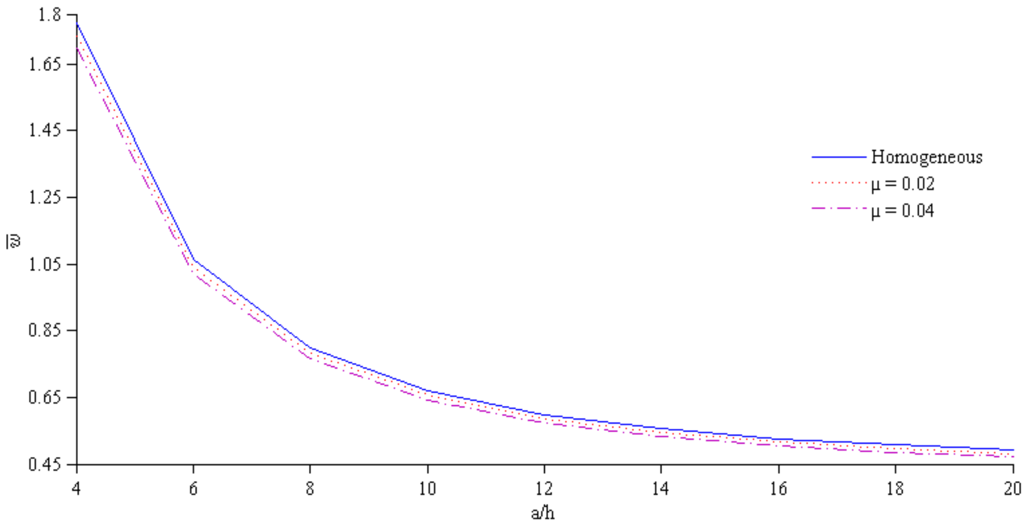

Fig. 1 Non-dimensional center deflection ($\overline{w}$) versus side-to-thickness ratio of a (0/90/0) square plate under sinusoidal load for various values of μ

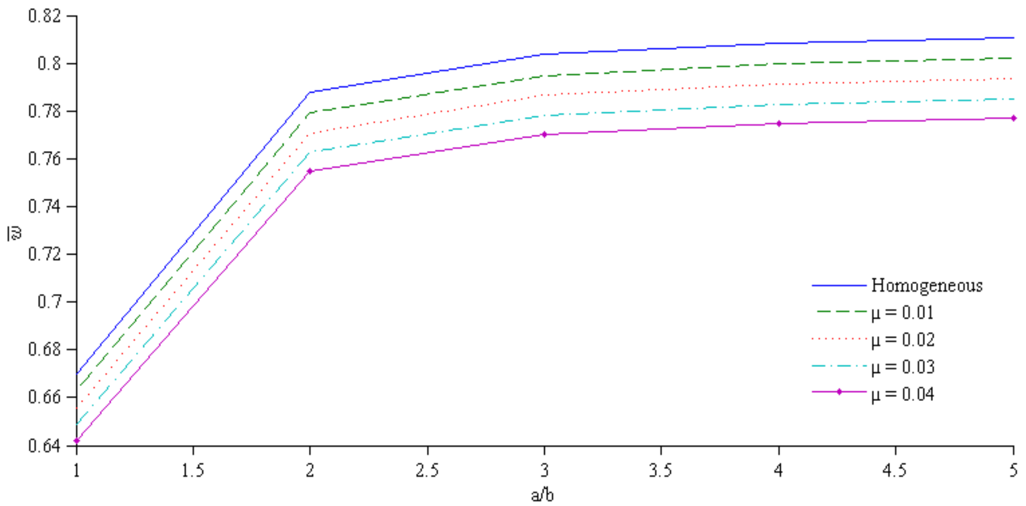

Fig. 2 Effect of the aspect ratio on the center deflection ($\overline{w}$) of a (0/90/0) plate under sinusoidal load for various values of μ (*a*/*h*=10)

non-homogeneity coefficient. These results imply that the laminated composite plates become more rigid due to inclusion of non-homogeneous elastic properties.

The results in Tables 1-2 demonstrate that the differences between sequences of elasticity solution and non-homogeneous effect on the composite plate are significant for all investigated side-to-thickness ratios (see Figs. 1, 3, 5 and 7). For *a/h*=4 (thick plates) these differences in deflection reach to about 13%. However, these differences become negligible when the thickness ratio *a/h* is greater than 4. Due to the decreasing thickness ratio, the FSDT provides acceptable results for the laminated composite plates.

The results in Tables 1-2 show that the discrepancy between sequences of higher order shear

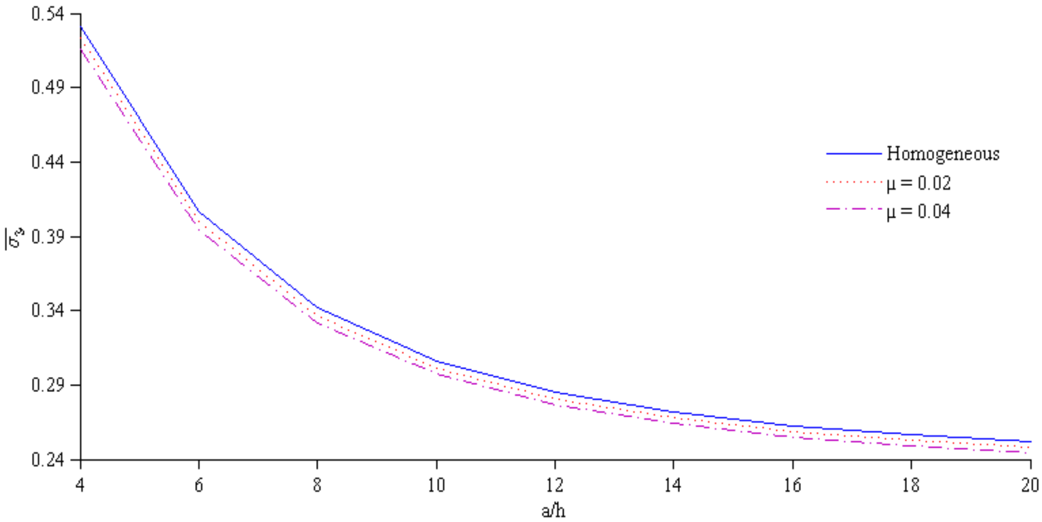

Fig. 3 Non-dimensional normal stress ($\bar{\sigma}_y$) versus side-to-thickness ratio of a (0/90/0) plate under sinusoidal load for various values of μ

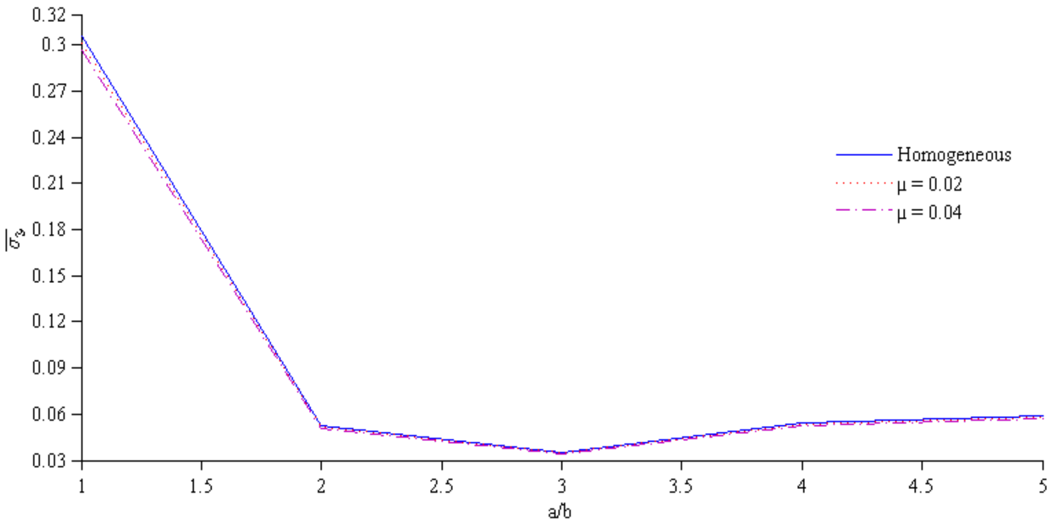

Fig. 4 Effect of the aspect ratio on the normal stress ($\bar{\sigma}_y$) of a (0/90/0) plate under sinusoidal load for various values of μ (*a/h*=10)

deformation theory solution (Zenkour and Fares 1999) and the results of first order shear deformation theory solution (present) is getting smaller with increasing of *a/h* ratio.

It is understood from Figs. 2, 4, 6 and 8 that the effect of non-homogeneity is substantial for rectangular plates due to the high aspect ratio, while it becomes less remarkable for symmetric and antisymmetric square plates.

Fig. 4 shows that variation of $\bar{\sigma}_y$ versus aspect ratio a/b. It is to be noted that the deviation of $\bar{\sigma}_y$ is minimum for aspect ratio of 3. Also, deviation of $\bar{\sigma}_{xy}$ is maximum for square plate (Fig. 6).

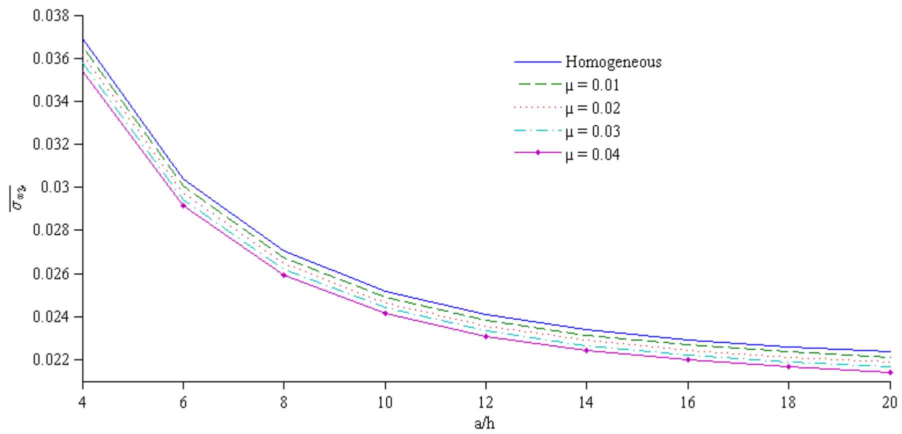

Fig. 5 Non-dimensional tangential stress ($\overline{\sigma}_{xy}$) versus side-to-thickness ratio of a (0/90/0) square plate under sinusoidal load for various values of μ

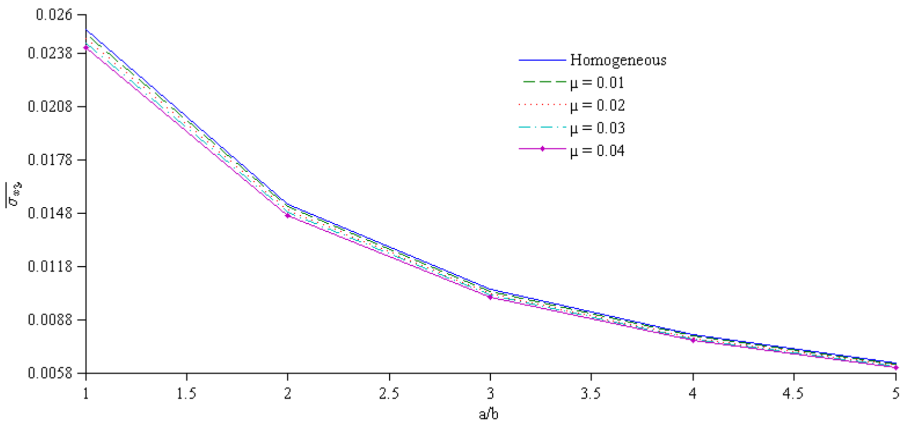

Fig. 6 Effect of the aspect ratio on the tangential stress ($\overline{\sigma}_{xy}$) of a (0/90/0) plate under sinusoidal load for various values of μ (*a/h*=10)

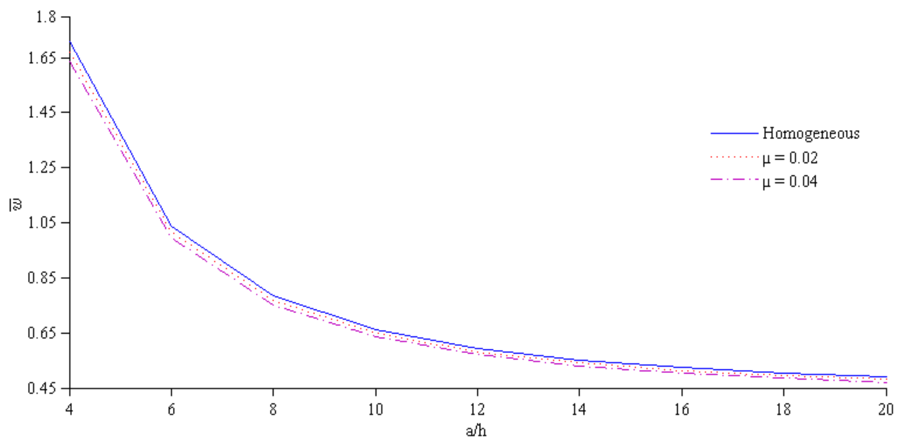

Fig. 7 Non-dimensional center deflection ($\overline{w}$) versus side-to-thickness ratio of a (0/90/90/0) square plate under sinusoidal load for various values of μ

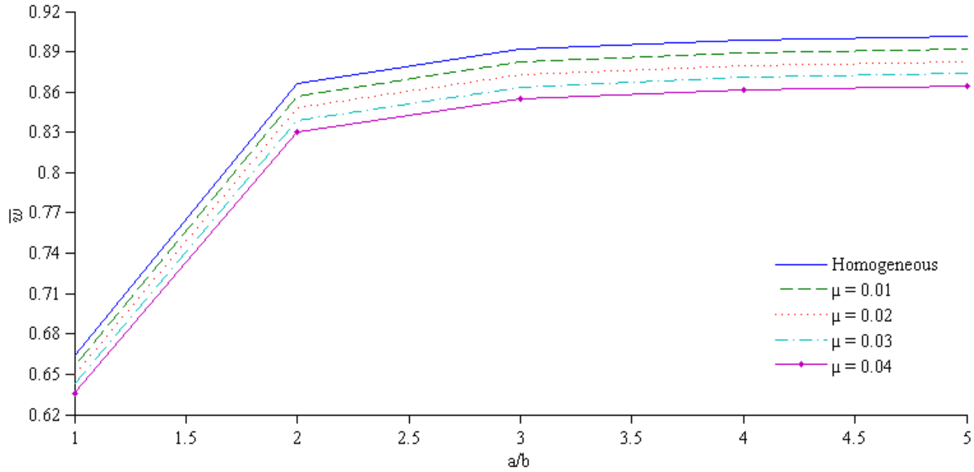

Fig. 8 Effect of the aspect ratio on the center deflection ($\overline{w}$) of a (0/90/90/0) plate under sinusoidal load for various values of μ (a/h=10)

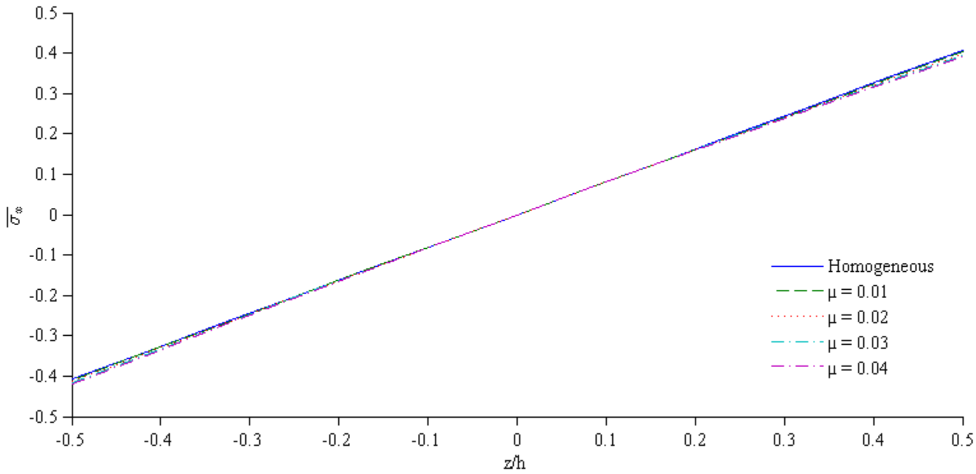

Fig. 9 Variation of non-dimensional normal stress ($\overline{\sigma}_x$) through the laminate thickness of a (0/90/90/0) square plate under sinusoidal load for various values of μ (a/h=4)

Fig. 9 demonsrates that variation of $\overline{\sigma}_x$ through the thickness of symmetric cross-ply square plate for a/h=4. It is seen in Fig. 9 that $\overline{\sigma}_x$ is more pronounced for the outer layers of the plate.

6. Conclusions

An appropriate first order shear deformation theory for the non-homogeneous laminated composite plates is investigated and evaluated by comparing the results with the other previous studies available in literature. The obtained results imply that effect of non-homogeneity on deflections and stresses are not negligible.

When the deflection and stress results are examined in detail, it can be observed that 0/90/90/0 lamination scheme is more effective than 0/90/0 lamination schemes in economic aspect of view for composite plate. The most significant point for the design of a laminated composite plate is that more appropriate and economical approach can be obtained by changing the orientation angle or increasing the number of layer.

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