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Analyse of the behavior of functionally graded beams based on neutral surface position

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Abstract. In this paper, a simple n-order refined theory based on neutral surface position is developed for bending and frees vibration analyses of functionally graded beams. The present theory is variationally consistent, uses the n-order polynomial term to represent the displacement field, does not require shear correction factor, and gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness satisfying shear stress free surface conditions. The governing equations are derived by employing the Hamilton's principle and the physical neutral surface concept. The accuracy of the present solutions is verified by comparing the obtained results with available published ones.

Keywords: mechanical properties; vibration; functionally graded; deformation; modeling

1. Introduction

Composite materials have been successfully used in aircraft and other engineering applications for many years because of their excellent strength to weight and stiffness to weight ratios. Recently, advanced composite materials known as functionally graded material have attracted much attention in many engineering applications due to their advantages of being able to resist high temperature gradient while maintaining structural integrity (Koizumi 1997). The functionally graded materials (FGMs) are microscopically inhomogeneous, in which the mechanical properties vary smoothly and continuously from one surface to the other. They are usually made from a mixture of ceramics and metals to attain the significant requirement of material properties.

Due to the increased relevance of the FGMs structural components in the design of engineering structures, many studies have been reported on the static, and vibration analyses of functionally graded (FG) beams. Sankar (2001) investigated an elasticity solution for bending of functionally graded beams (FG beams) based on Euler-Bernoulli beam theory. Li (2008) investigated static bending and transverse vibration of FGM Timoshenko beams, in which by introducing a new function, the governing equations for bending and vibration of FGM beams were decoupled and the deflection, rotational angle and the resultant force and moment were expressed only in the terms of this new function. Sallai *et al*. (2009) investigated the static responses of a sigmoid FG

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thick beam by using different beam theories. Benatta *et al*. (2009) presented a mathematical solution for bending of short hybrid composite beams with variable fibers spacing. Şimşek (2010a) studied the free vibration analysis of an FG beam using different higher order beam theories. In a recent study, Şimşek (2010b) has studied the dynamic deflections and the stresses of an FG simply-supported beam subjected to a moving mass by using Euler-Bernoulli, Timoshenko and the parabolic shear deformation beam theory. Bachir Bouiadjra *et al*. (2012) used a four-variable refined plate theory for the buckling response of FG plates under thermal loads. Bourada *et al*. (2012) developed a new four-variable refined plate theory for thermal buckling of FG sandwich plates. Fekrar *et al*. (2012) analyzed the buckling response of FG hybrid composite plates using a new four variable refined plate theory. Bouremana *et al*. (2013) proposed a novel first shear deformation beam theory based on neutral surface position for FG beams.

Simsek (2013) presented an analytical solutions for bending and buckling of functionally graded nanobeams based on the nonlocal Timoshenko beam theory. Thai *et al.* (2012) has studied the bending and free vibration of functionally graded beams using various higher-order shear deformation beam theories. Ghiasian (2013) investigated the dynamic buckling of suddenly heated or compressed FGM beams resting on nonlinear elastic foundation. Li *et al.* (2013) studied the bending solutions of FGM Timoshenko beams from those of the homogenous Euler-Bernoulli beams. Allahverdizadeh (2013) investigated the nonlinear vibration analysis of FGER sandwich beams. Hadji (2014) studied the static and free vibration of FGM beam using a higher order shear deformation theory. Hebali *et al*. (2014) analyzed the bending and free vibration behaviour of FG plates using a novel quasi-3D hyperbolic shear deformation theory.

Since, the material properties of functionally graded beam vary through the thickness direction, the neutral surface of such beam may not coincide with its geometric middle surface. Therefore, stretching and bending deformations of FG beam are coupled. Some researchers (Zhang and Zhou 2008, Saidi and Jomehzadeh 2009) have shown that there is no stretching-bending coupling in constitutive equations if the reference surface is properly selected. Recently, Ould Larbi Latifa *et al.* (2012) investigated an efficient shear deformation beam theory based on neutral surface position for bending and frees vibration analysis of functionally graded beams.

In this paper, a *n*-order refined theory is used to analyze the static and vibration characteristics of functionally graded beams. The present *n*-order refined theory is based on assumption that the in-plane and transverse displacements consist of bending and shear components, in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. The most interesting feature of this theory is that it accounts for a parabolic variation of the transverse shear strains across the thickness and satisfies the zero traction boundary conditions on the top and bottom surfaces of the beam without using shear correction factors. The material properties of FG beam are assumed to vary according to a power law distribution of the volume fraction of the constituents. To simplify the governing equations for the FG beams, the coordinate system is located at the physical neutral surface of the beam. This is due to the fact that the stretching-bending coupling in the constitutive equations of an FG beam does not exist when the physical neutral surface is considered as a coordinate system (Yahoobi and Feraidoon 2010, Ould Larbi *et al*. 2013, Bouremana *et al*. 2013, Klouche Djedid *et al*. 2014). Thus, the present n-order four variable refined theory based on the exact position of neutral surface together with Hamilton principle are employed to extract the motion equations of the FG beams. Analytical solutions are obtained for simply supported beam, and its accuracy is verified by comparing the obtained results with those reported in the literature.

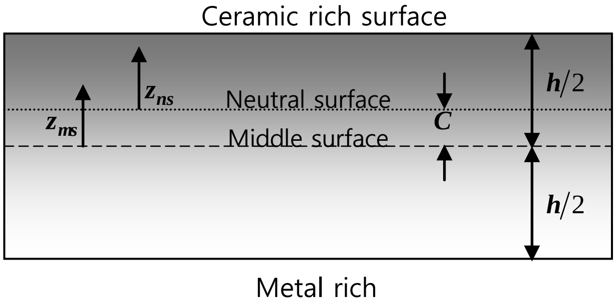

Fig. 1 The position of middle surface and neutral surface for a functionally graded beam.

2. Mathematical formulation

Consider a functionally graded beam with length L and rectangular cross section $b\times h$, with b being the width and h being the height. Since in functionally graded beams the condition of mid-plane symmetry does not exist, the stretching and bending equations are coupled. But, if the origin of the coordinate system is suitably selected in the thickness direction of the FG beam so as to be the neutral surface, the analysis of the FG beams can easily be treated with the homogenous isotropic beam theories, because the stretching and bending equations of the beam are not coupled. In order to determine the position of neutral surface of FG beams, two different datum planes are considered for the measurement of z, namely, z_{ms} and z_{ns} measured from the middle surface and the neutral surface of the beam, respectively, as shown in Fig. 1.

Following the power law distribution in the thickness direction, the volume fractions of ceramic constituent V_C, and metal constituent V_M, may be written in the form

$$V_C = \left(\frac{z_{ms}}{h} + \frac{1}{2}\right)^k = \left(\frac{z_{ns} + C}{h} + \frac{1}{2}\right)^k \tag{1}$$

Material non-homogeneous properties of a functionally graded material beam may be obtained by means of the Voigt rule of mixture (Suresh and Mortensen, 1998). Thus, using Eq. (1), the material non-homogeneous properties of FG beam P, as a function of thickness coordinate, become

$$P(z) = P_M + P_{CM}\left(\frac{z_{ns} + C}{h} + \frac{1}{2}\right)^k, \quad P_{CM} = P_C - P_M \tag{2}$$

where P_M and P_C are the corresponding properties of the metal and ceramic, respectively, and k is the material parameter which takes the value greater or equal to zero. Also, the parameter C is the distance of neutral surface from the middle surface. In the present work, we assume that the elasticity modules E is described by Eq. (2), while Poisson's ratio ν, is considered to be constant across the thickness. The position of the neutral surface of the FG beam is determined to satisfy the first moment with respect to Young's modulus being zero as follows (Ould Larbi *et al*. 2013, Bouremana *et al*. 2013)

$$\int_{-h/2}^{h/2} E(z_{ms})(z_{ms} - C)dz_{ms} = 0 \tag{3}$$

Consequently, the position of neutral surface can be obtained as

$$C = \frac{\int_{-h/2}^{h/2} E(z_{ms}) z_{ms} dz_{ms}}{\int_{-h/2}^{h/2} E(z_{ms}) dz_{ms}} \tag{4}$$

It is clear that the parameter *C* is zero for homogeneous isotropic beams, as expected.

2.1 Basic assumptions

The assumptions of the present theory are as follows:

- The origin of the Cartesian coordinate system is taken at the neutral surface of the FG beam.

-The displacements are small in comparison with the height of the beam and, therefore, strains involved are infinitesimal.

-The transverse displacement w includes two components of bending w_b, and shear w_s. These components are functions of coordinates x, y only.

$$w(x, z_{ns}, t) = w_b(x, t) + w_s(x, t) \tag{5}$$

- The transverse normal stress σ_z is negligible in comparison with in-plane stresses σ_x. The axial displacement u in x-direction, consists of extension, bending, and shear components.

$$u = u_0 + u_b + u_s\,, \tag{6}$$

- The bending component u_b is assumed to be similar to the displacements given by the classical beam theory. Therefore, the expression for u_b can be given as

$$u_b = -z_{ns} \frac{\partial w_b}{\partial x} \tag{7}$$

- The shear component u_s gives rise, in conjunction with w_s, to the hyperbolic variation of shear strain γ_{xz} and hence to shear stress τ_{xz} through the thickness of the beam in such a way that shear stress τ_{xz} is zero at the top and bottom faces of the beam. Consequently, the expression for u_s can be given as

$$u_s = -f(z_{ns}) \frac{\partial w_s}{\partial x} \tag{8}$$

where

$$f(z_{ns}) = \frac{1}{n}\left(\frac{2}{h}\right)^{n-1} (z_{ns} + C)^n \tag{9}$$

2.2 Kinematics and constitutive equations

Based on the assumptions made in the preceding section, the displacement field can be obtained using Eqs. (5)-(9) as

$$u(x, z_{ns}, t) = u_0(x,t) - z_{ns}\frac{\partial w_b}{\partial x} - f(z_{ns})\frac{\partial w_s}{\partial x} \tag{10a}$$

$$w(x, z_{ns}, t) = w_b(x,t) + w_s(x,t) \tag{10b}$$

The strains associated with the displacements in Eq. (10) are

$$\varepsilon_x = \varepsilon_x^0 + z_{ns}\, k_x^b + f(z_{ns})\, k_x^s \tag{11a}$$

$$\gamma_{xz} = g(z_{ns})\, \gamma_{xz}^s \tag{11b}$$

Where

$$\boldsymbol{\varepsilon}_x^0 = \frac{\partial \boldsymbol{u}_0}{\partial \boldsymbol{x}}, \quad k_x^b = -\frac{\partial^2 w_b}{\partial x^2}, \quad k_x^s = -\frac{\partial^2 w_s}{\partial x^2}, \quad \gamma_{xz}^s = \frac{\partial w_s}{\partial x}, \tag{11c}$$

$$g(z_{ns}) = 1 - \frac{df(z_{ns})}{dz_{ns}} = 1 - \frac{(z_{ns}+C)^n}{(z_{ns}+C)}\left(\frac{2}{h}\right)^{n-1} \tag{11d}$$

By assuming that the material of FG beam obeys Hooke’s law, the stresses in the beam become

$$\sigma_x = Q_{11}(z_{ns})\,\varepsilon_x \quad \text{and} \quad \tau_{xz} = Q_{55}(z_{ns})\,\gamma_{xz} \tag{12a}$$

Where

$$Q_{11}(z_{ns}) = E(z_{ns}) \quad \text{and} \quad Q_{55}(z_{ns}) = \frac{E(z_{ns})}{2(1+\nu)} \tag{12b}$$

2.3 Governing equations

Hamilton’s principle is used herein to derive equations of motion. The principle can be stated in an analytical form as

$$0 = \int_0^T (\delta U + \delta V - \delta K)\,dt \tag{13}$$

where δU is the virtual variation of the strain energy; δV is the virtual variation of the potential energy; and δK is the virtual variation of the kinetic energy. The variation of the strain energy of the beam can be stated as

$$\begin{aligned}\delta U &= \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (\sigma_x \delta\varepsilon_x + \tau_{xz}\delta\gamma_{xz})\,dz_{ns}\,dx \\ &= \int_0^L \left(N\frac{d\delta u_0}{dx} - M_b\frac{d^2\delta w_b}{dx^2} - M_s\frac{d^2\delta w_s}{dx^2} + Q\frac{d\delta w_s}{dx}\right)dx\end{aligned} \tag{14}$$

where N, M_b, M_s and Q are the stress resultants defined as

$$(N, M_b, M_s) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (1, z_{ns}, f)\,\sigma_x\,dz_{ns} \quad \text{and} \quad Q = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} g\tau_{xz}\,dz_{ns} \tag{15}$$

The variation of the potential energy by the applied transverse load q can be written as

$$\delta V = -\int_0^L q(\delta w_b + \delta w_s)dx \tag{16}$$

The variation of the kinetic energy can be expressed as

$$\begin{aligned}
\delta K &= \int_0^L \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} \rho(z_{ns})[\dot{u}\delta\dot{u} + \dot{w}\delta\dot{w}]dz_{ns}dx \\
&= \int_0^L \Bigg\{ I_0[\dot{u}_0\delta\dot{u}_0 + (\dot{w}_b + \dot{w}_s)(\delta\dot{w}_b + \delta\dot{w}_s)] - I_1\left(\dot{u}_0\frac{d\delta\dot{w}_b}{dx} + \frac{d\dot{w}_b}{dx}\delta\dot{u}_0\right) \\
&\quad + I_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_b}{dx}\right) - J_1\left(\dot{u}_0\frac{d\delta\dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx}\delta\dot{u}_0\right) + K_2\left(\frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_s}{dx}\right) \\
&\quad + J_2\left(\frac{d\dot{w}_b}{dx}\frac{d\delta\dot{w}_s}{dx} + \frac{d\dot{w}_s}{dx}\frac{d\delta\dot{w}_b}{dx}\right)\Bigg\}dx
\end{aligned} \tag{17}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; $\rho(z_{ns})$ is the mass density; and (I_0, I_1, J_1, I_2, J_2, K_2) are the mass inertias defined as

$$(I_0, I_1, J_1, I_2, J_2, K_2) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} (1, z_{ns}, f, z_{ns}^2, z_{ns}f, f^2)\rho(z_{ns})dz_{ns} \tag{18}$$

Substituting the expressions for δU, δV, and δK from Eqs. (14), (16), and (17) into Eq. (13) and integrating by parts versus both space and time variables, and collecting the coefficients of δu_0, δw_b, and δw_s, the following equations of motion of the functionally graded beam are obtained

$$\delta u_0: \ \frac{dN}{dx} = I_0\ddot{u}_0 - I_1\frac{d\ddot{w}_b}{dx} - J_1\frac{d\ddot{w}_s}{dx} \tag{19a}$$

$$\delta w_b: \ \frac{d^2M_b}{dx^2} + q = I_0(\ddot{w}_b + \ddot{w}_s) + I_1\frac{d\ddot{u}_0}{dx} - I_2\frac{d^2\ddot{w}_b}{dx^2} - J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{19b}$$

$$\delta w_s: \ \frac{d^2M_s}{dx^2} + \frac{dQ}{dx} + q = I_0(\ddot{w}_b + \ddot{w}_s) + J_1\frac{d\ddot{u}_0}{dx} - J_2\frac{d^2\ddot{w}_b}{dx^2} - K_2\frac{d^2\ddot{w}_s}{dx^2} \tag{19c}$$

Eq. (19) can be expressed in terms of displacements (u_0, w_b, w_s) by using Eqs. (10), (11), (12) and (15) as follows

$$A_{11}\frac{\partial^2 u_0}{\partial x^2} - B_{11}^s\frac{\partial^3 w_s}{\partial x^3} = I_0\ddot{u}_0 - I_1\frac{d\ddot{w}_b}{dx} - J_1\frac{d\ddot{w}_s}{dx} \tag{20a}$$

$$-D_{11}\frac{\partial^4 w_b}{\partial x^4} - D_{11}^s\frac{\partial^4 w_s}{\partial x^4} + q = I_0(\ddot{w}_b + \ddot{w}_s) + I_1\frac{d\ddot{u}_0}{dx} - I_2\frac{d^2\ddot{w}_b}{dx^2} - J_2\frac{d^2\ddot{w}_s}{dx^2} \tag{20b}$$

$$\begin{aligned} &B_{11}^s\frac{\partial^3 u_0}{\partial x^3} - D_{11}^s\frac{\partial^4 w_b}{\partial x^4} - H_{11}^s\frac{\partial^4 w_s}{\partial x^4} + A_{55}^s\frac{\partial^2 w_s}{\partial x^2} + q \\ &= I_0(\ddot{w}_b + \ddot{w}_s) + J_1\frac{d\ddot{u}_0}{dx} - J_2\frac{d^2\ddot{w}_b}{dx^2} - K_2\frac{d^2\ddot{w}_s}{dx^2} \end{aligned} \tag{20c}$$

where A_{11}, D_{11}, etc., are the beam stiffness, defined by

$$\left(A_{11}, D_{11}, B_{11}^s, D_{11}^s, H_{11}^s\right) = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{11}\left(1, z^2, f(z_{ns}), z_{ns}\, f(z_{ns}), f^2(z_{ns})\right)dz_{ns} \tag{21a}$$

and

$$A_{55}^s = \int_{-\frac{h}{2}-C}^{\frac{h}{2}-C} Q_{55}\left[g(z_{ns})\right]^2 dz_{ns}, \tag{21b}$$

3. Analytical solution

The equations of motion admit the Navier solutions for simply supported beams. The variables u_0, w_b, w_s can be written by assuming the following variations

$$\begin{Bmatrix} u_0 \\ w_b \\ w_s \end{Bmatrix} = \sum_{m=1}^{\infty} \begin{Bmatrix} U_m \cos(\lambda x)\, e^{i\omega t} \\ W_{bm} \sin(\lambda x)\, e^{i\omega t} \\ W_{sm} \sin(\lambda x)\, e^{i\omega t} \end{Bmatrix} \tag{22}$$

where U_m, W_{bm}, and W_{sm} are arbitrary parameters to be determined, ω is the eigenfrequency associated with m th eigenmode, and $\lambda = m\pi/L$. The transverse load q is also expanded in Fourier series as

$$q(x) = \sum_{m=1}^{\infty} Q_m \sin(\lambda x) \tag{23}$$

where Q_m is the load amplitude calculated from

$$Q_m = \frac{2}{L}\int_0^L q(x)\sin(\lambda\, x)dx \tag{24}$$

The coefficients Q_m are given below for some typical loads. For the case of a sinusoidally distributed load, we have

$$m = 1 \ \text{ and } \ Q_1 = q_0 \tag{25a}$$

and for the case of uniform distributed load, we have

$$Q_m = \frac{4q_0}{m\pi}, \ \ (m = 1,3,5....) \tag{25b}$$

Substituting the expansions of u_0, w_b, w_s, and q from Eqs. (22) and (23) into the equations of motion Eq. (20), the analytical solutions can be obtained from the following equations

$$\left(\begin{bmatrix} a_{11} & 0 & a_{13} \\ 0 & a_{22} & a_{23} \\ a_{13} & a_{23} & a_{33} \end{bmatrix} - \omega^2 \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{12} & m_{22} & m_{23} \\ m_{13} & m_{23} & m_{33} \end{bmatrix}\right) \begin{Bmatrix} U_m \\ W_{bm} \\ W_{sm} \end{Bmatrix} = \begin{Bmatrix} 0 \\ Q_m \\ Q_m \end{Bmatrix} \tag{26}$$

where

$$a_{11} = A_{11}\lambda^2, \quad a_{13} = -B_{11}^s\lambda^3, \quad a_{22} = D_{11}\lambda^4, \quad a_{23} = D_{11}^s\lambda^4, \quad a_{33} = H_{11}^s\lambda^4 + A_{55}^s\lambda^2 \tag{27a}$$

$$m_{11} = I_0, \quad m_{12} = -I_1\lambda, \quad m_{13} = -J_1\lambda, \quad m_{22} = I_0 + I_2\lambda^2, \quad m_{23} = I_0 + J_2\lambda^2,$$

$$m_{33} = I_0 + K_2\lambda^2 \tag{27b}$$

4. Numerical results

In this section, various numerical examples are presented and discussed to verify the accuracy of the present theory in predicting the bending and free vibration responses of simply supported FG beams. The FG beam is taken to be made of aluminum and alumina with the following material properties:

Ceramic (P_C : Alumina, Al2O3): $E_c = 380$ GPa; $\nu = 0.3$; $\rho_c = 3960$ kg/m3.

Metal (P_M : Aluminium, Al): $E_m = 70$ GPa; $\nu = 0.3$; $\rho_m = 2707$ kg/m3.

4.1 Bending analysis

For bending analysis, a beam subjected to a uniform load is considered. For convenience, the following dimensionless forms are used

$$\bar{w} = 100\frac{E_m h^3}{q_0 L^4} w\left(\frac{L}{2}\right), \quad \bar{u} = 100\frac{E_m h^3}{q_0 L^4} u\left(0, -\frac{h}{2} - C\right), \quad \bar{\sigma}_x = \frac{h}{q_0 L}\sigma_x\left(\frac{L}{2}, \frac{h}{2} - C\right), \quad \bar{\tau}_{xz} = \frac{h}{q_0 L}\tau_{xz}(0, -C),$$

Table 1 Nondimensional deflections and stresses of FG beams under uniform load *L/h*=5

k	Method	*L/h*=5				*L/h*=20			
		$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$	$\bar{w}$	$\bar{u}$	$\bar{\sigma}_x$	$\bar{\tau}_{xz}$
0	Li *et al.* (2010)	3.1657	0.9402	3.8020	0.7500	2.8962	0.2306	15.0130	0.7500
	Ould Larbi *et al.* (2013)	3.1651	0.9406	3.8043	0.7489	2.8962	0.2305	15.0136	0.7625
	Present *n*=3	3.1653	0.9398	3.8019	0.7329	2.8962	0.2306	15.0129	0.7437
	Present *n*=5	3.1597	0.9349	3.7880	0.6337	2.8959	0.2305	15.0094	0.6381
	Present *n*=7	3.1528	0.9319	3.7796	0.5954	2.8954	0.2304	15.0073	0.5976
	Present *n*=9	3.1474	0.9299	3.7742	0.5743	2.8951	0.2304	15.0060	0.5755
0.5	Li *et al.* (2010)	4.8292	1.6603	4.9925	0.7676	4.4645	0.4087	19.7005	0.7676
	Ould Larbi *et al.* (2013)	4.8282	1.6608	4.9956	0.7660	4.4644	0.4087	19.7013	0.7795
	Present *n*=3	4.8285	1.6596	4.9923	0.7501	4.4644	0.4087	19.7003	0.7606
	Present *n*=5	4.8213	1.6527	4.9731	0.6508	4.4640	0.4086	19.6955	0.6551
	Present *n*=7	4.8124	1.6484	4.9614	0.6130	4.4634	0.4085	19.6926	0.6151
	Present *n*=9	4.8056	1.6456	4.9539	0.5924	4.4630	0.4085	19.6908	0.5935
1	Li *et al.* (2010)	6.2599	2.3045	5.8837	0.7500	5.8049	0.5686	23.2054	0.7500
	Ould Larbi *et al.* (2013)	6.2590	2.3052	5.8875	0.7489	5.8049	0.5685	23.2063	0.7625
	Present *n*=3	6.2594	2.3038	5.8835	0.7329	5.8049	0.5685	23.2051	0.7437
	Present *n*=5	6.2499	2.2956	5.8601	0.6337	5.8043	0.5684	23.1993	0.6381
	Present *n*=7	6.2382	2.2905	5.8459	0.5954	5.8036	0.5683	23.1958	0.5976
	Present *n*=9	6.2291	2.2872	5.8367	0.5743	5.8030	0.5683	23.1935	0.5755
2	Li *et al.* (2010)	8.0602	3.1134	6.8812	0.6787	7.4415	0.7691	27.0989	0.6787
	Ould Larbi *et al.* (2013)	8.0683	3.1146	6.8878	0.6870	7.4421	0.7691	27.1005	0.7005
	Present *n*=3	8.0677	3.1129	6.8824	0.6704	7.4421	0.7691	27.0989	0.6812
	Present *n*=5	8.0465	3.1032	6.8514	0.5692	7.4407	0.7689	27.0912	0.5737
	Present *n*=7	8.0255	3.0969	6.8328	0.5291	7.4394	0.7689	27.0866	0.5314
	Present *n*=9	8.0099	3.0929	6.8207	0.5068	7.4384	0.7688	27.0836	0.5081
5	Li *et al.* (2010)	9.7802	3.7089	8.1030	0.5790	8.8151	0.9133	31.8112	0.5790
	Ould Larbi *et al.* (2013)	9.8345	3.7128	8.1187	0.6084	8.8186	0.9134	31.8151	0.6218
	Present *n*=3	9.8281	3.7100	8.1104	0.5904	8.8182	0.9134	31.8127	0.6013
	Present *n*=5	9.7627	3.6937	8.0639	0.4844	8.8141	0.9131	31.8012	0.4889
	Present *n*=7	9.7122	3.6839	8.0365	0.4413	8.8109	0.9129	31.7944	0.4437
	Present *n*=9	9.6768	3.6776	8.0189	0.4171	8.8087	0.9128	31.7901	0.4185
10	Li *et al.* (2010)	10.8979	3.8860	9.7063	0.6436	9.6879	0.9536	38.1372	0.6436
	Ould Larbi *et al.* (2013)	10.9413	3.8898	9.7203	0.6640	9.6907	0.9537	38.1408	0.6788
	Present *n*=3	10.9381	3.8863	9.7119	0.6465	9.6905	0.9536	38.1382	0.6586
	Present *n*=5	10.8809	3.8659	9.6639	0.5398	9.6869	0.9533	38.1263	0.5451
	Present *n*=7	10.8247	3.8534	9.6336	0.4943	9.6833	0.9531	38.1188	0.4971
	Present *n*=9	10.7819	3.8452	9.6137	0.4677	9.6806	0.9530	38.1139	0.4694

Table 1 contains nondimensional stresses and displacements of a simply supported FG beam for different values of power law index *k* and span-to-depth ratio *L/h*. The obtained results are compared with the analytical solutions given by Ould Larbi (2013), Li *et al.* (2010). It can be found that the present n-order refined theory produces the close results to those of Ould Larbi

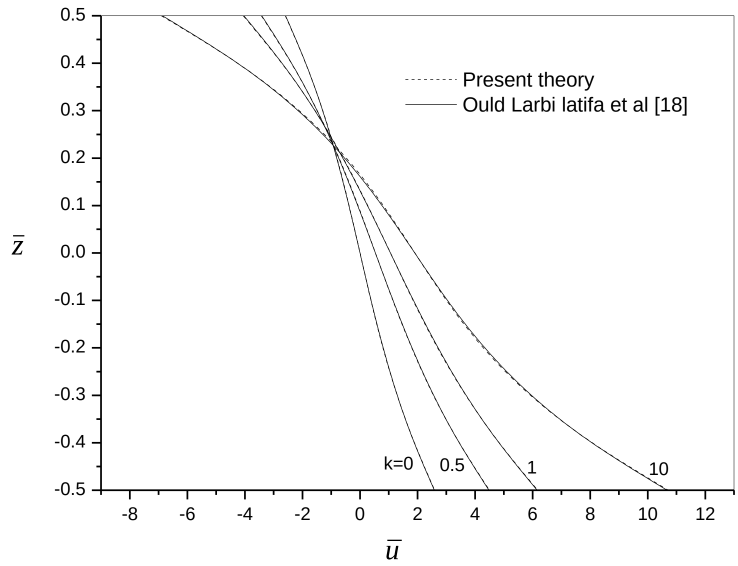

Fig. 2 The variation of the axial displacement $\bar{u}$ through-the-thickness of a FG beam (L=2h)

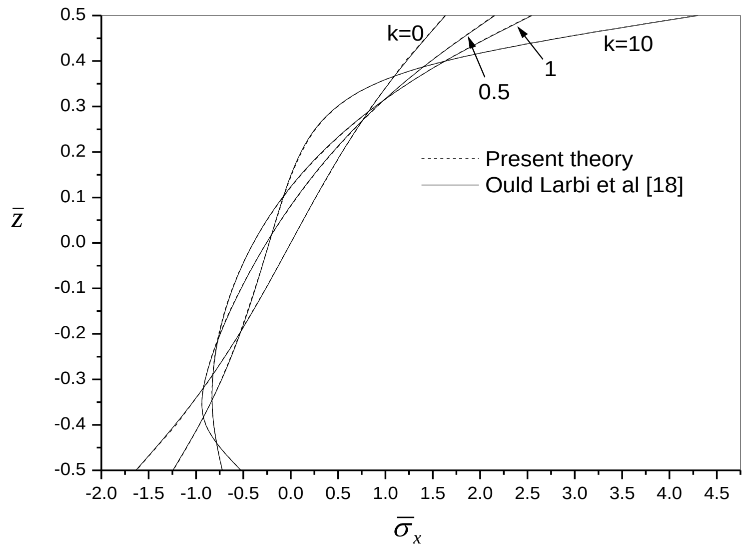

Fig. 3 The variation of the axial stress $\bar{\sigma}_x$ through-the-thickness of a FG beam (L=2h)

(2013), Li *et al.* (2010).

In Figs. 2-4 we present the evolution of the axial displacement $\bar{u}$, axial stresses $\bar{\sigma}_x$ and transverse shear stress $\bar{\tau}_{xz}$ across the depth of the FG beam under uniform load. A comparison with the analytical solutions developed by Ould Larbi (2013) is also shown in these figures using different values of the power law index k. It is seen that there is a good agreement between the present theory and those of Ould Larbi (2013). It can be seen from Fig. 2 that the increase of the power law index k leads to an increase of the axial displacement $\bar{u}$ and especially at the top and

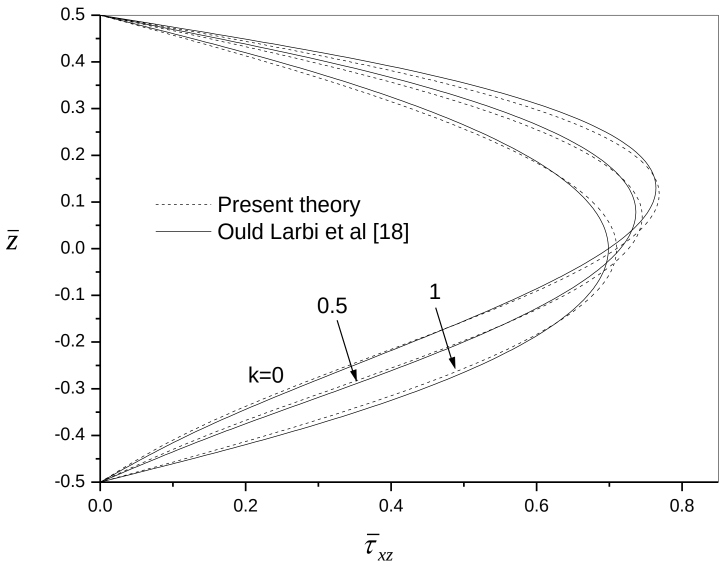

Fig. 4 The variation of the transverse shear stress $\bar{\tau}_{xz}$ through-the-thickness of a FG beam (L=2h)

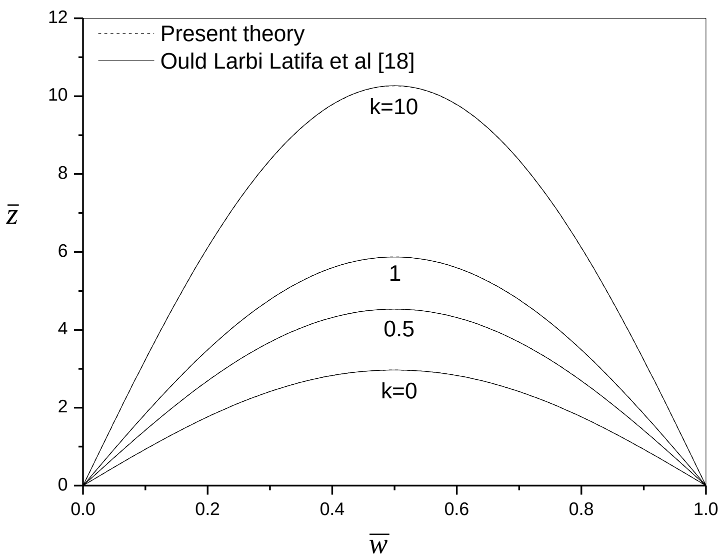

Fig. 5 Variation of the transverse displacement $\overline{w}$ versus non-dimensional length of a FG beam (L=5h)

bottom of the beam. In Fig. 3, the axial stress $\bar{\sigma}_x$ is tensile at the top surface and compressive at the bottom surface. The homogeneous ceramic beam (k=0) yields the maximum compressive stresses at the bottom surface and the minimum tensile stresses at the top surface of the beam. In Fig. 4 we have plotted the through-the-thickness distributions of the transverse shear stress $\bar{\tau}_{xz}$. The through-the-thickness distributions of the transverse shear stresses for FG beams are not parabolic as in the case of homogeneous metal or ceramic beams.

Fig. 5 illustrates the variation of the non-dimensional transversal displacement $\overline{w}$ versus non-dimensional length for different power law index *k*. It can be seen also that the present beam theory based on neutral surface position gives almost identical results to Ould Larbi. In addition, the results show that the increase of the power law index *k* leads to an increase of transversal displacement $\overline{w}$.

4.2 Free vibration analysis

In this section, various numerical examples are presented and discussed to verify the accuracy of the present theory in predicting the natural frequency of simply supported beams. For convenience, following natural frequency parameter is used in presenting the numerical results in tabular and graphical forms

$$\bar{\omega} = \frac{\omega L^2}{h}\sqrt{\frac{\rho_m}{E_m}}$$

For the verification purpose, the nondimensional fundamental frequencies $\bar{\omega}$ obtained by the present theory are compared with those given by Ould Larbi (2013) and Simsek (2010) of FG beams for different values of power law index *k* and span-to-depth ratio *L/h* and the results are presented in Table 2. It can be seen that the present theory and the theories used by Ould Larbi (2013), Simsek (2010) give almost identical results specially for *n*=3.

Fig. 6 shows the non-dimensional fundamental natural frequency $\bar{\omega}$ versus the power law index *k* for different values of span-to-depth ratio *L/h* using both the present theory and theory developed by Ould Larbi (2013). An excellent agreement between the present theory and Ould Larbi is showed from Fig. 6. It can be observed that the frequency decreases with increasing the power law index. The full ceramic beams (*k*=0) lead to a highest frequency. However, the lowest frequency values are obtained for full metal beams (*k*→∞). This is due to the fact that an increase in the value of the power law index results in a decrease in the value of elasticity modulus.

Table 2 Variation of fundamental frequency $\bar{\omega}$ with the power-law index for FG beam

L/h	Theory	*k*					
		0	0.5	1	2	5	10
5	Simsek (2010)	5.1527	4.4111	3.9904	3.6264	3.4012	3.2816
	Ould Larbi *et al*. (2013)	5.1529	4.4108	3.9905	3.6263	3.4001	3.2812
	Present *n*=3	5.1527	4.4107	3.9904	3.6264	3.4012	3.2816
	Present *n*=5	5.1572	4.4139	3.9934	3.6309	3.4119	3.2898
	Present *n*=7	5.1627	4.4178	3.9969	3.6355	3.4204	3.2979
	Present *n*=9	5.1669	4.4209	3.9998	3.6389	3.4264	3.3042
20	Simsek (2010)	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
	Ould Larbi *et al*. (2013)	5.4603	4.6511	4.2051	3.8361	3.6484	3.5389
	Present *n*=3	5.4603	4.6511	4.2051	3.8361	3.6485	3.5390
	Present *n*=5	5.4607	4.6514	4.2053	3.8365	3.6494	3.5397
	Present *n*=7	5.4611	4.6517	4.2056	3.8368	3.6500	3.5403
	Present *n*=9	5.4614	4.6519	4.2058	3.8371	3.6505	3.5408

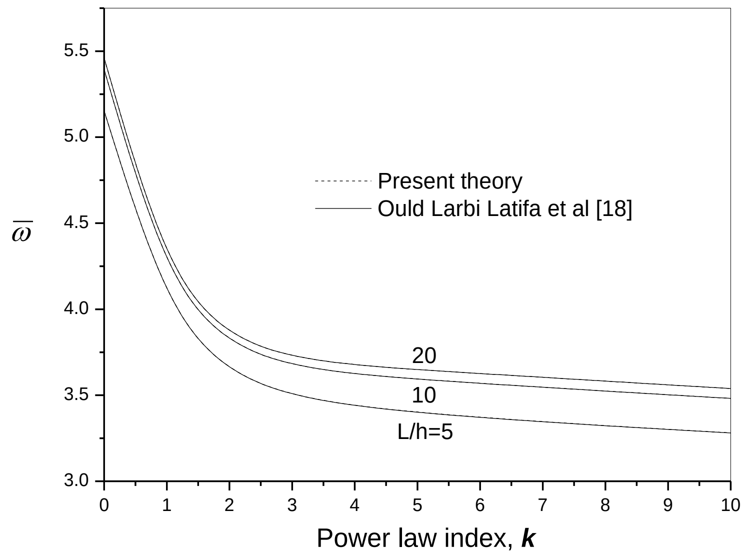

Fig. 6 Variation of the nondimensional fundamental frequency $\overline{\varpi}$ of FG beam with power law index k and span-to-depth ratio L/h

5. Conclusions

A n-order four variable refined theory is proposed to analyze the bending and free vibration of functionally graded beams. The present theory is variationally consistent, uses the n-order polynomial term to represent the displacement field, does not require shear correction factor, and gives rise to transverse shear stress variation such that the transverse shear stresses vary parabolically across the thickness satisfying shear stress free surface conditions. It is based on the assumption that the transverse displacements consist of bending and shear components in which the bending components do not contribute toward shear forces and, likewise, the shear components do not contribute toward bending moments. Based on the present beam theory and the neutral surface concept, the equations of motion are derived from Hamilton's principle. Numerical examples show that the proposed theory gives solutions which are almost identical with those obtained using other shear deformation theories.

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