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Effects of rotary inertia shear deformation and non-homogeneity on frequencies of beam

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Abstract. In the present study, separate and combined effects of rotary inertia, shear deformation and material non-homogeneity (MNH) on the values of natural frequencies of the simply supported beam are examined. MNH is characterized considering the parabolic variations of the Young's modulus and density along the thickness direction of the beam, while the value of Poisson's ratio is assumed to remain constant. At first, the equation of the motion including the effects of the rotary inertia, shear deformation and MNH is provided. Then the solutions including frequencies of the first three modes for various combinations of the parameters of the MNH, depth to length ratios, and shear corrections factors are reported. To show the accuracy of the present results, two comparisons are carried out and good agreements are found.

Keywords: rotary inertia; shear deformation; material non-homogeneity; natural frequency; beam

1. Introduction

Beam is one of the most important components in the engineering structures due to it can be used in various applications. In addition, different structures such as helicopter rotor blades, space craft antennae, flexible satellites, airplane wings, gun barrels, robot arms, high-rise buildings, long-span bridges, and subsystems of more complex structures can be modelled as a beam-like slender member. Therefore, examining the vibration behavior of this simple structural component considering different conditions would be helpful in understanding and explaining the behavior of more complex and real structures subjected similar conditions (Wang *et al.* 2000, Aghababaei *et al.* 2009a, b, Civalek and Gurses 2009, Carrera *et al.* 2011).

It is seen that the dynamics of beams has received an extensive research for a long period, and the most of these works are mainly focused on thin beams, which are analyzed on the basis of Euler-Bernoulli beam theory also known as classical beam theory (CBT). However, CBT slightly overestimates the natural frequencies and the error increases with the increase of modes and thickness of the beam, due to rotary inertia and shear deformation effects are neglected. To enhance the accuracy, different theories are developed, i.e. including rotary inertia effects to CBT, which is called Reissner beam theory (RBT), including shear deformation to the CBT, which is called shear model beam theory (SMBT), and including both rotary inertia and shear deformation

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effects which is called Timoshenko beam theory (TBT) (Timoshenko 1937, Han *et al.* 1999, De Silva 2000, Rao 2007, Labuschagne *et al.* 2009, Leissa and Qatu 2011, Wang and Wang 2014). Numerous studies have been published on the vibration of beams using different theories and methods, some of them are given in Refs. (Hurty and Moshe 1964, Grant 1978, Chang and Yuan 1985, Horr and Schmidt 1995, Al-Ansary 1998, Yıldırım and Kıral 2000, Şimşek and Kocatürk 2007, Civalek and Kiracioglu 2009, Demir *et al.* 2010, Irvine 2010, Saffari *et al*. 2012, Berrabah *et al.* 2013, Liu *et al.* 2013, Avcar 2014, Avcar and Saplioglu 2015, Bagdatli 2015, Mao 2015, Szyiko-Bigus and Sniady 2015, Yesilce 2015).

All of the above mentioned studies are carried out for homogeneous (H) structures in sense that mechanical properties of the structure are taken to be constant throughout. However, plenty of materials exist in the nature which are non-homogeneous (NH), such as plywood, delta wood, timber, fiber reinforced plastic are examples of naturally NH materials, and glass epoxy and boron epoxy in steel alloys are examples of artificially NH materials. The mechanical properties of these materials may vary with the space coordinates, either continuously or discontinuously, in an arbitrary specified way (Chakraverty and Petyt 1997, Lal and Sharma 2004, Chakraverty *et al*. 2007, Gupta *et al.* 2007, Civalek 2009, Gupta and Kumar 2010, Gupta *et al.* 2010, Lal and Kumar 2013, Civalek 2013, Tounsi *et al.* 2013, Belabed *et al.* 2014, Hebali *et al.* 2014, Ait Amar Meziane *et al.* 2014, Ait Yahia *et al.* 2015, Mahi *et al.* 2015). As the use of these advanced materials in various fields of engineering and technology has been increased, the vibration problems of NH beams have been also received the attention of numerous researchers for long years (Nayfeh 1972, Chaudhuri and Datta 1989, Ji andYeh 1994, Tong *et al.* 1995, Elishakoff and Becquet 2000, Becquet and Elishakoff 2001, Elishakoff and Candan 2001, Elishakoff and Guede 2001, Elishakoff 2005, Ece *et al.* 2007, Taha and Abohadima 2008, Avcar 2010, Lin 2010, Şimşek 2010, Mazzei and Scott 2012, Nandi *et al.* 2012, Akgöz and Civalek 2013, Mohammadnejad *et al.* 2014, Bourada *et al.* 2015, Gan *et al.*2015, Zemri *et al.* 2015).

From the review of available literature it is observed that the separate and combined effects of rotary inertia, shear deformation and the MNH on the values of natural frequencies of the simply supported beam in which the MNH is characterized with the parabolic variation of the Young's modulus and density along the thickness direction has not been dealt yet. The objective of the present study is to address this problem. For this aim, firstly the equation of the motion including the effects of MNH, the rotary inertia and shear deformation is provided. Then the solutions including natural frequencies of the first three modes for various combinations of the parameters of the non-homogeneity, depth to length ratios and shear corrections factors are reported. The present analysis may be of beneficial to the designers, researchers, scientists and engineers, dealing with NH beams for finding the required natural frequency by changing the different parameters considered here.

2. Fundamental equations

Consider an elastic a beam of length *L*, Young' modulus *E*, and mass density *ρ*, subjected to the effects of bending, shear deformation and rotary inertia as shown in Fig. 1 (De Silva 2000).

As one can see from Fig. 1(b), the total slope of the beam can be expressed as

$$\frac{\partial w}{\partial x} = \theta + \psi \tag{1}$$

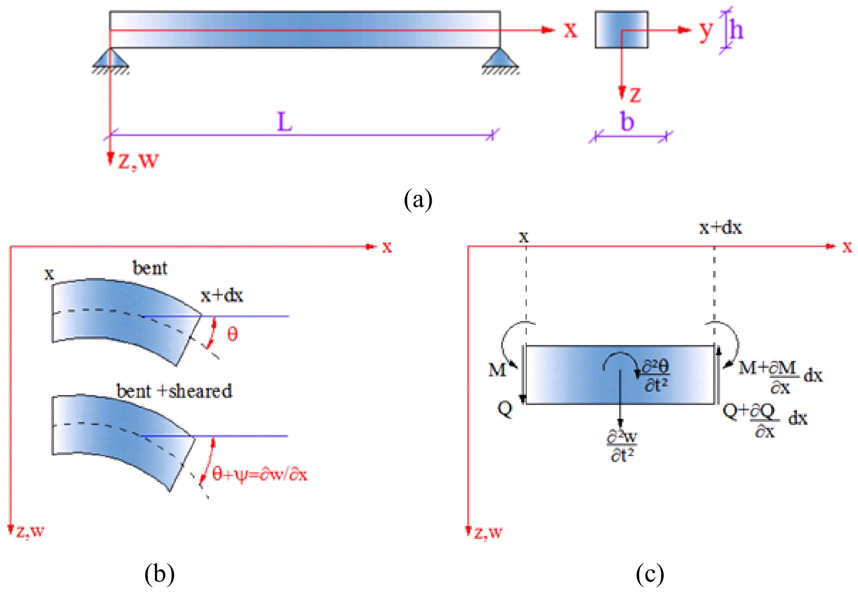

Fig. 1 (a) Geometry of the beam, beam element under the effects of (b) bending and shear (c) rotary inertia

where, θ, ψ, x and w are angle of rotation of beam element due to bending, increase in the slope due to shear deformation, longitudinal coordinate and the transverse displacement of beam, respectively.

The linear shear stress-shear strain relation can be expressed using the sign convention given in Fig. 1(c), as

$$Q = -kGA\psi \tag{2}$$

here, Q, k, G and A are shear force, shear correction factor, shear modulus and area, respectively and following definition apply

$$G = \frac{E}{2(1+\nu)} \tag{3}$$

where ν is the Poisson's ratio.

The equation of translational motion is

$$\frac{\partial Q}{\partial x} = -\rho A \frac{\partial^2 w}{\partial t^2} \tag{4}$$

here t is time.

The equation of rotational motion of beam element including the rotary inertia is

$$\frac{\partial M}{\partial x} - Q = I\rho \frac{\partial^2 \theta}{\partial t^2} \tag{5}$$

where I is the area moment of inertia of the beam.

The relation between moment and curvature is

$$M = EI\frac{\partial\theta}{\partial x} \tag{6}$$

Manipulations of above mentioned equations yields the equation of motion for the free vibration of a H beam including shear deformation and rotary inertia effects as follows (Timoshenko 1937, De Silva 2000, Rao 2007, Leissa and Qatu 2011)

$$EI\frac{\partial^4 w}{\partial x^4} + \rho A\frac{\partial^2 w}{\partial t^2} - \left[\rho I + \frac{EI\rho}{kG}\right]\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho^2 I}{kG}\frac{\partial^4 w}{\partial t^4} = 0 \tag{7}$$

It is assumed that the MNH of the beam stems from the variation of Young's modulus and mass density along the thickness direction (Chakraverty and Petyt 1997, Lal and Sharma 2004, Chakraverty *et al*. 2007, Gupta *et al.* 2007, Gupta and Kumar 2010, Gupta *et al.* 2010, Lal and Kumar 2013)

$$E_1 = E(1 + \beta_1 \phi(\bar{z})) \tag{8}$$

$$\rho_1 = \rho(1 + \beta_2 \phi(\bar{z})) \tag{9}$$

where $\bar{z} = z/h$, $\phi(\bar{z})$ is the continuous function of MNH defining the variation of Young's modulus and mass density, E_1 and ρ_1 are the Young's modulus and mass density of non-homogenous material and β_1 and β_2 are the non-homogeneity and density parameters ($-0.5 \le \beta_i \le 0.5$, i=1,2). It should be noted that the value of Poisson's ratio is assumed to remain constant.

The MNH function of the beam is taken to be parabolic function (Chakraverty and Petyt 1997, Chakraverty *et al*. 2007)

$$\phi(\bar{z}) = \bar{z}^2 \tag{10}$$

Considering Eqs. (8)-(10) in Eq.(7), the governing equation for the free vibration of a NH beam including shear deformation and rotary inertia effects is obtained as follow

$$D\frac{\partial^4 w}{\partial x^4} + \rho_1 A\frac{\partial^2 w}{\partial t^2} - \left[\rho_1 I + \frac{D\rho_1}{kG_1}\right]\frac{\partial^4 w}{\partial x^2 \partial t^2} + \frac{\rho_1^2 I}{kG_1}\frac{\partial^4 w}{\partial t^4} = 0 \tag{11}$$

where G_1 is shear modulus and D is the flexural rigidity of the NH beam, and the following definitions apply

$$\begin{aligned} D &= Ebh^3 \int_{-1/2}^{1/2} \bar{z}^2 (1 + \beta_1 \phi(\bar{z})) d\bar{z} \\ G_1 &= \frac{E}{2(1+\nu)} \int_{-1/2}^{1/2} (1 + \beta_1 \phi(\bar{z})) d\bar{z} \\ \rho_1 &= \rho \int_{-1/2}^{1/2} (1 + \beta_2 \phi(\bar{z})) d\bar{z} \end{aligned} \tag{12}$$

Note that, the first two terms represent the classical beam theory, plus three additional terms show the rotary inertia, shear deformation and their combined effects respectively in Eq. (11).

3. Solution of the fundamental equations

The solution of the Eq. (11) is sought by separation of variables. Assume that the displacement can be separated into spatial and temporal variables

$$w(x,t) = \Gamma(x)T(t) \tag{13}$$

where Γ and T are independent of time and position, respectively.

Substituting Eq. (13) into Eq. (11) and after some mathematical operations, the following equation is obtained

$$D\Gamma^{IV}(x) + \rho_1 A\Gamma(x)\frac{T^{II}(t)}{T(t)} - \left[\rho_1 I + \frac{D\rho_1}{kG_1}\right]\Gamma^{II}(x)\frac{T^{II}(t)}{T(t)} + \frac{\rho_1^2 I}{kG_1}\frac{T^{IV}(t)}{T(t)}\Gamma(x) = 0 \tag{14}$$

The temporal function can be represented as

$$T(t) = c_1 \cos\omega t + c_2 \sin\omega t \tag{15}$$

Substituting Eq. (15) into Eq. (14) and after some mathematical rearrangements, the following equation is gotten

$$D\Gamma^{IV}(x) - \omega^2\rho_1 A\Gamma(x) + \omega^2\left[\rho_1 I + \frac{D\rho_1}{kG_1}\right]\Gamma^{II}(x) + \omega^4\frac{\rho_1^2 I}{kG_1}\Gamma(x) = 0 \tag{16}$$

The solution of the Eq. (16) is sought as follow

$$\Gamma(x) = d_1\sinh(\delta x) + d_2\cosh(\delta x) + d_3\sin(\delta x) + d_4\cos(\delta x) \tag{17}$$

The both boundary conditions of the beam are assumed as simply supported, therefore the following expressions are satisfied

$$\Gamma(0) = \Gamma^{II}(0) = \Gamma(L) = \Gamma^{II}(L) = 0 \tag{18}$$

Considering Eq. (18) in Eq. (17) and after some mathematical operations the following coefficient matrix is obtained

$$\begin{bmatrix} \sinh\delta L & \sin\delta L \\ \sinh\delta L & -\sin\delta L \end{bmatrix}\begin{bmatrix} d_1 \\ d_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{19}$$

The non-trivial solution of the determinant of the coefficient matrix is as follows

$$\sin\delta L \sinh\delta L = 0 \tag{20}$$

From Eq. (20) the following result is obtained

$$\delta_n = \frac{n\pi}{L},\ n = 1,2,\ldots \tag{21}$$

where the subscript, n, is an integer index.

Thus the displacement function becomes

$$\Gamma(x) = d_5\left[\sin(\delta L)\sinh(\delta x) + \sinh(\delta L)\sin(\delta x)\right] \tag{22}$$

Substituting Eqs. (22) and then (21) into Eq. (16) respectively, after some mathematical rearrangements the following equation is found

$$D\left(\frac{n\pi}{L}\right)^4 - \omega^2\left\{\left(\frac{n\pi}{L}\right)^2\left[\rho_1 I + \frac{D\rho_1}{kG_1}\right] + \rho_1 A\right\} + \frac{\rho_1^2 I}{kG_1}\omega^4 = 0 \tag{23}$$

Consequently, from Eq. (23), the expression for the free vibration frequency of the NH beam including rotary inertia and shear deformation effects, which corresponds to the TBT, is gotten

$$\omega_{RIS} = \sqrt{\left\{\frac{kG_1}{2\rho_1^2 I}\right\}\left(\left[\rho_1 I + \frac{D\rho_1}{kG_1}\right]\left(\frac{n\pi}{L}\right)^2 + \rho_1 A - \sqrt{\left(\left[\rho_1 I + \frac{D\rho_1}{kG_1}\right]\left(\frac{n\pi}{L}\right)^2 + \rho_1 A\right)^2 - 4\frac{\rho_1^2 I}{kG_1} D\left(\frac{n\pi}{L}\right)^4}\right)} \tag{24}$$

One can obtain the equation for the free vibration the NH beam including only the shear deformation effect by setting the terms including $\rho_1 I$ equal to zero in Eq. (11)

$$D\frac{\partial^4 w}{\partial x^4} + \rho_1 A\frac{\partial^2 w}{\partial t^2} - \frac{D\rho_1}{kG_1}\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \tag{25}$$

After repeating the above mentioned solution procedure for the Eq. (25), the expression for the free vibration frequency of NH beam including only shear deformation effect, which corresponds to the SMBT, is gotten as

$$\omega_S = \sqrt{\frac{D\left(\frac{n\pi}{L}\right)^4}{\left[\rho_1 A + \frac{D\rho_1}{kG_1}\left(\frac{n\pi}{L}\right)^2\right]}} \tag{26}$$

Similarly, one can obtain the equation for the free vibration the NH beam including only the rotary inertia effect by letting $G_1 \rightarrow \infty$ in Eq. (11)

$$D\frac{\partial^4 w}{\partial x^4} + \rho_1 A\frac{\partial^2 w}{\partial t^2} - \rho_1 I\frac{\partial^4 w}{\partial x^2 \partial t^2} = 0 \tag{27}$$

Reapplying the above mentioned solution procedure for the Eq. (27), the expression for the free vibration frequency of NH beam including only rotary inertia effect, which corresponds to RBT, is found as

$$\omega_{RI} = \sqrt{\frac{D\left(\frac{n\pi}{L}\right)^4}{\left(\rho_1 A + \rho_1 I\left(\frac{n\pi}{L}\right)^2\right)}} \tag{28}$$

Additionally, one can obtain the equation for the free vibration the NH classical beam which neglects the effects of rotary inertia and shear deformation by setting the terms including $\rho_1 I$ equal to zero and letting $G_1\rightarrow\infty$ in Eq. (11)

$$D\frac{\partial^4 w}{\partial x^4}+\rho_1 A\frac{\partial^2 w}{\partial t^2}=0 \tag{29}$$

As the above mentioned solution procedure employed for Eq. (29), the expression for the free vibration frequency of NH beam neglecting the both shear deformation and rotary inertia effects, which corresponds CBT, is gotten as

$$\omega_C=\sqrt{\frac{D\left(\frac{n\pi}{L}\right)^4}{\rho_1 A}} \tag{30}$$

4. Numerical results and discussion

In this section illustrative studies are given to examine the present problem. At first, two comparative studies are presented to show the accuracy of the present formulation for homogenous beams, by taking $\beta_1=\beta_2=0$. Then, three different studies are exhibited to show the separate and combined effects of rotary inertia, shear deformation and MNH versus various non-homogeneity coefficients, depth to length ratios and shear corrections factors on the values of natural frequencies of beams. Here the natural frequency (Hz) is defined as $\Omega_i=\omega_i/2\pi$, i=C, RI, S, RIS, material properties are taken to be E=210 GPa; ρ=7850 kg/m^3, ν=0.3

Comparative Study 1. Separate and combined contributions of rotary inertia and shear deformation on the frequencies of simply supported rectangular beam versus mode number, n, are compared with those of Leissa and Qatu (2011) in Table 1. Here, ω_i/ω_C i=RI, S, RIS, is the frequency ratio and h/L=0.1, G/E=0.4, k=2/3 are taken into account.

Table 1 Comparisons of frequency ratios of H beam with results of Leissa and Qatu (2011)

n	ω_i/ω_C					
	Leissa and Qatu (2011)			Present Study		
	ω_{RI}	ω_S	ω_{RIS}	ω_{RI}	ω_S	ω_{RIS}
1	0.9959	0.9849	0.9811	0.9959	0.9849	0.9811
2	0.9839	0.9435	0.9314	0.9839	0.9435	0.9314
3	0.9649	0.8847	0.8651	0.9649	0.8847	0.8651
4	0.9401	0.8183	0.7945	0.9401	0.8183	0.7945
5	0.9107	0.7514	0.7268	0.9107	0.7514	0.7268
6	0.8784	0.6884	0.6650	0.8784	0.6884	0.6650
7	0.8442	0.6310	0.6100	0.8442	0.6310	0.6100
8	0.8094	0.5799	0.5614	0.8094	0.5799	0.5614
9	0.7747	0.5347	0.5187	0.7747	0.5347	0.5187
10	0.7407	0.4948	0.4816	0.7407	0.4948	0.4816

Table 2 Comparisons of values of natural frequencies of H beam with results of Rao (2008)

	Natural Frequency (rad/s)					
	ω_C		ω_{RI}		ω_{RIS}	
n	Rao (2008)	Present Study	Rao (2008)	Present Study	Rao (2008)	Present Study
1	703.0149	702.9992	696.5987	696.5834	677.8909	677.9541
2	2812.0598	2811.9968	2713.4221	2713.3651	2473.3691	2474.1605
3	6327.1348	6326.9929	5858.0654	5857.9512	4948.0063	4950.8366

Comparative Study 2. The values of natural frequencies (rad/s) of homogeneous beam are compared with results of Rao (2008) in Table 2. Here, L=1 m, b=0.05m, h=0.15 m, $E=207\times10^9$ Pa, $G=79.3\times10^9$ Pa, $\rho=76.5\times10^3$ N/m^3, k=5/6 are considered.

It is clear that from above given comparisons, the both results are in good agreements, which shows the accuracy of the present expressions (24), (26), (28) and (30).

Study 1. Table 3 shows the natural frequencies Ω_C, Ω_{RI}, Ω_S and Ω_{RIS} (Hz) of simply supported H and NH beams for β_i i=1,2; h=0.5 m; h/L=0.15; k=5/6 taking the values β_1=−0.5, 0, 0.5 and β_2=−0.5, 0, 0.5. It is found that the natural frequencies, Ω_i, increase with the increase in non-homogeneity parameter, β_1, while they decrease with the increase in density parameter β_2. On the other hand, the effects of the MNH on the values of natural frequencies vary according to the non-homogeneity parameter, β_1 and density parameter β_2. For example the highest effect is observed as the MNH arises due to β_1=0.5; β_2=−0.5, while the lowest effect is observed as the MNH arises due to β_1=0.5; β_2=0.5. Besides, the efficiency of MNH decreases with increasing mode numbers for Ω_S and Ω_{RIS} while it remains constant for Ω_{RI} and Ω_C. Moreover, it is observed that as MNH arises from the variation of density parameter, β_2, only, it has constant effects on the values of natural frequencies in the all number of modes. However as MNH arises from the variation of non-homogeneity parameter, β_1, only, it has variable effect on the values of natural frequencies according to number of modes for Ω_S and Ω_{RIS} while it remains constant for Ω_{RI}, Ω_C.

Table 3 Variation of natural frequencies, Ω_i, (Hz) of H and NH beam versus non-homogeneity parameters

β_1		n=1			n=2			n=3		
	β_2	−0.5	0	0.5	−0.5	0	0.5	−0.5	0	0.5
−0.5	Ω_C	103.688	101.505	99.454	414.753	406.020	397.817	933.193	913.545	895.088
	Ω_{RI}	102.742	100.579	98.547	400.205	391.779	383.863	864.013	845.821	828.732
	Ω_S	100.914	98.790	96.794	375.051	367.155	359.737	761.553	745.519	730.456
	Ω_{RIS}	100.086	97.979	95.999	366.009	358.303	351.064	733.995	718.541	704.023
0	Ω_C	107.810	105.540	103.408	431.239	422.160	413.630	970.288	949.859	930.668
	Ω_{RI}	106.826	104.577	102.464	416.113	407.352	399.122	898.358	879.443	861.674
	Ω_S	104.826	102.619	100.546	388.685	380.502	372.814	787.102	770.530	754.962
	Ω_{RIS}	103.969	101.780	99.724	379.430	371.441	363.936	759.245	743.259	728.242
0.5	Ω_C	111.780	109.426	107.215	447.118	437.704	428.861	1006.016	984.835	964.937
	Ω_{RI}	110.759	108.427	106.237	431.436	422.352	413.818	931.437	911.825	893.403
	Ω_S	108.591	106.305	104.157	401.793	393.334	385.387	811.656	794.567	778.514
	Ω_{RIS}	107.706	105.439	103.308	392.333	384.073	376.313	783.506	767.009	751.513

Table 4 Variation of natural frequencies, Ω_i, (Hz) of H and NH beam versus the depth to length ratio

n	$\beta_1=\beta_2=0$ (H Case)							
	h/L=0.1				*h/L*=0.15			
	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}
1	46.907	46.715	46.316	46.136	105.540	104.577	102.619	101.780
2	187.626	184.614	178.680	176.304	422.160	407.352	380.502	371.441
3	422.160	407.352	380.502	371.441	949.859	879.443	770.530	743.259
n	*h/L*=0.2				h/L=0.25			
	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}
1	187.626	184.614	178.680	176.304	293.166	285.910	272.153	267.073
2	750.506	705.519	631.911	611.502	1172.665	1067.996	915.273	880.750
3	1688.638	1483.267	1217.467	1168.351	2638.497	2181.666	1687.939	1619.441
	$\beta_1=0.5$; $\beta_2=-0.5$ (NH Case)							
n	*h/L*=0.1				*h/L*=0.15			
	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}
1	49.680	49.477	49.035	48.844	111.780	110.759	108.591	107.706
2	198.719	195.529	188.963	186.465	447.118	431.436	401.793	392.333
3	447.118	431.436	401.793	392.333	1006.016	931.437	811.656	783.506
n	*h/L*=0.2				*h/L*=0.25			
	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}
1	198.719	195.529	188.963	186.465	310.499	302.813	287.608	282.285
2	794.877	747.230	666.175	645.029	1241.995	1131.137	963.380	927.873
3	1788.473	1570.960	1279.652	1229.476	2794.490	2310.650	1771.071	1701.712

Furthermore, the separate and combined effects of rotary inertia and shear deformation increase with increasing number of modes. Additionally, as the MNH is considered; the separate effect of the shear deformation and the combined effect of shear deformation and rotary inertia become more pronounced while the separate effect of rotary inertia remains constant.

Study 2. Table 4 shows the natural frequencies Ω_C, Ω_{RI}, Ω_S and Ω_{RIS} (Hz) of simply supported H and NH beams for $\beta_1=0.5$; $\beta_2=-0.5$, $k=5/6$ versus the depth to length ratio, *h/L*. It is seen that the natural frequencies increase with the increase in the ratio, *h/L*. On the other hand, the effect of the MNH decrease with the increase of depth to length ratio, *h/L*, as well as it becomes more inefficient with increasing number of modes for Ω_S and Ω_{RIS} while it remains constant for Ω_C and Ω_{RI} in all modes. Besides, it is found that the MNH has lowest effect on the values of Ω_S. Additionally, the separate effect of the shear deformation is higher than the separate effect of the rotary inertia, while their combined effect is the most pronounced one and the significances of all of these three effects increase with the increase in the ratio *h/L* and number of modes. Furthermore, the separate effect of shear deformation and combined effects of rotary inertia and shear deformation become more pronounced with the consideration of MNH.

Study 3. Table 5 shows the natural frequencies Ω_C, Ω_{RI}, Ω_S and Ω_{RIS} (Hz) of simply supported H and NH beams for $\beta_1=-0.5$; $\beta_2=0.5$, $h/L=0.25$ versus two well-known shear correction factor, k, for rectangular beam. It is observed that natural frequencies, Ω_S and Ω_{RIS} increase with the increase

Table 5 Variation of natural frequencies, Ω_i, (Hz) of H and NH beam versus the shear correction factor

β_1=β_2=0 (H Case)												
k	n=1				n=2				n=3			
	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}
2/3	293.166	285.910	267.570	262.890	1172.665	1067.996	873.591	845.919	2638.497	2181.666	1575.598	1526.516
5/6	293.166	285.910	272.153	267.073	1172.665	1067.996	915.273	880.750	2638.497	2181.666	1687.939	1619.441
β_1=−0.5; β_2=0.5 (NH Case)												
k	n=1				n=2				n=3			
	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}	Ω_C	Ω_{RI}	Ω_S	Ω_{RIS}
2/3	276.262	269.424	252.877	248.404	1105.047	1006.412	829.664	802.599	2486.355	2055.865	1501.643	1452.767
5/6	276.262	269.424	257.079	252.236	1105.047	1006.412	868.418	834.821	2486.355	2055.865	1607.206	1539.372

of k, while the natural frequencies Ω_C and Ω_{RI} remains constant due to they are independent from it. Besides, the rate of increase of natural frequencies, Ω_S and Ω_{RIS} according the shear correction factor, k, in H case slightly higher than those in NH case. On the other hand, the effect of the MNH increases with the increase of k for Ω_S and Ω_{RIS}, and the effect of MNH on the values of Ω_{RIS} is more pronounced. Moreover, it is observed that the influence of variation of shear correction factor, k, on Ω_S is higher than that for Ω_{RIS} in both H and NH cases. The separate effect of the shear deformation and the combined effect of shear deformation and rotary inertia are more pronounced than the separate effect of the rotary inertia, however the differences between them decrease with the increase of k and consideration of MNH.

5. Conclusions

In the present work, the free vibration of simply supported beams composed of NH materials was analyzed. Several examples were carried out to demonstrate the separate and combined effects of rotary inertia, shear deformation and MNH on the values of natural frequencies considering various non-homogeneity and density parameters, depth to length ratios, and shear correction factors.

The following interesting results are found

• Non-homogeneity parameter, β_1, and density parameter, β_2, have contrary effects on the values of natural frequencies, and so the effect of MNH on the values of natural frequencies change according to variations of the non-homogeneity parameter, β_1, and density parameter, β_2. The percentage variation in the value of natural frequencies Ω_C, Ω_{RI}, Ω_S and Ω_{RIS} for the first mode vary in the interval from −5.912 to 5.766 versus β_1 and β_2 ($-0.5 \le \beta_i \le 0.5$, i=1,2)

• MNH has highest on the values of natural frequencies Ω_C and Ω_{RI} while it has least effect on the values of natural frequency Ω_S

• The effect of MNH on the values of natural frequencies Ω_C and Ω_{RI} are exactly same, as well as it is independent from the variation of the number of the modes and the depth to length ratios

• The effect of the MNH on the values of natural frequencies Ω_S and Ω_{RIS} decreases with increase of number of modes and depth to length ratios while it increase with increase of shear correction factor which validates that the effect of MNH decreases as the shear deformation effect is taken into account

• The effects of variations of the depth to length ratio, shear correction factor on the values of natural frequencies decrease as the MNH is taken into account

• The orders of values of frequencies are always $\Omega_C > \Omega_{RI} > \Omega_S > \Omega_{RIS}$ in the all number of modes and cases. And the differences between them increase with the increase of depth to length ratios and number of modes

Consequently, it is concluded that the effects of MNH, rotary inertia and shear deformation on the natural frequencies of beams are considerable. Besides, the effects of rotary inertia and shear deformation should be necessarily taken into account while examining the vibrations of thick beams. The present analysis may be beneficial to designers, researchers, scientists and engineers, dealing with NH beams for finding the required natural frequency by changing the different parameters considered here.

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