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Small scale effect on the vibration of non-uniform nanoplates

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Abstract. Free vibration of non-uniform embedded nanoplates based on classical (Kirchhoff's) plate theory in conjunction with nonlocal elasticity theory has been studied. The nanoplate is assumed to be rested on two-parameter Winkler-Pasternak elastic foundation. Non-uniform material properties of nanoplates have been considered by taking linear as well as quadratic variations of Young's modulus and density along the space coordinates. Detailed analysis has been reported for all possible cases of such variations. Trial functions denoting transverse deflection of the plate are expressed in simple algebraic polynomial forms. Application of the present method converts the problem into generalised eigen value problem. The study aims to investigate the effects of non-uniform parameter, elastic foundation, nonlocal parameter, boundary condition, aspect ratio and length of nanoplates on the frequency parameters. Three- dimensional mode shapes for some of the boundary conditions have also been illustrated. One may note that present method is easier to handle any sets of boundary conditions at the edges.

Keywords: classical plate theory; Rayleigh-Ritz method; nonlocal elasticity theory; mode shapes

1. Introduction

Because of excellent properties of nanostructures (Ruud *et al.* 1994), they have been used in nanoelectronics, nano-devices, nano-sensors, nano-oscillators and nanoactuators. Seeing its practical applications, static and dynamic analyses of nanostructures have been carried out by the researchers. Among these studies, vibration analysis of nanostructures is of great importance in nanotechnology. It plays an important role in the design of many nanoelectromechanicalsystem devices like oscillators, clocks and sensors. Since conducting experiments at nanoscale are quite difficult and expensive, so development of mathematical model is an important issue in the field of nanomechanics. One of the important nanostructures is graphene sheets. Generally, three approaches have been used to model nanostructures. These are (a) atomistic modeling (b) hybrid atomistic-continuum mechanics and (c) continuum mechanics. Continuum mechanics may be of two types i.e., classical continuum mechanics and nonclassical continuum mechanics. Among nonclassical continuum mechanics, nonlocal elasticity theory pioneered by Eringen (1972, 1983) has been widely accepted by the researchers. In nonlocal elasticity theory, the stress at a point depends on the strain tensors of the entire body. At nanoscale size, the system will no longer remain continuum due to the significant influence of interatomic and intermolecular cohesive

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forces on the static and dynamic responses of the nanostructures. One should incorporate small scale effects and atomic forces in the design of nanostructures (viz., nanoresonantors (Peng *et al.* 2006), nanoactuators (Dubey *et al.* 2004), nanomachines and nanooptomechanical systems) to achieve solutions with accuracy. Ignoring small scale effects in nanodesigning fields may cause improper designs. Recent literature shows that nonlocal elasticity theory (includes small scale effect) is being increasingly used for reliable and quick analysis of nanostructures. Nonlocal parameter plays an important role which has been determined by some of the authors. Liang and Han (2014) has derived explicit expression of the nonlocal scaling parameter from the nonlocal model and the solutions of nonlocal parameter are verified by molecular dynamics simulations. Authors found that nonlocal parameter is not a constant but a scale-related variable dependent on the size of the structures and independent of the value of the loads. Similarly, Narendar *et al.* (2011) obtained an expression of the nonlocal parameter as a function of the geometric and electronic properties of the rolled graphene sheet in single-walled CNTs. Liang and Han (2012) extracted the proper values of nonlocal scale parameter by molecular dynamics (MD) simulations for various radii and lengths of armchair and zigzag CNTs. Authors have developed numerical and analytical methods to study bending, buckling and vibration of nanoplates based on nonlocal elasticity theory. Some of them are discussed below:

Analytical solution of simply supported nanoplates based on classical and first order shear deformation plate theories shows that effect of nonlocal parameter becomes more significant as the size of the plate decreases (Pradhan and Phadikar 2009). Semi inverse method (Adali 2012) is one of the powerful tool for deriving variational principles which helps in solving the problems by different numerical methods such as Ritz (Anjomshoa 2013), finite difference (Ravari and Shahidi 2013) and finite element. Refined plate theory in conjunction with nonlocal elasticity theory has been used by Malekzadeh and Shojaee (2013) to study free vibration of nanoplates. Wang and Wang (2011) have shown the importance of surface energy in the prediction of lower natural frequencies. Free vibration of orthotropic quadrilateral nanoplates has been investigated by Malekzadeh *et al.* (2011) using differential quadrature method. Thermal effect on the free vibration of piezoelectric nanoplates has been investigated by Liu *et al.* (2013) based on Kirchhoff's plate theory. Small scale effect on the vibration of thin nanoplates subjected to a moving nanoparticle has been analyzed by Kiani (2011). Bending and free vibration of simply supported rectangular plate have been investigated by Aghababaei and Reddy (2009) based on third order shear deformation plate theory. Kiani (2011a, 2011b) has carried out theoretical formulation and parametric studies when a nanoparticle is subjected to a moving nanoparticle. Again Kiani (2014) studied free vibration of conducting nanoplates exposed to unidirectional in-plane magnetic fields based on nonlocal shear deformable plate theories. Free vibration of orthotropic nonprismatic skew nanoplates has been analyzed by Beni and Malekzadeh (2012). Further some of the studies have also been carried out for functionally graded nanoplates (Salehipour *et al.* 2015, Nami and Janghorban 2014, Ansari *et al.* 2015, Natarajan *et al.* 2012, Belkorissat *et al.* 2015). Vibration response of double piezoelectric nanoplate system under an external electric voltage has been investigated and the study shows that natural frequencies are are quite sensitive to the vibrational mode and non-local parameter (Jomehzadeh and Saidi 2012). Temperature, surrounding medium and other aspect also affects vibration response of the nanoplates (Narendar 2011).

Structural members with variable cross sections are frequently used in civil, mechanical and aeronautical engineering to satisfy architectural requirements. For efficient design of nanostructures, non-uniform geometries of the nanocomponents should be employed. Non-uniform geometries provide efficient vibration control in nanosize sensors and actuators. Unfortunately, design engineers are lack of proper knowledge on design of non-uniform structural elements since most of the design specifications are available for uniform elements. The literature reveals that previous studies done in nanoplates are mostly with constant parameters like Young's modulus and density. But in actual practice, there may be a variation in these parameters. Hence for proper practical applications of nanoplates, one should investigate geometrically non-uniform plate model of graphene sheets. Though previous studies show some of the investigations on variable cross sections in case of nanostructures (Murmu and Pradhan 2009, Pradhan and Murmu 2010, Farajpour *et al.* 2011) but to the best of authors' knowledge, free vibration of non-uniform nanoplates using Rayleigh-Ritz method has not yet been investigated. Moreover, nanoplates may be used as reinforcement in nano-composites. As such, the nanoplate is considered to be rested on an elastic medium. Therefore, present article also includes Winkler-Pasternak model for considering both the springy and shear effects of the elastic foundation.

Researchers throughout the globe are trying to develop efficient analytical and numerical methods to obtain desired results with ease. However, it is not always possible to find analytical solutions for complicated problems. Previous researchers have applied numerical methods such as Navier and Levy (Aksencer and Aydogdu 2011), Finite Element (Phadikar and Pradhan 2010) and Differential Quadrature (Murmu and Pradhan 2009). One should note that Levy type method is applied for plates with two opposite edges as simply supported and remaining ones as any arbitrary boundary condition. Similarly, Navier type method is applied only for simply supported plates. One may also note that in finite element method, we need to take very large number of elements for smooth shape of deflection curve which is quite difficult to handle for complicated problems. In this regard, one may use Rayleigh-Ritz method which is quite easy to handle any set of classical boundary conditions at the edges with ease. Though application of Rayleigh-Ritz method has been done in vibration of classical plates (Bhat 1985, 1991, Singh and Chakraverty 1994a, Chakraverty *et al.* 2007, Rajalingham *et al.* 1996, Behera and Chakraverty 2013, Chakraverty and Behera 2014) but least work has been done in the vibration of nanoplates. In this paper, effects of non-uniformity, elastic foundation, aspect ratio, boundary condition and length of nanoplates on the frequency parameters have been investigated. Three-dimensional mode shapes for specified nanoplates have also been presented. Detailed analysis has been given which may help researcher of nanotechnology for the production of nanomaterial.

2. Theoretical formulation of nanoplates based on classical plate theory

Let us consider a rectangular nanoplate in the domain $a \le x \le b$, $a \le y \le b$ in xy plane. Here a and b are the length and width of the nanoplate respectively. One may note that x and y axes are taken along the edges of the nanoplate and z axis is perpendicular to xy plane. Middle surface is taken along z=0 and origin is considered at one of the corners of the nanoplate. For applying the present method, one should know about the energies. As such, we have given a brief overview of the energies.

Maximum kinetic energy (T) of isotropic nanoplates in cartesian coordinate is given by (Adali 2012, Anjomshoa 2013)

$$T = \frac{1}{2} m_0 \omega^2 \int_0^a \int_0^b \left\{ W^2 + \mu \left(\left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 \right) \right\} dx\,dy \tag{1}$$

where W is the amplitude of motion, ω the natural frequency of the nanoplate and $\mu=(e_0 l_{int})^2$ the nonlocal parameter. It may be noted that e_0 is a material constant which could be determined from experiments or by matching dispersion curves of plane waves with those of atomic lattice dynamics, l_{int} is an internal characteristic length such as lattice parameter, C-C bond length or granular distance. The term nonlocal parameter signifies scale effect in models or it reveals the nanoscale effect on the response of structures.

In Eq. (1), m_0 is mass per unit area of the nanoplate and is defined as $m_0=\int_{-h/2}^{h/2}\rho dz$

Here ρ denotes the mass per unit volume of the nanoplate and h is the height.

Maximum potential energy U of isotropic nanoplates in cartesian coordinate may be written as (Adali 2012, Anjomshoa 2013)

$$U=\frac{1}{2}D\int_0^a\int_0^b\left\{\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2\nu\left(\frac{\partial^2 W}{\partial X^2}\frac{\partial^2 W}{\partial Y^2}\right)+\left(\frac{\partial^2 W}{\partial y^2}\right)^2+2(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right\}dxdy \tag{2}$$

where υ is the Poisson's ratio and $D=\dfrac{Eh^3}{12(1-\upsilon^2)}$ is the flexural rigidity of isotropic nanoplate with E as the Young's modulus.

Equating maximum kinetic and potential energies, one may obtain Rayleigh Quotient

$$\lambda^2=\frac{\int_0^a\int_0^b\left\{\left(\frac{\partial^2 W}{\partial x^2}\right)^2+2\nu\left(\frac{\partial^2 W}{\partial x^2}\frac{\partial^2 W}{\partial y^2}\right)+\left(\frac{\partial^2 W}{\partial y^2}\right)^2+2(1-\nu)\left(\frac{\partial^2 W}{\partial x\partial y}\right)^2\right\}dxdy}{\int_0^a\int_0^b\left\{W^2+\mu\left(\left(\frac{\partial W}{\partial x}\right)^2+\left(\frac{\partial W}{\partial y}\right)^2\right)\right\}dxdy} \tag{3}$$

where $\lambda^2=\dfrac{\rho h\omega^2}{D}$ and $m_0=\rho h$**.** We have introduced the non-dimensional variables as $X=\dfrac{x}{a}$ and $Y=\dfrac{y}{b}$

In this investigation, graphene sheets with non-uniform material properties have been considered. The non-uniform material properties are assumed as per the following relations:

$$E=E_0(1+\alpha X+\beta X^2)\text{ and }\rho=\upsilon_0(1+\gamma X+\delta X^2)$$

Eq. (3) may be written in non-dimensional form as

$$\lambda^2=\frac{\int_0^1\int_0^1 A\left\{\left(\frac{\partial^2 W}{\partial X^2}\right)^2+2\nu R^2\left(\frac{\partial^2 W}{\partial X^2}\frac{\partial^2 W}{\partial Y^2}\right)+R^4\left(\frac{\partial^2 W}{\partial Y^2}\right)^2+2(1-\nu)R^2\left(\frac{\partial^2 W}{\partial X\partial Y}\right)^2\right\}dXdY}{\int_0^1\int_0^1 B\left\{W^2+\frac{\mu}{a^2}\left(\left(\frac{\partial W}{\partial X}\right)^2+\left(\frac{\partial W}{\partial Y}\right)^2\right)\right\}dXdY} \tag{4}$$

where $\lambda^2=\dfrac{\rho_0 h a^4\omega^2}{D_0}$, $R=\dfrac{a}{b}$, $A=(1+\alpha X+\beta X^2)$ and $B=(1+\gamma X+\delta X^2)$

Now, if the nanoplate is embedded in elastic medium, then maximum kinetic energy is given by Eq. (1) but maximum potential energy may be written as (Adali 2012, Anjomshoa 2013)

$$U = \frac{1}{2} D \int_0^a \int_0^b \left\{ \left(\frac{\partial^2 W}{\partial x^2} \right)^2 + 2\nu \left(\frac{\partial^2 W}{\partial x^2} \frac{\partial^2 W}{\partial y^2} \right) + \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 2(1-\nu) \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 + \right.$$

$$\left. k_w \left[W^2 + \mu \left(\left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 \right) \right] + k_p \left[\left(\frac{\partial W}{\partial x} \right)^2 + \left(\frac{\partial W}{\partial y} \right)^2 + \mu \left(\left(\frac{\partial^2 W}{\partial x^2} \right)^2 + \left(\frac{\partial^2 W}{\partial y^2} \right)^2 + 2 \left(\frac{\partial^2 W}{\partial x \partial y} \right)^2 \right) \right] \right\} dx dy \tag{5}$$

Here k_w and k_p denote the Winkler and Pasternak coefficients of the elastic foundation respectively.

Equating maximum kinetic and potential energies, one may obtain Rayleigh Quotient for λ^2 in nondimensional form as

$$\lambda^2 = \frac{\int_0^1 \int_0^1 A \left\{ \left(\frac{\partial^2 W}{\partial X^2} \right)^2 + 2\nu R^2 \left(\frac{\partial^2 W}{\partial X^2} \frac{\partial^2 W}{\partial Y^2} \right) + R^4 \left(\frac{\partial^2 W}{\partial Y^2} \right)^2 + 2(1-\nu) R^2 \left(\frac{\partial^2 W}{\partial X \partial Y} \right)^2 + c_1 + c_2 \right\} dX dY}{\int_0^1 \int_0^1 B \left\{ W^2 + \frac{\mu}{a^2} \left(\left(\frac{\partial W}{\partial X} \right)^2 + \left(\frac{\partial W}{\partial Y} \right)^2 \right) \right\} dX dY} \tag{7}$$

where $c_1 = k_w \left[W^2 + \frac{\mu}{a^2} \left(\left(\frac{\partial W}{\partial X} \right)^2 + R^2 \left(\frac{\partial W}{\partial Y} \right)^2 \right) \right]$ and

$$c_2 = k_p \left[\left(\frac{\partial W}{\partial X} \right)^2 + R^2 \left(\frac{\partial W}{\partial Y} \right)^2 + \frac{\mu}{a^2} \left(\left(\frac{\partial^2 W}{\partial X^2} \right)^2 + R^2 \left(\frac{\partial^2 W}{\partial X \partial Y} \right)^2 \right) + \frac{\mu}{b^2} \left(\left(\frac{\partial^2 W}{\partial X \partial Y} \right)^2 + R^2 \left(\frac{\partial^2 W}{\partial Y^2} \right)^2 \right) \right]$$

with $K_w = \frac{k_w a^4}{D_0}$ and $K_p = \frac{k_p a^2}{D_0}$

3. Solution methodology

Amplitude of the motion may be expressed as

$$W(X,Y) = \sum_{i=1}^{n} c_i \phi_i (X,Y) \tag{8}$$

where n is the number of terms taken for computation, c_i are the unknowns and ϕ_i are the polynomials consisting of a boundary polynomial (χ) and two dimensional simple polynomials (θ_i). That is $\phi_i = \chi\theta_i$ $\quad i=1, 2, 3,..., n$

where

$$\chi = X^p (1-X)^q Y^r (1-Y)^s \tag{9}$$

and

$$\theta_i = \{1, X, Y, X^2, XY, Y^2, X^3, X^2Y, XY^2, Y^3 ....\} \tag{10}$$

In Eq. (9), p takes the values 0, 1 and 2 as the edge X=0 is free, simply supported and clamped respectively.

Same justification can be given to q, r and s for the edges X=1, Y=0, and Y=1 respectively.

Substituting Eq. (8) into Eq. (7), a generalized eigen value problem is obtained as

$$[K]\{Y\} = \lambda^2 [M]\{Y\} \tag{11}$$

where Y is a column vector of constants, K and M are the matrices given as follows

$$K_{ij} = A[\psi_{ij}{}^{2020} + \upsilon R^2(\psi_{ij}{}^{2002} + \psi_{ij}{}^{0220}) + R^4\psi_{ij}{}^{0202} + 2(1-\upsilon)R^2\psi_{ij}{}^{1111} + K_w[\psi_{ij}{}^{0000} + \frac{\mu}{a^2}(\psi_{ij}{}^{1010} + R^2\psi_{ij}{}^{0101})]$$

$$+ K_p[\psi_{ij}{}^{1010} + R^2\psi_{ij}{}^{0101} + \frac{\mu}{a^2}(\psi_{ij}{}^{2020} + R^2\psi_{ij}{}^{1111}) + \frac{\mu}{b^2}(\psi_{ij}{}^{1111} + R^2\psi_{ij}{}^{0202})]]$$

$$M_{ij} = B\left[\psi_{ij}{}^{0000} + \frac{\mu}{a^2}\left(\psi_{ij}{}^{1010} + R^2\psi_{ij}{}^{0101}\right)\right]$$

With $\psi_{ij}{}^{klmn} = \int_0^1\int_0^1 \left[\frac{\partial^{k+l}\phi_i}{\partial X^k \partial Y^l}\right]\left[\frac{\partial^{m+n}\phi_i}{\partial X^m \partial Y^n}\right] dXdY$

4. Results and discussion

Generalised eigen value problem (Eq. (11)) has been solved by using computer programme developed by authors in MATLAB. Eigen values of Eq. (11) correspond to the frequency parameters. Different sets of boundary conditions (B.cs) have been considered here with Poisson's ratio as 0.3. Letters C, S and F refer to clamped, simply supported and free edge conditions respectively. Edge conditions are taken in anticlockwise direction starting at the edge X=0 and are obtained by taking various values to p, q, r, and s as 0, 1, 2 for free, simply supported and clamped edge conditions respectively.

4.1 Convergence

Convergence study of first three frequency parameters of SSSS and CCCC nanoplates has been shown in Table 1 taking α=0.2, β=0.3, γ=0.4, δ=0.5, aspect ratio (R)=1, nonlocal parameter (μ)=2 nm^2 and length (a)=5 nm. Results have been shown for K_w=0 and K_p=0. One may see that n=37 is sufficient for computing converged results for nanoplates without elastic foundation. Again, convergence of embedded nanoplates has been shown in Table 2 for SSSS edge condition. Numerical values of parameters are taken as α=β=γ=δ=0.1, K_w=200, K_p=5, μ=1 nm^2, a=10 nm. It is observed that converged results for embedded nanoplates are obtained at n=46. It is also noticed that frequency parameters increase with mode number.

4.2 Validation

Table 1 Convergence of first three frequency parameters of SSSS and CCCC nanoplates

	SSSS			CCCC		
n	λ_1	λ_2	λ_3	λ_1	λ_2	λ_3
5	11.8785	24.3785	24.4884	19.7881	29.8410	29.9750
10	11.5401	20.9273	24.2786	19.7317	29.0494	29.1283
15	11.5366	20.9013	20.9271	19.7210	29.0469	29.1277
20	11.5366	20.8621	20.8924	19.7210	29.0402	29.1049
25	11.5366	20.8404	20.8620	19.7203	29.0265	29.1048
30	11.5366	20.8404	20.8617	19.7196	29.0264	29.1047
35	11.5366	20.8403	20.8615	19.7196	29.0248	29.1030
36	11.5366	20.8400	20.8615	19.7196	29.0247	29.1030
37	11.5366	20.8400	20.8615	19.7196	29.0247	29.1030

Table 2 Convergence of first three frequency parameters of SSSS nanoplates

n	λ_1	λ_2	λ_3
10	45.6173	63.9934	107.8843
15	45.6044	63.9885	90.5544
20	45.6044	63.8730	90.5418
25	45.6044	63.8729	89.7499
30	45.6044	63.8726	89.7465
35	45.6044	63.8721	89.7464
40	45.6044	63.8721	89.7328
45	45.6044	63.8721	89.7325
46	45.6044	63.8721	89.7325

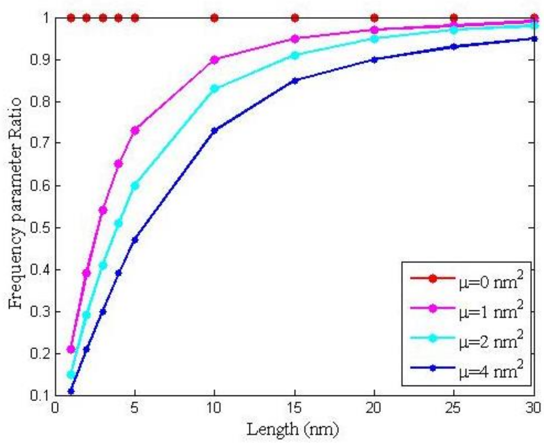

Fig. 1 Variation of first frequency parameter ratio with length

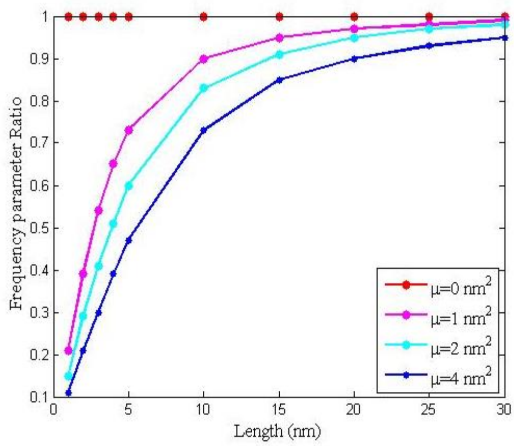

Fig. 2 Variation of second frequency parameter ratio with length

Figs. 1 and 2 show graphical comparison of SCSC nanoplates with Aksencer and Aydogdu (2011). For this comparison, we have taken $\alpha=\beta=\gamma=\delta=K_w=K_p=0$ with $R=1$. In these graphs, variation of frequency parameter ratio (associated with first two modes) with length has been

Table 3 Comparison of first three frequency parameters for SSSS nanoplate

μ=0		μ=1		μ=2		μ=3		μ=4	
Present	Ref.[2]	Present	Ref. [2]	Present	Ref. [2]	Present	Ref. [2]	Present	Ref. [2]
0.0963	0.0963	0.0880	0.0880	0.0816	0.0816	0.0763	0.0763	0.0720	0.0720
0.3874	0.3853	0.2884	0.288	0.2402	0.2399	0.2102	0.2099	0.1892	0.1889
0.8608	0.8669	0.5167	0.5202	0.4045	0.4063	0.3435	0.3446	0.3037	0.3045

given for different nonlocal parameters (0 nm^2, 1 nm^2, 2 nm^2, 4 nm^2). Here we have calculated frequency parameter ratio (F_r) as follows:

$$F_r = \frac{\text{frequency parameter calculated using nonlocal theory}}{\text{frequency parameter calculated using local theory}} .$$

One may notice that increase in nonlocal parameter decreases frequency parameter ratio. Same observation may also be seen in Ref. Aksencer and Aydogdu (2011) and we may found a close agreement of the results. Next tabular comparison has been given with Ref. Aghababaei *et al.* (2009) for the first three frequency parameters with R=1 and a=10 nm. Results have been verified for different nonlocal parameters with the consideration of SSSS edge condition.

4.3 Effect of non-uniform parameter

In this sub-section, we have studied the effects of non-uniform parameters on the frequency parameters in the absence of elastic foundation. First, we have shown the effects of non-uniform parameters when density and Young's modulus vary quadratically. This case may be achieved by assigning zero to α and γ. Variation of first three frequency parameters with β has been illustrated in Fig. 3 keeping δ constant (0.3). Similarly, effect of δ on the first three frequency parameters has been shown in Fig. 4 keeping β constant (0.2). In these graphs, CCCC edge condition is taken into consideration and aspect ratio is taken as 2 and 3 respectively. One may see that frequency parameters decrease with δ and increase with β. This may be due to the fact that $\sqrt{\lambda}$ is directly proportional to β and inversely proportional to δ in the Eq. (11). It is also observed that frequency parameters increase with increase in mode number.

In this paragraph, we have presented the effects non-uniform parameters when density and Young's modulus vary linearly. This is achieved by taking β and δ to zero. Graphical variation of frequency parameters with α taking γ constant (0.3) has been shown in Fig. 5. Similarly, graphical variation of frequency parameters with γ taking α constant (0.2) has been shown in Fig. 6. In these graphs, we have considered FFFF boundary condition and aspect ratio as 1. It is observed that frequency parameters decrease with γ and increase with α. This behavior is due to the fact that $\sqrt{\lambda}$ is directly proportional to α and inversely proportional to γ in Eq. (11).

Here, we have considered the effects of non-uniform parameters when Young's modulus varies linearly and density varies quadratically. This is the situation which is obtained by taking β and γ as zero. Figs. 7 and 8 depict variation of frequency parameters with α and δ respectively. In these graphs, CCCC nanoplates with R=2 are taken into consideration. It is noticed that frequency parameters decrease with δ and increase with α.

Next, we have analyzed the effects of non-uniform parameter when Young's modulus varies quadratically and density varies linearly. For this, we have taken α and δ as zero. Variations of first three frequency parameters with β and γ have been illustrated in Figs. 9 and 10. Results have been shown for FFFF nanoplates with R=1 In these graphs, One may see frequency parameters increase with β and decrease with γ.

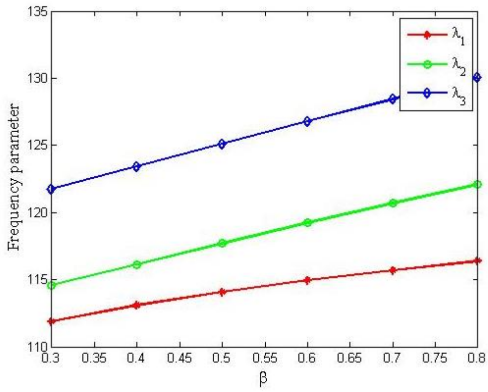

Fig. 3 Variation of frequency parameter with β

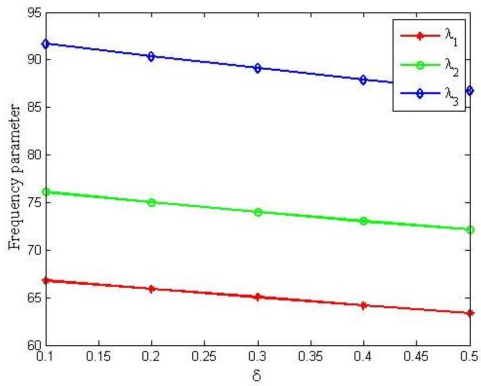

Fig. 4 Variation of frequency parameter with δ

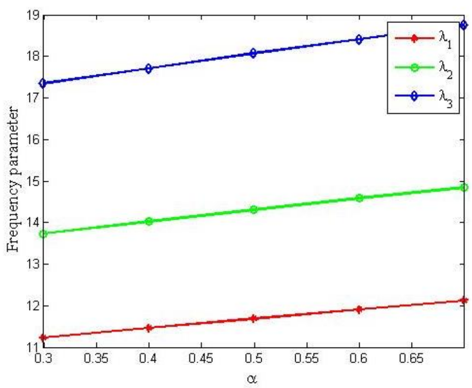

Fig. 5 Variation of frequency parameter with α

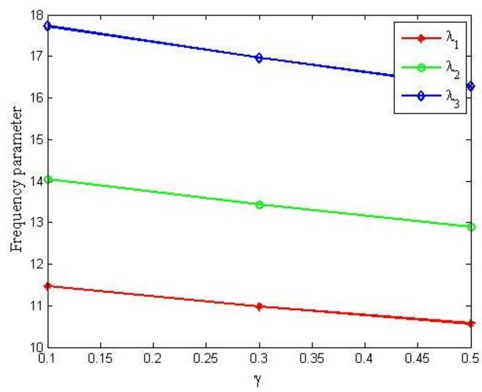

Fig. 6 Variation of frequency parameter with γ

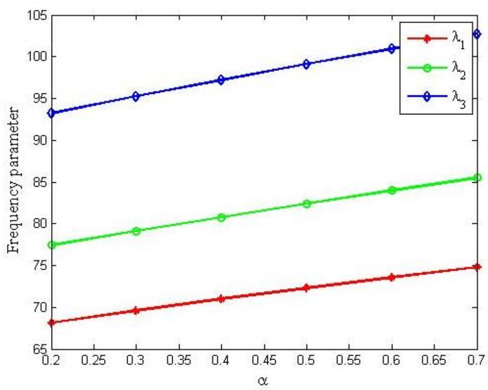

Fig. 7 Variation of frequency parameter with α

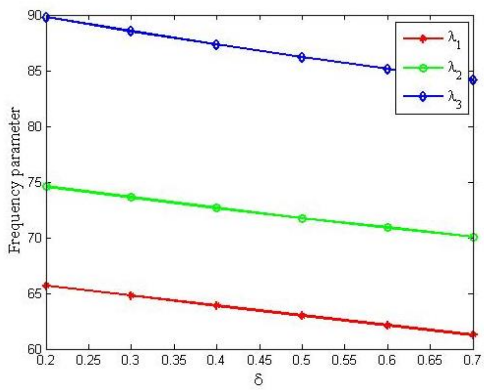

Fig. 8 Variation of frequency parameter with δ

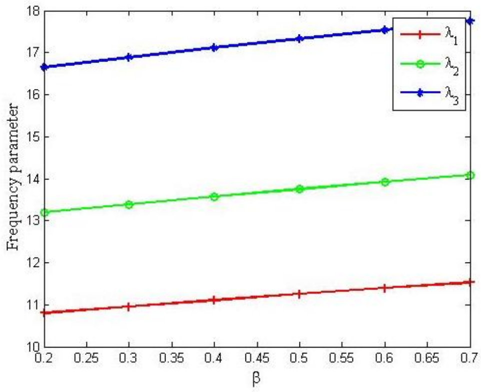

Fig. 9 Variation of frequency parameter with β

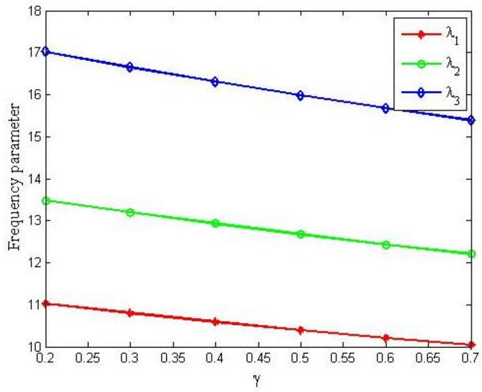

Fig. 10 Variation of frequency parameter with γ

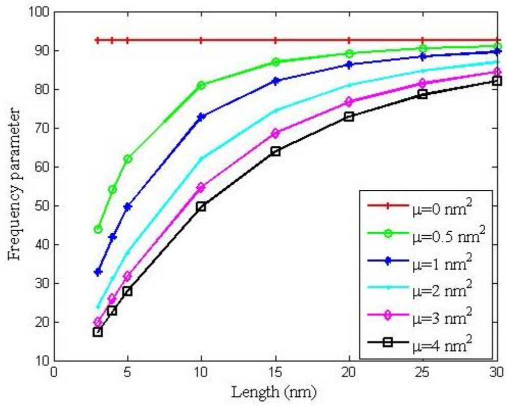

Fig. 11 Variation of frequency parameter with length

4.4 Effect of length

To investigate the effect of length on the frequency parameters, variation of fundamental frequency parameter with length has been shown in Fig. 11 for CCCC nanoplates with R=2, α=0.2, β=0.3, γ=0.4, δ=0.5 in the absence of elastic foundation. Results have been shown for different values of nonlocal parameters (0 nm^2, 0.5 nm^2, 1 nm^2, 2 nm^2, 3 nm^2, 4 nm^2). It is seen that frequency parameter increases with increase in length. This observation may be explained as follows. Assuming l_{int} as constant, increasing length (a) would lead to decrease in small scale effect$\left(\frac{\mu}{a^2}\right)$. It is also noticed that frequency parameters are highest in case of μ=0 and goes on decreasing with increase in nonlocal parameter. This fact may also been explained in terms of relative error percent. Let us define the relative error percent (REP) as

$$\text{REP}=\frac{\left|\text{Local Result - Nonlocal Result}\right|}{\left|\text{Local Result}\right|}\times 100$$

Neglecting nonlocal effect, relative error percents of fundamental frequency parameter for a=3

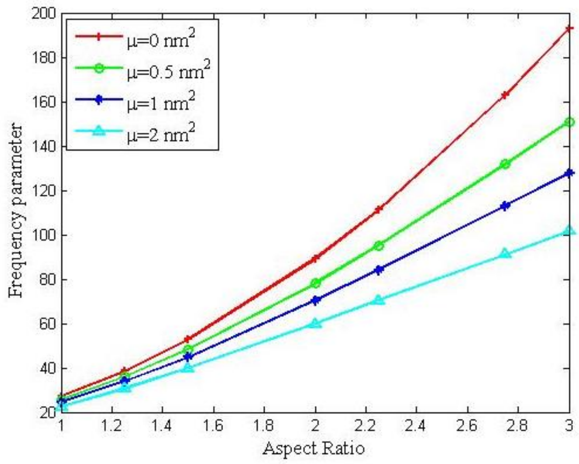

Fig. 12 Variation of frequency parameter with aspect ratio

nm and a=25 nm with μ=3 nm^2 are 78.6753% and 12.0950% respectively. From this, we may also say that nonlocal theory should be taken into account for vibration analysis of small enough nanoplates.

4.5 Effect of aspect ratio

In this subsection, we have considered the effect of aspect ratio on the frequency parameters in the absence of elastic foundation. Fig. 12 shows the effect of fundamental frequency parameter of SCSC nanoplate with aspect ratio taking a=10 nm, α=0.1, β=0.2, γ=0.3, δ=0.4. It is seen that nonlocal effect on the frequency parameters is more prominent in greater values of aspect ratio. This is due to the fact that for a particular length of nanoplates, increase in aspect ratio would lead to smaller nanoplates which in turn lead to increase in small scale effect. It is also observed that frequency parameter increases with aspect ratio. One may notice that frequency parameter decreases with increase in nonlocal parameter. As the nonlocal parameter increases, frequency parameters obtained from nonlocal plate theory become smaller than those of its local counterpart. This reduction is clearly seen in case of higher vibration modes. The reduction is due to the fact that nonlocal model may be viewed as atoms linked by elastic springs while in case of local continuum model, the spring constant is assumed to take an infinite value. So small scale effect makes the nanoplates more flexible and hence nonlocal impact cannot be neglected. Effect of nonlocal parameter is seen more in case of higher vibration modes. This fact may also be explained in terms of relative error percent. Neglecting nonlocal effect, the relative error percents for aspect ratios 1 and 3 with μ=3 nm^2 are 22.5041% and 54.9963% respectively. This shows that, nonlocal theory should be considered for free vibration of nanoplates with high aspect ratios.

4.6 Effect of nonlocal parameter

Here we have examined the effect of nonlocal parameter on the frequency parameters in the absence of elastic foundation. Variation of frequency ratio (associated with first four mode) with nonlocal parameter has been illustrated in Fig. 13 for SCSS edge condition. In this graph, we have

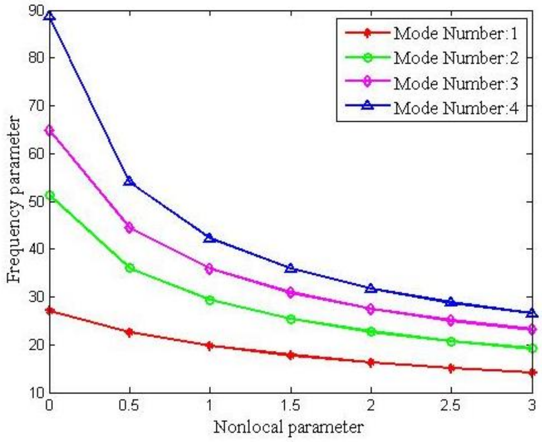

Fig. 13 Variation of frequency ratio with nonlocal parameter

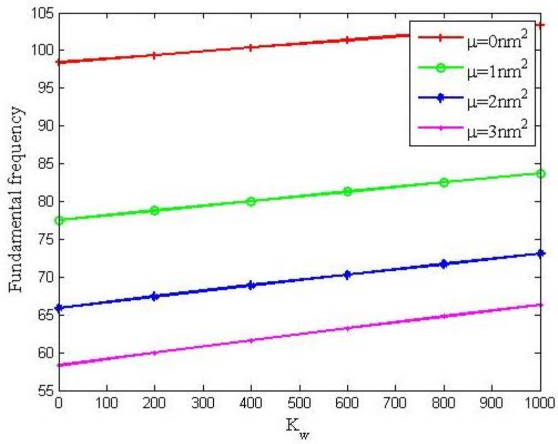

Fig. 14 Effect of Winkler coefficient on frequency parameter

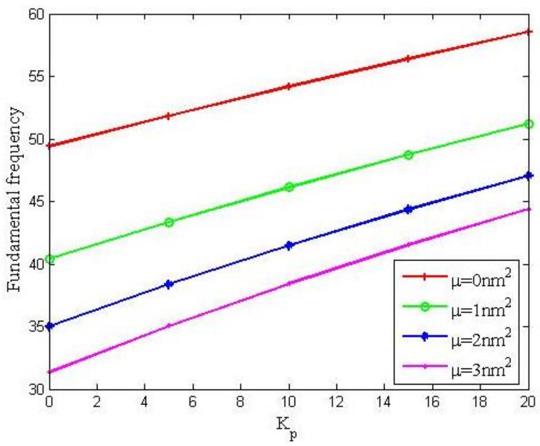

Fig. 15 Effect of Pasternak coefficient on frequency parameter

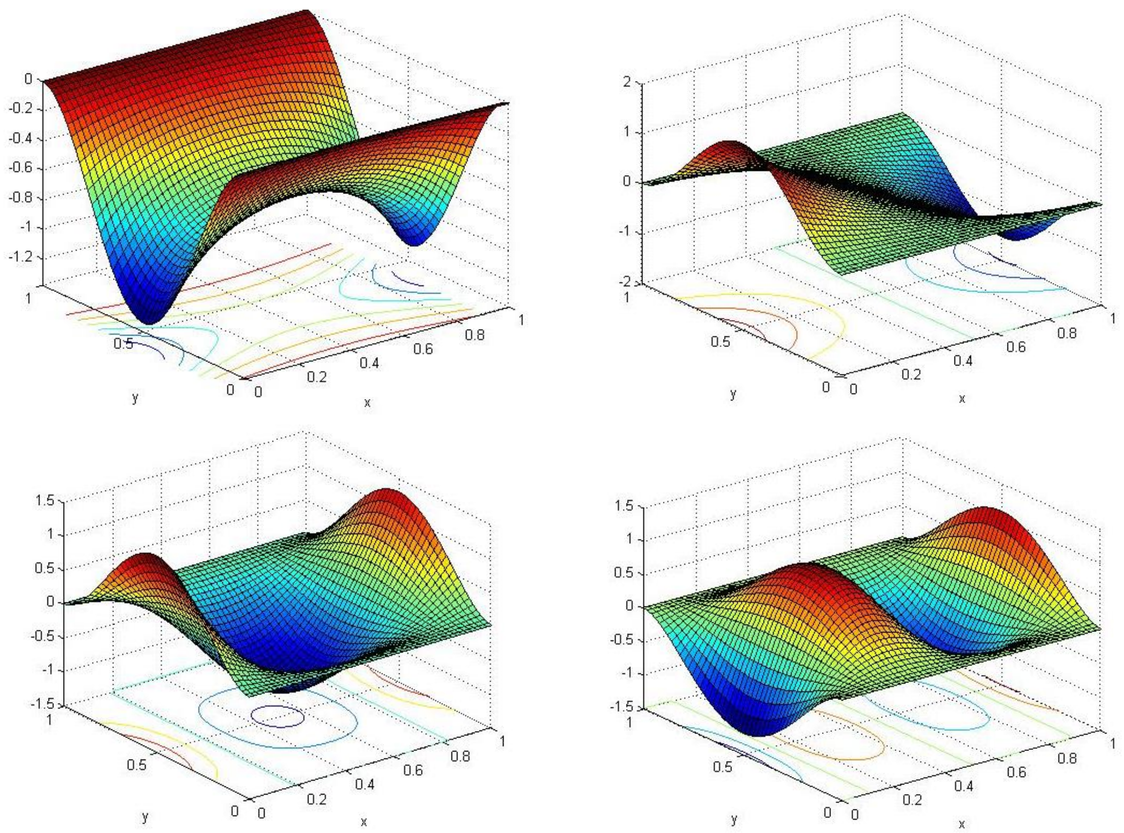

Fig. 16 First four deflection shapes of FCFC nanoplates

taken a=5 nm, R=1, α=0.1, β=0.2, γ=0.3, δ=0.4. It is clearly seen from the figure that frequency ratio is less than unity. This implies that application of local beam model for vibration analysis of graphene sheets would lead to overprediction of the frequency. Hence, nonlocal beam theory should be used for better predictions of frequencies of nanoplates. Neglecting nonlocal effect, the relative error percents for the first and fourth mode number of SCSC nanoplates with μ=0.5 nm^2, R=1, a=5 nm are found to be 15.3605% and 37.8513% respectively. This shows nonlocal effect on the frequency parameters is more in higher modes.

4.7 Effect of elastic foundation

In this subsection, effect of two-parameter Winkler-Pasternak elastic foundation on the fundamental frequency parameter of embedded nanoplate has been investigated. One may note from Eq. (11) that the effects of elastic foundation enter through the stiffness matrix of the nanoplate i.e., [K]. Therefore, total stiffness of the embedded nanoplate increases as the stiffness of the elastic foundation increases. This trend has been shown in Figs. 14 and 15 for for the springy and shear effect of the elastic foundation respectively. Numerical values of the parameters are taken as α=β=γ=δ=0.1, a=10 nm, R=2. In Fig. 14, we have considered K_p=0 with CCCC edge condition while in Fig. 15, we have taken K_w=0with SSSS edge condition. Results have been given

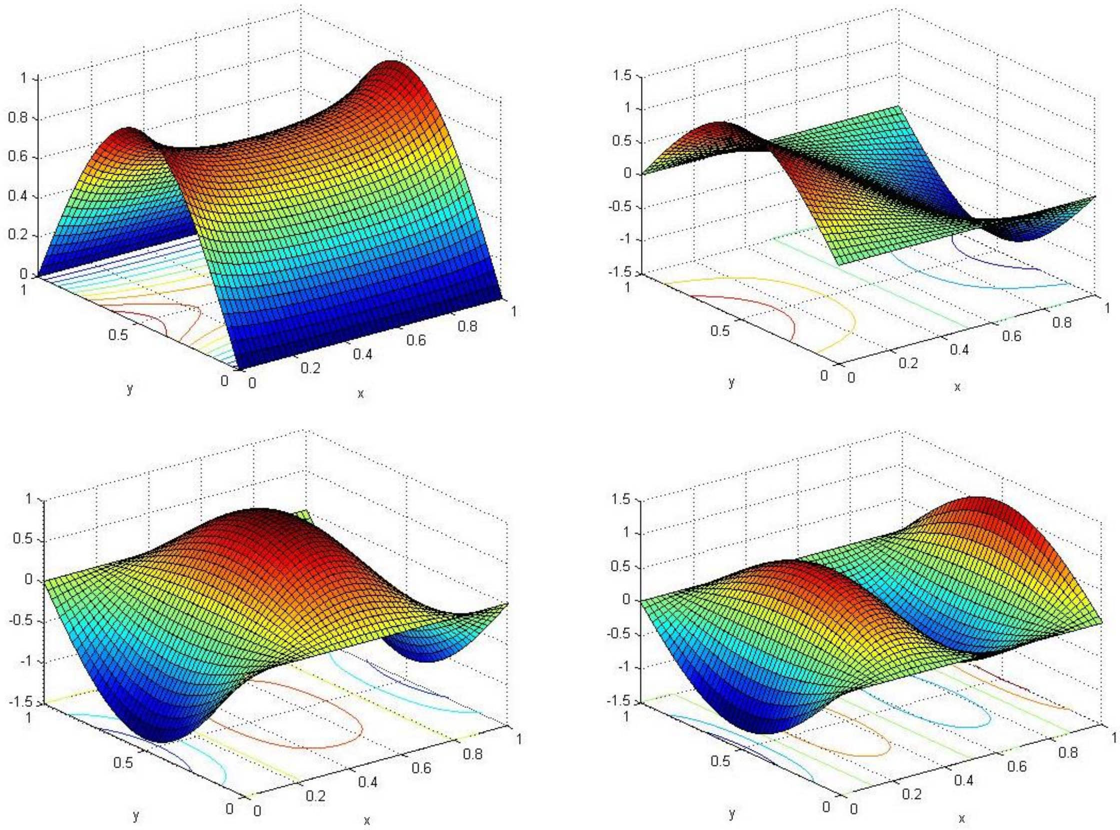

Fig. 17 First four deflection shapes of FSFS nanoplates

for different values of nonlocal parameters. It is observed from these figures that fundamental frequency parameter increases linearly by increasing the stiffness of the elastic foundation either through the springy (Winkler coefficient) or the shear effect (Pasternak coefficient).

Knowledge of higher modes is necessary before finalizing design of engineering systems. As such, first four mode shapes of FCFC and FSFS nanoplates are given respectively in Figs. 16-17 with μ=1 nm^2, a=10 nm, R=2, $\alpha=\beta=\gamma=\delta$=0.1. Results have been given for without considering elastic foundation. This mode shapes will help design engineers and also researchers of nanotechnology.

5. Conclusions

Two-dimensional polynomials have been used as shape functions in the Rayleigh-Ritz method to study free vibration of embedded isotropic rectangular nanoplates based on classical plate theory. Effects of non-uniform parameters, elastic foundation, boundary condition, aspect ratio and length of nanoplates on the frequency parameters have been investigated. It is found that frequency parameters decrease with increase in nonlocal parameter. It is also observed that frequency parameters increase with length, Winkler and Pasternak coefficients and also with aspect ratio.

Nonlocal elasticity theory should be considered for vibration of nanoplates having high aspect ratio. Similarly, nonlocal elasticity theory should also be considered for vibration of nanoplates having small length. One of the important observation seen that nonlocal effect is more in higher modes. Present method may be applicable to any sets of boundary conditions at the edges and may be extended to other plate theories.

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