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A new analytical approach for determination of flexural, axial and torsional natural frequencies of beams

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Abstract. In this paper, a new and simplified method is presented in which the natural frequencies of the uniform and non-uniform beams are calculated through simple mathematical relationships. The various vibration problems such as: Rayleigh beam under variable axial force, axial vibration of a bar with and without end discrete spring, torsional vibration of a bar with an attached mass moment of inertia, flexural vibration of the beam with laterally distributed elastic springs and also flexural vibration of the beam with effects of viscose damping are investigated. The governing differential equations are first obtained and then; according to a harmonic vibration, are converted into single variable equations in terms of location. Through repetitive integrations, the governing equations are converted into weak form integral equations. The mode shape functions of the vibration are approximated using a power series. Substitution of the power series into the integral equations results in a system of linear algebraic equations. The natural frequencies are determined by calculation of a non-trivial solution for system of equations. The efficiency and convergence rate of the current approach are investigated through comparison of the numerical results obtained with those obtained from other published references and results of available finite element software.

Keywords: natural frequency; Rayleigh beam; axial vibration; torsional vibration; flexural vibration; weak form integral equation

1. Introduction

The vibration of continuous systems is always encountered in engineering practices. According to the history of structural dynamics, Bernoulli-Euler, Rayleigh and Timoshenko beams theories has been proposed for characterization of elastic beams vibration. The classical Bernoulli-Euler theory of flexural vibrations is characterized by giving higher natural frequencies than those obtained by experiments on thick beams, especially for higher modes. Lord Rayleigh (1877) improved the classical theory by considering the effect of rotational inertia of the cross-section. Later, Timoshenko (1921, 1922) introduced the effect of transverse shear deformation. For slender beams the effects of the shear deformation and rotational inertia are small and can be neglected. Bernoulli-Euler theory provides good results for the fundamental frequency of slender beams; nevertheless Timoshenko and Rayleigh theories also provide good results. But for short beams the effects of shear deformation and rotational inertia are significant. For such beams, Timoshenko

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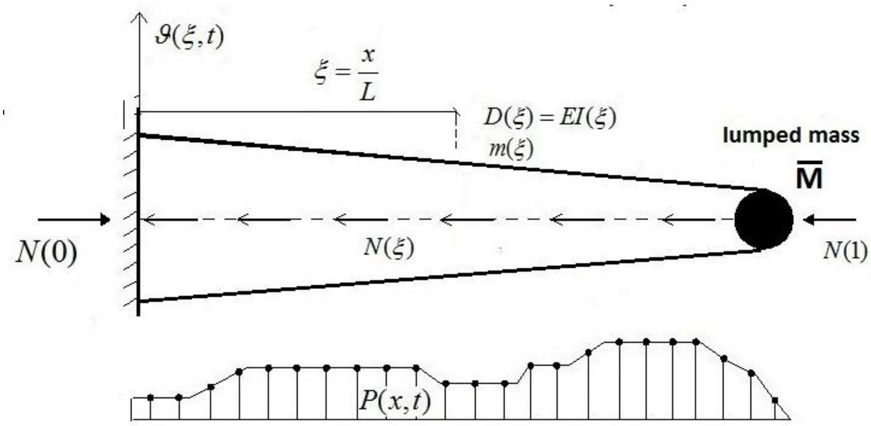

Fig. 1 Non-uniform Rayleigh beam with a lumped mass under variable axial forces and transverse forces

beam theory yields the best predictive results. The subject of free vibration analysis of Timoshenko and Bernoulli beams has been paid attention by many researchers. However, few of them have investigated Rayleigh beam theory. Lateral vibration of a beam is governed by a partial differential equation that is of the fourth order in the spatial variable and second order in time. Axial and torsional vibrations of the beam are governed by a partial differential equation which is called the "wave equation" that is of the second order in the spatial variable and second order in time. In this paper, a new analytical approach is presented for conversion of the governing differential equations of all mentioned vibration problems into solvable ones. Corresponding vibration frequencies are determined as well.

By seeking a non-trivial solution of the integral equation, the natural frequencies of a cantilever Rayleigh beam with axial force and tip mass has been calculated (Li *et al.* 2013). By conversion of the governing differential equations into weak form integral equations, the natural frequencies of the non-prismatic Bernoulli beam under variable axial force has been obtained (Saffari *et al.* 2012). Using Differential Transform Method and Numerical Assembly Technique, the exact natural frequencies and mode shapes of the axial-loaded Timoshenko multiple-step beam has been calculated by Yesilce (2015). The vibration analysis of rotating Timoshenko beams by means of the differential quadrature method has been investigated (Bambill *et al.* 2010). A boundary element method has been developed for the general flexural-torsional buckling analysis of Timoshenko beams of arbitrarily shaped cross section by Sapountzakis and Dourakopoulos (2010). Transverse vibration of Euler-Bernoulli beams carrying concentrated masses with rotatory inertia has been studied (Maiz *et al.* 2007). The natural frequencies of a shaft with non-uniform cross section and various kinds of end conditions have been calculated by Pouyet and Lataillade (1981). Using Rayleigh quotient, the axial vibration frequencies of the non-prismatic beams have been calculated (Chalah *et al.* 2014). The first five natural frequencies and mode shapes of a Timoshenko multi-span beam subjected to the axial force have been obtained and the effects of attached spring-mass systems on the free vibration characteristics multi-span beams have been studied by Yesilce and Demirdag (2008). Using the dynamic stiffness method, the natural frequencies of a rotating tapered Rayleigh beam have been calculated by Banerjee and Jackson (2013). A boundary element method (BEM) has been developed for the non-uniform torsion of simply or multiply connected cylindrical bars of arbitrary cross-section by Saapountzakis (2000). The analysis of a rotating tapered cantilever with centrifugal force has been formulated (Wright *et al.* 1982, Yan *et al.* 2011). Torsional vibration of multi-step non-uniform rods with various concentrated elements has been investigated by Li (2002). The free vibration of Bernoulli, Rayleigh and Timoshenko uniform/non-uniform beams with/without effects of axial forces have been investigated (Hijmissen and Horssen 2008, Pradhan and Chakraverty, 2013, Stojanovic and Kozic 2012, Huang *et al.* 2013, Huang and Li 2010).

2. Cantilever Rayleigh beam with a lumped mass under variable axial force

2.1 Conversion of the governing differential equation to its weak form

The governing differential equation for vibration of a non-uniform Rayleigh beam under transverse forces (Fig. 1) is given by (Li *et al.* 2013)

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2}{\partial x^2}\vartheta(x,t)\right]+m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t)+\frac{\partial}{\partial x}\left[N(x)\frac{\partial}{\partial x}\vartheta(x,t)\right]-\frac{\partial}{\partial x}\left[\rho I(x)\frac{\partial}{\partial x}\left(\frac{\partial^2\vartheta(x,t)}{\partial t^2}\right)\right]=P(x,t) \quad (1)$$

In which $D(x)=EI(x)$ is bending stiffness which depends on both young's modulus E and the inertial moment of cross-sectional area $I(x)$. $N(x)$, $m(x)$, $\vartheta(x,t)$, ρ and $P(x,t)$ are the axial force, the mass per unit length, transverse displacement, density of the beam and transverse force, respectively. Axial force includes a concentrated axial force at free end of the beam and variably distributed axial force. Setting $p(x,t)=0$, the free vibration equation is obtained. If motion is represented by a harmonic vibration, the transverse displacement is obtained using the following relation

$$\vartheta(x,t)=\phi(x)e^{\mathrm{i}\omega t} \quad (2)$$

Where $\phi(x)$ and ω are mode shape function and natural frequency of the beam, respectively. It is assumed that the beam has a constant cross section. Hence, the functions $I(x)\equiv constant=I$ and $A(x)\equiv constant=A$ are applied. Substitution of relationship (2) into Eq. (1) leads to a single-variable equation in terms of location, as follows

$$EI\frac{\mathrm{d}^4\phi}{\mathrm{d}x^4}-\rho A\omega^2\phi(x)+\frac{\mathrm{d}}{\mathrm{d}x}[N(x)\frac{\mathrm{d}\phi}{\mathrm{d}x}]+\rho\mathrm{I}\omega^2\frac{\mathrm{d}^2\phi}{\mathrm{d}x^2}=0 \qquad 0\le x\le L \quad (3)$$

In which L is the beam length. For further convenience, the following variables are introduced

$$\xi=\frac{x}{L}, \qquad \Omega=\omega L^2\sqrt{\frac{\rho A}{EI}}, \quad r=\frac{1}{L}\sqrt{\frac{I}{A}}, \quad \alpha(\xi)=\frac{N(\xi)L^2}{EI} \quad (4)$$

Substitution of variables (4) into Eq. (3) leads to

$$\frac{\mathrm{d}^4\phi}{\mathrm{d}\xi^4}-\Omega^2\phi(\xi)+\frac{\mathrm{d}}{\mathrm{d}\xi}[\alpha(\xi)\frac{\mathrm{d}\phi}{\mathrm{d}\xi}]+\Omega^2r^2\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}=0 \qquad 0\le\xi\le 1 \quad (5)$$

In which $\phi(\xi)=\frac{\phi(x)}{L}$ is applied. Eq. (5) is, in fact, the free vibration equation of a Rayleigh beam under variable axial forces based on the non-dimensional variable ξ. In order to transform Eq. (5) to its weak form, both sides of Eq. (5) are integrated twice with respect to ξ within the range 0 to ξ. The resulting integral equations are as follows

$$\frac{\mathrm{d}^3\phi}{\mathrm{d}\xi^3} - \Omega^2 \int_0^{\xi} \phi(s)\mathrm{d}s + \Omega^2 r^2 \frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \alpha(\xi)\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = C_1 \tag{6}$$

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \alpha(\xi)\phi(\xi) + \Omega^2 r^2 \phi(\xi) - \int_0^{\xi} \left[\Omega^2(\xi - s) + \alpha'(s)\right]\phi(s)\mathrm{d}s = C_1\xi + C_2 \tag{7}$$

Further, integration from both sides of Eq. (7) twice with respect to ξ from 0 to ξ yields

$$\begin{aligned}\frac{\mathrm{d}\phi}{\mathrm{d}\xi} + \int_0^{\xi}\left[-\frac{\Omega^2}{2}(\xi - s)^2 + \alpha(s) - (\xi - s)\alpha'(s) + \Omega^2 r^2\right]\phi(s)\mathrm{d}s \\ = \frac{C_1}{2}\xi^2 + C_2\xi + C_3\end{aligned} \tag{8}$$

$$\begin{aligned}\phi(\xi) + \int_0^{\xi}\left[-\frac{\Omega^2}{6}(\xi - s)^3 + (\xi - s)\alpha(s) - \frac{(\xi - s)^2}{2}\alpha'(s) + \Omega^2 r^2(\xi - s)\right]\phi(s)\mathrm{d}s \\ = \frac{C_1}{6}\xi^3 + \frac{C_2}{2}\xi^2 + C_3\xi + C_4\end{aligned} \tag{9}$$

In Eq. (9) C_1, C_2, C_3 and C_4 are the integration constants which are determined through boundary conditions of both ends of the beam. Eq. (9) is the integral equation of the weak form for free vibration of a Rayleigh beam under variable axial force. Eqs. (6)-(9) are applicable for determination of the integration constants. Further substitution of the resulting integration constants into Eq. (9) yields an integral equation in $\phi(\xi)$.

2.2 Boundary conditions

For a Rayleigh beam under variable axial force, including a concentrated axial force at the end of the beam and a variably distributed axial force, the beam rotation (θ), the bending moment (M) and the shear force (V) can be stated by the following relations (Li *et al.* 2013)

$$\begin{cases} \theta(x,t) = \dfrac{\partial}{\partial x}\vartheta(x,t) \\ M(x,t) = EI\dfrac{\partial^2}{\partial x^2}\vartheta(x,t) \\ V(x,t) = EI\dfrac{\partial^3}{\partial x^3}\vartheta(x,t) + N(x)\dfrac{\partial}{\partial x}\vartheta(x,t) - \rho I\dfrac{\partial^3}{\partial x\partial t^2}\vartheta(x,t) \end{cases} \tag{10}$$

Regarding the relationship (2), the relations (10) can be expressed as follows

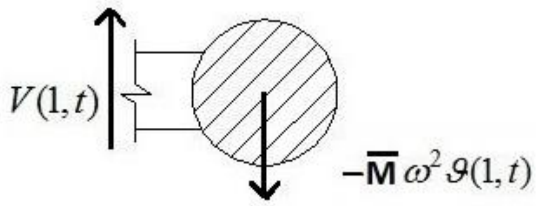

Fig. 2 shear force acting on the end mass of the cantilever beam

$$\begin{cases} \theta(\xi,t)=\dfrac{\mathrm{d}}{L\mathrm{d}\xi}\phi(\xi)e^{i\omega t} \\ M(\xi,t)=\left[\dfrac{EI}{L^2}\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)\right]e^{i\omega t} \\ V(\xi,t)=\left[\dfrac{EI}{L^3}\dfrac{\mathrm{d}^3}{\mathrm{d}\xi^3}\phi(\xi)+\dfrac{N(\xi)}{L}\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)+\dfrac{\omega^2\rho I}{L}\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]e^{i\omega t} \end{cases} \tag{11}$$

it has been assumed that the Rayleigh cantilever beam has a lumped mass at free end. Therefore, the shear force is of non-zero value at the free end of the beam (Fig. 2).

Regarding the variables introduced in relations (4), the boundary condition of the shear force at free end of the beam is obtained as follows (Clough and Penzien 1975)

$$\xi=1,\qquad V(1,t)=\left[\overline{M}\,\frac{\partial^2\vartheta(\xi,t)}{\partial t^2}\right]_{\xi=1}$$

or

$$\left[\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\phi(\xi)+\alpha(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)+\Omega^2r^2\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]_{\xi=1}e^{i\omega t}=\left[-\beta\Omega^2\phi(1)\right]e^{i\omega t} \tag{12}$$

$$e^{i\omega t}\neq 0\quad\rightarrow\quad\left[\frac{\mathrm{d}^3}{\mathrm{d}\xi^3}\phi(\xi)+\alpha(\xi)\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)+\Omega^2r^2\frac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]_{\xi=1}=-\beta\Omega^2\phi(1)$$

In which $\overline{M}$ is the lumped mass at free end of the beam and $\beta=\dfrac{\overline{M}}{\rho AL}$ is applied. The other boundary conditions are established as follows

$$\begin{cases} \xi=0,\qquad \vartheta(0,t)=0,\quad \phi(0)e^{i\omega t}=0\ \rightarrow\ \phi(0)=0 \\ \xi=0,\qquad \theta(0,t)=0\quad or\quad \left[\dfrac{\mathrm{d}}{L\mathrm{d}\xi}\phi(\xi)\right]_{\xi=0}e^{i\omega t}=0\rightarrow\left[\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]_{\xi=0}=0 \\ \xi=1,\qquad M(1,t)=0\quad or\quad \left[\dfrac{EI}{L^2}\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)\right]_{\xi=1}e^{i\omega t}=0\ \rightarrow\left[\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)\right]_{\xi=1}=0 \end{cases} \tag{13}$$

Substituting ϕ=0 into (9) as well as $\frac{d\phi}{d\xi}=0$ into (8) and setting ξ=0 leads to

$$C_3 = C_4 = 0 \tag{14}$$

Similarly, Substitution of $V(1,t)=\left[\overline{M}\,\frac{\partial^2 \vartheta(\xi,t)}{\partial t^2}\right]_{\xi=1}$ into (6) and M=0 into (7) as well as setting ξ=1, yields, respectively

$$-\beta\Omega^2\phi(1)-\Omega^2\int_0^1 \phi(s)\mathrm{d}s = C_1 \tag{15}$$

$$\left[\alpha(1)+\Omega^2 r^2\right]\phi(1)-\int_0^1\left[\Omega^2(1-s)+\alpha'(s)\right]\phi(s)\mathrm{d}s = C_1 + C_2 \tag{16}$$

As it’s obvious in Eqs. (15) and (16), $\phi(1)$ is initially unknown. Therefore, it necessitates extra equation for uniquely determination of C_1 and C_2. Setting C_3=C_4=0and ξ=1 in Eq. (9) yields

$$\phi(1)=\int_0^1\left[\frac{\Omega^2}{6}(1-s)^3-(1-s)\alpha(s)+\frac{(1-s)^2}{2}\alpha'(s)-\Omega^2 r^2(1-s)\right]\phi(s)\mathrm{d}s+\frac{C_1}{6}+\frac{C_2}{2} \tag{17}$$

Elimination of $\phi(1)$ from Eqs. (15), (16) and (17), results in the coefficients C_1 and C_2 to be determined by the following relations

$$C_1=\int_0^1\left[g_2(s)-\left(\frac{\psi_1}{\psi_2\psi}\right)g_1(s)\right]\phi(s)ds \tag{18}$$

$$C_2=\int_0^1\left[\frac{\mathrm{g}_1(s)}{\psi}\right]\phi(s)ds \tag{19}$$

In which

$$\begin{aligned}
&g_1(s)=g_3(s)+\frac{\psi_3}{6}g_2(s)-g_2(s)\\
&g_2(s)=\frac{\beta\Omega^2 g_5(s)-\Omega^2}{\psi_2}\\
&g_3(s)=-g_4(s)-\psi_3 g_5(s)\\
&g_4(s)=\Omega^2(1-s)+\alpha'(s)\\
&g_5(s)=-\frac{\Omega^2}{6}(1-s)^3+(1-s)\alpha(s)-\frac{(1-s)^2}{2}\alpha'(s)+\Omega^2 r^2(1-s)
\end{aligned} \tag{20}$$

where

$$\begin{cases} \psi = \dfrac{\psi_1\psi_3}{6\psi_2} - \dfrac{\psi_3}{2} - \dfrac{\psi_1}{\psi_2} + 1 \\ \psi_1 = \dfrac{\beta\Omega^2}{2} \\ \psi_2 = \dfrac{\beta\Omega^2}{6} + 1 \\ \psi_3 = \Omega^2 r^2 + \alpha(1) \end{cases} \tag{21}$$

Substitution of the integration constants into (9) yields an integral equation as follows

$$\phi(\xi) + \int_0^{\xi} f_1(\xi,s)\phi(s)\mathrm{d}s + \int_0^{1} f_2(\xi,s)\phi(s)\mathrm{d}s = 0 \tag{22}$$

In Eq. (22), functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations

$$\begin{cases} f_1(\xi,s) = -\dfrac{\Omega^2}{6}(\xi-s)^3 + (\xi-s)\alpha(s) - \dfrac{(\xi-s)^2}{2}\alpha'(s) + \Omega^2 r^2(\xi-s) \\ f_2(\xi,s) = \left(\dfrac{\psi_1}{6\psi_2\psi}\right)\mathrm{g}_1(s)\xi^3 - \dfrac{g_2(s)}{6}\xi^3 - \dfrac{\mathrm{g}_1(s)}{2\psi}\xi^2 \end{cases} \tag{23}$$

3. Free vibration in axial deformation

3.1 Case1: cantilever beam

The analysis of free vibration associated with axial motion of a bar can be carried out in a manner similar to the case of flexural vibration of Rayleigh beam. The free vibration equation of motion is

$$\frac{\partial}{\partial x}[EA(x)\frac{\partial}{\partial x}\vartheta(x,t)] - m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t) = 0 \tag{24}$$

In which $EA(x)$ and $\vartheta(x,t)$ are the axial stiffness and longitudinal displacement of the beam. Using the solution

$$\vartheta(x,t) = \phi(x)e^{i\omega t} \tag{25}$$

And introducing the following variables

$$\xi = \frac{x}{L}, \qquad \lambda = \omega^2 L^2 \tag{26}$$

Eq. (24) can be written in the form

$$\frac{\mathrm{d}}{\mathrm{d}\xi}[EA(\xi)\frac{\mathrm{d}\phi}{\mathrm{d}\xi}] + \lambda m(\xi)\phi(\xi) = 0, \qquad 0 \le \xi \le 1 \tag{27}$$

In order to transform Eq. (27) to its weak form, both sides of Eq. (27) are integrated twice with respect to ξ within the range 0 to ξ. The resulting integral equations are as follows

$$EA(\xi)\frac{\mathrm{d}\phi}{\mathrm{d}\xi}+\lambda\int_0^{\xi} m(s)\phi(s)ds = C_1 \tag{28}$$

$$EA(\xi)\phi(\xi)+\int_0^{\xi}\left[\lambda(\xi-s)m(s)-EA'(\mathrm{s})\right]\phi(s)ds = C_1\xi + C_2 \tag{29}$$

3.1.1 Boundary conditions

For axial vibration of a non-prismatic cantilever beam the following boundaries conditions are introduced (Clough and Penzien 1975)

$$\begin{cases} \xi = 0, \quad \vartheta(0,t)=0, \quad or \quad \phi = 0 \\ \xi = 1, \quad N(1,t)=0 \quad or \quad \left[EA(\xi)\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]_{\xi=1} e^{i\omega t} = 0 \quad \rightarrow \left[EA(\xi)\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]_{\xi=1} = 0 \end{cases} \tag{30}$$

In which $N(1,t)$ is the axial force acting on the free end of the beam. Substituting ϕ=0 into (29) and setting ξ=0 as well as substituting $N(1,t)$=0 into (28) and setting ξ=1 yields, respectively

$$C_1 = \lambda\int_0^1 m(s)\phi(s)ds \tag{31}$$

$$C_2 = 0 \tag{32}$$

Substitution of the integration constants into (29) yields an integral equation as follows

$$EA(\xi)\phi(\xi)+\int_0^{\xi} f_1(\xi,s)\phi(s)\mathrm{d}s+\int_0^1 f_2(\xi,s)\phi(s)\mathrm{d}s = 0 \tag{33}$$

In Eq. (33), functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations

$$\begin{cases} f_1(\xi,s) = \lambda(\xi-s)m(s)-EA'(s) \\ f_2(\xi,s) = -\lambda\xi m(s) \end{cases} \tag{34}$$

3.2 Case2: cantilever beam with end discrete spring

In this section, it is assumed that there is a discrete spring at the free end of the cantilever beam (Fig. 2). The analysis method is exactly the same as what was stated in section 3.1. The difference, however, is that, the axial force is of non-zero value at the free end of the beam. Other boundary condition is assumed unchanged. In this case, the boundary conditions are stated as follows (Kelly 2007)

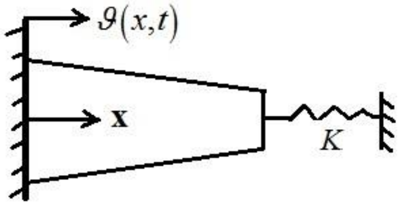

Fig. 2 axial vibration of the non-prismatic cantilever beam with end discrete spring

$$\begin{cases} \xi = 0, \qquad \vartheta(0,t) = 0, \quad or \quad \phi = 0 \\ x = L, \qquad \left[EA(x) \dfrac{\partial}{\partial x} \vartheta(x,t) \right]_{x=L} + K\vartheta(L,t) = 0 \quad or \quad \left[EA(\xi) \dfrac{\mathrm{d}}{\mathrm{d}\xi} \phi(\xi) \right]_{\xi=1} = -KL\phi(1) \end{cases} \tag{35}$$

In which K is spring stiffness. If the boundary conditions introduced in (35) are applied to Eqs. (28)-(29), the integration constants are determined. Introducing the integration constants into Eq. (29) yields the integral equation as follows

$$EA(\xi)\phi(\xi) + \int_0^{\xi} f_1(\xi,s)\phi(s)\mathrm{d}s + \int_0^1 f_2(\xi,s)\phi(s)\mathrm{d}s = 0 \tag{36}$$

In which the functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations

$$\begin{cases} f_1(\xi,s) = \lambda(\xi - s)m(s) - EA'(s) \\ f_2(\xi,s) = \left(\dfrac{KLE}{EA(1)+KL} \right) A'(s)\xi - \left(\dfrac{\lambda EA(1)}{EA(1)+KL} \right) m(s)\,\xi - \left(\dfrac{KL\lambda}{EA(1)+KL} \right) (1-s)m(s)\xi \end{cases} \tag{37}$$

4. Torsional vibration with an attached mass moment of inertia

The governing differential equation for free torsional vibration of a bar is as follows

$$\frac{\partial}{\partial x}\left[JG \frac{\partial}{\partial x} \theta(\mathrm{x},\mathrm{t}) \right] - \rho J \frac{\partial^2}{\partial t^2} \theta(\mathrm{x},\mathrm{t}) = 0 \qquad 0 \le x \le L \tag{38}$$

Where G, J, ρ and $\theta(x,t)$ are the shear modulus, polar moment of inertia of the cross section, density of the bar and angular displacement of the bar, respectively. Using the solution $\theta(x,t)=\phi(x)e^{i\omega t}$ and introducing the variables: $\xi = \dfrac{x}{L}$ and $\Omega = \dfrac{\omega^2 L^2 \rho}{G}$ the following differential equation is obtained

$$\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2} + \Omega\phi(\xi) = 0, \qquad 0 \le \xi \le 1 \tag{39}$$

Both sides of Eq. (39) are integrated twice with respect to ξ within the range 0 to ξ. The resulting integral equations are as follows

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi}+\Omega\int_0^{\xi}\phi(s)ds=C_1 \tag{40}$$

$$\phi(\xi)+\Omega\int_0^{\xi}(\xi-s)\phi(s)ds=C_1\xi+C_2 \tag{41}$$

The boundary conditions of a cantilever bar with an attached mass moment of inertia at the end are as follows (Kelly 2007)

$$\begin{cases} \xi=0, \quad \theta(0,t)=0, \quad or \quad \phi=0 \\ \xi=1, \quad \left[GJ\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]_{\xi=1}=I_m\omega^2 L\phi(1) \end{cases} \tag{42}$$

In which I_m is attached mass moment of inertia. By introducing $\eta=\dfrac{I_m}{\rho JL}$ and applying the boundary conditions into Eqs. (40)-(41), the following equations are obtained for C_1 and C_2

$$\begin{cases} C_1=\eta\Omega\phi(1)+\Omega\displaystyle\int_0^1\phi(s)ds \\ C_2=0 \\ \phi(1)+\Omega\displaystyle\int_0^1(1-s)\phi(s)ds=C_1 \end{cases} \tag{43}$$

By elimination of $\phi(1)$ from relations (43), the coefficients C_1 and $C_2$are determined. Substitution of the integration constants into Eq. (41) yields an integral equation as follows

$$\phi(\xi)+\int_0^{\xi}f_1(\xi,s)\phi(s)\mathrm{d}s+\int_0^1 f_2(\xi,s)\phi(s)\mathrm{d}s=0 \tag{44}$$

In which the functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations

$$\begin{cases} f_1(\xi,s)=\lambda(\xi-s) \\ f_2(\xi,s)=\left(\dfrac{\eta\lambda^2}{1-\eta\lambda}\right)(1-s)\xi-\left(\dfrac{\lambda}{1-\eta\lambda}\right)\xi \end{cases} \tag{45}$$

5. Flexural vibration with distributed elastic support

In this section, the free vibration of flexural beam which supported transversely by distributed elastic springs of the type shown in Fig. (3) is investigated.

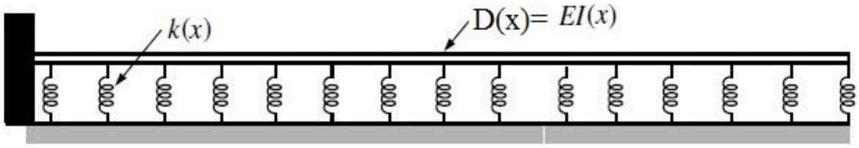

Fig. 3 Flexural cantilever beam supported transversely by distributed elastic springs

The free vibration equation of motion for this system is given by (Clough and Penzien 1975)

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2}{\partial x^2}\vartheta(x,t)\right]+m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t)+K(x)\vartheta(x,t)=0 \tag{46}$$

In which $D(x)=EI(x)$ and $K(x)$ are bending stiffness and stiffness of the distributed springs, respectively. Similar to previous sections, the transverse displacement of the beam is assumed as

$$\vartheta(x,t)=\phi(x)e^{i\omega t} \tag{47}$$

And also for further convenience, the following variables are introduced

$$\xi=\frac{x}{L},\qquad \lambda=\omega^2L^4 \tag{48}$$

Substituting relations (47)-(48) into Eq. (46) leads to

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[D(\xi)\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}\right]+\overline{m}(\xi)\phi(\xi)=0,\qquad 0\le\xi\le1 \tag{49}$$

In which: $\overline{m}(\xi)=L^4K(\xi)-\lambda m(\xi)$.

Both sides of Eq. (49) are integrated twice with respect to ξ within the range 0 to ξ. The resulting integral equations are as follows

$$\frac{d}{d\xi}\left[D(\xi)\frac{d^2}{d\xi^2}\phi(\xi)\right]+\int_0^\xi\overline{m}(s)\phi(s)ds=C_1 \tag{50}$$

$$D(\xi)\frac{d^2}{d\xi^2}\phi(\xi)+\int_0^\xi(\xi-s)\overline{m}(s)\phi(s)ds=C_1\xi+C_2 \tag{51}$$

Further, integration from both sides of Eq. (51) twice with respect to ξ from 0 to ξ yields

$$D(\xi)\frac{d}{d\xi}\phi(\xi)-D'(\xi)\phi(\xi)+\int_0^\xi\left[D''(s)+\frac{(\xi-s)^2}{2}\overline{m}(s)\right]\phi(s)ds=\frac{C_1}{2}\xi^2+C_2\xi+C_3 \tag{52}$$

$$D(\xi)\phi(\xi)+\int_0^\xi\left[(\xi-s)D''(s)-2D'(s)+\frac{(\xi-s)^3}{6}\overline{m}(s)\right]\phi(s)ds=\frac{C_1}{6}\xi^3+\frac{C_2}{2}\xi^2+C_3\xi+C_4 \tag{53}$$

Eqs. (50)-(53) are used for determination of the integration constants C_1, C_2, C_3 and C_4.

5.1 Boundary conditions

The boundary conditions are established as follows

$$\begin{cases} \xi = 0, \quad \phi = 0 \\ \xi = 0, \quad \theta = 0 \quad \textit{or} \quad \left[\dfrac{\mathrm{d}}{\mathrm{d}\xi}\phi(\xi)\right]_{\xi=0} = 0 \\ \xi = 1, \quad V = 0 \quad \textit{or} \quad \left[\dfrac{\mathrm{d}}{\mathrm{d}\xi}\left(D(\xi)\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)\right)\right]_{\xi=1} = 0 \\ \xi = 1, \quad M = 0 \quad \textit{or} \quad \left[D(\xi)\dfrac{\mathrm{d}^2}{\mathrm{d}\xi^2}\phi(\xi)\right]_{\xi=1} = 0 \end{cases} \tag{54}$$

Substituting ϕ=0 into (53) as well as ϕ=θ=0 into (52) and setting ξ=0 leads to

$$C_3 = C_4 = 0 \tag{55}$$

Similarly, Substitution of V=0 into (50) and M=0 into (51) as well as setting ξ=1, yields, respectively

$$\int_0^1 \overline{m}(s)\phi(s)\mathrm{d}s = C_1 \tag{56}$$

$$\int_0^1 \left[-s\overline{m}(s)\right]\phi(s)\mathrm{d}s = C_2 \tag{57}$$

Substitution of the integration constants into (53) yields an integral equation as follows

$$D(\xi)\phi(\xi) + \int_0^\xi f_1(\xi,s)\phi(s)\mathrm{d}s + \int_0^1 f_2(\xi,s)\phi(s)\mathrm{d}s = 0 \tag{58}$$

In Eq. (58), functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are expressed by the following relations

$$\begin{cases} f_1(\xi,s) = (\xi - s)D''(s) - 2D'(s) + \dfrac{(\xi - s)^3}{6}\overline{m}(s) \\ f_2(\xi,s) = \left(\dfrac{s\xi^2}{2} - \dfrac{\xi^3}{6}\right)\overline{m}(s) \end{cases} \tag{59}$$

6. Flexural vibration including viscose damping

In this section, the governing differential equation for free vibration of a non-prismatic cantilever beam including viscose damping is converted to its weak form. The governing differential equation is obtained as follows

$$\frac{\partial^2}{\partial x^2}\left[D(x)\frac{\partial^2}{\partial x^2}\vartheta(x,t)\right]+m(x)\frac{\partial^2}{\partial t^2}\vartheta(x,t)+C(x)\frac{\partial}{\partial t}\vartheta(x,t)=0 \tag{60}$$

In Eq. (60) $C(x)=a_0\ m(x)$ is damping resistance per unit velocity which depends on both coefficient a_0 and mass per unit length $m(x)$. a_0 is called mass proportional damping coefficient and can be calculated as follows

$$\mathrm{a}_0 = 2\zeta_n\omega_n \tag{61}$$

In relation (61) ζ_n and ω_n are damping ratio and natural frequency corresponding to nth mode. The lateral displacement of the beam is assumed as a harmonic vibration as follows

$$\vartheta(x,t)=\phi(x)e^{\alpha t} \tag{62}$$

In which α is an unknown complex coefficient. Substitution of relationship (62) into Eq. (60) and assumption of $\xi=\dfrac{x}{L}$ leads to

$$\frac{\mathrm{d}^2}{\mathrm{d}\xi^2}\left[D(\xi)\frac{\mathrm{d}^2\phi}{\mathrm{d}\xi^2}\right]+\overline{m}(\xi)\phi(\xi)=0, \qquad 0\le\xi\le 1 \tag{63}$$

In which $\overline{m}(\xi)=\alpha^2L^4m(\xi)+\alpha L^4C(\xi)$. As can be seen, Eq. (63) is similar to Eq. (49) and also, since viscose damping not changes the boundary conditions of flexural beam, the relations (59) calculated in section (5) can be used for determination of unknown coefficient α. Of course it must be noted that $\overline{m}(\xi)=L^4K(\mathrm{s})-\lambda m(s)$ in section (5) is replaced by $\overline{m}(\xi)=\alpha^2L^4m(\xi)+\alpha L^4C(\xi)$ obtained in this section.

7. Establish the system of linear algebraic equations

In the previous sections, the governing differential equations for free lateral, axial and torsional vibration of the beams were converted into the following integral equations

$$\begin{cases} D(\xi)\phi(\xi)+\displaystyle\int_0^{\xi} f_1(\xi,s)\phi(s)\mathrm{d}s+\int_0^{1} f_2(\xi,s)\phi(s)\mathrm{d}s=0 & \text{for non-uniform beams} \\ \phi(\xi)+\displaystyle\int_0^{\xi} f_1(\xi,s)\phi(s)\mathrm{d}s+\int_0^{1} f_2(\xi,s)\phi(s)\mathrm{d}s=0 & \text{for uniform beams} \end{cases} \tag{64}$$

In which $D(\xi)$ is the stiffness function that is equal to: $EA(\xi)$ for axial vibration, and $EI(\xi)$for lateral vibration. The functions $f_1(\xi,s)$ and $f_2(\xi,s)$ were introduced for each case. The mode shape function $\phi(s)$ is the unknown parameter in the integral Eq. (64). In order to solve the integral Eq. (64) and to determine the corresponding natural frequencies, the mode shape function of the vibration is approximated by the following power series

$$\phi(\xi)=\sum_{r=0}^{R}c_r\xi^r \tag{65}$$

Where C_r are unknown coefficients and R is a given positive integer, which is adopted such that the accuracy of the results is sustained. Introducing Eq. (65) into integral Eq. (64) leads to

$$\sum_{r=0}^{R}\left[D(\xi)\xi^{r}+\int_{0}^{\xi} f_1(\xi,s)s^{r}\mathrm{d}s+\int_{0}^{1} f_2(\xi,s)s^{r}\mathrm{d}s\right]c_r=0 \tag{66}$$

Both sides of (66) are multiplied by ξ^m and integrated subsequently with respect to ξ between 0 and 1. This results in a system of linear algebraic equations in C_r

$$\sum_{r=0}^{R}\left[G(m,r)+F_1(m,r)+F_2(m,r)\right]c_r=0 \qquad m=0,1,2,...,R \tag{67}$$

In which functions $G(m,r)$, $F_1(m,r)$ and $F_2(m,r)$ are expressed as follows

$$\begin{cases} G(m,r)=\int_0^1 \xi^{r+m}D(\xi)\mathrm{d}\xi \\ F_1(m,r)=\int_0^1\int_0^{\xi} f_1(\xi,s)s^{r}\xi^{m}\mathrm{d}s\mathrm{d}\xi \\ F_2(m,r)=\int_0^1\int_0^1 f_2(\xi,s)s^{r}\xi^{m}\mathrm{d}s\mathrm{d}\xi \end{cases} \tag{68}$$

The system of linear algebraic Eq. (67) may be expressed in matrix notations as follows

$$\begin{cases} [A]_{(R+1)\times(R+1)}[C]_{(R+1)\times 1}=[0]_{(R+1)\times 1} \\ [A]=\begin{bmatrix} \left[G(0,0)+F_1(0,0)+F_2(0,0)\right] & \left[G(0,1)+F_1(0,1)+F_2(0,1)\right] & .... & \left[G(0,R)+F_1(0,R)+F_2(0,R)\right] \\ \left[G(1,0)+F_1(1,0)+F_2(1,0)\right] & \left[G(1,1)+F_1(1,1)+F_2(1,1)\right] & .... & \left[G(1,R)+F_1(1,R)+F_2(1,R)\right] \\ . & . & . & . \\ . & . & . & . \\ \left[G(R,0)+F_1(R,0)+F_2(R,0)\right] & \left[G(R,1)+F_1(R,1)+F_2(R,1)\right] & .... & \left[G(R,R)+F_1(R,R)+F_2(R,R)\right] \end{bmatrix} \\ [C]^T=\left[C_0 \quad C_1 \quad .... \quad C_R\right] \end{cases} \tag{69}$$

In which $[A]$ and $[C]^T$ are matrix coefficients and unknowns vector transpose, respectively. In order to obtain the natural frequencies of the beam, the functions $f_1(\xi,s)$ and $f_2(\xi,s)$ are first obtained. Introducing these functions into (68), the functions $G(m,r)$, $F_1(m,r)$ and $F_2(m,r)$ associated with the coefficients of matrix $[A]$ are obtained next. The unknown parameter in the coefficients matrix $[A]$ is therefore the natural frequency of the beam. $[C]=0$ is a trivial solution for the resulting system of equations introduced in (67). The natural frequencies are determined through calculation of a non-trivial solution for resulting system of equations. To achieve this, the determinant of the coefficients matrix of the system has to be vanished. Accordingly, a frequency equation in ω (which is a polynomial function of the order $2(R+1)$ is introduced. Given the fact that the mode shape function is approximated by the power series of (65), the results accuracy is improved if more number of the series sentences is taken into account. In this case, the order of polynomial is also increased accordingly. Hence, adoption of larger R yields more accurate results.

8. Numerical examples

In this section, several numerical examples are presented in order to verify the accuracy of the presented approach. The results of presented approach are compared with those obtained from other published references and finite elements software SAP-2000. For all following examples, the beam model is a cantilever beam.

8.1 Rayleigh beam

8.1.1 Effects of rotatory inertia and axial loading

In this example, the first three dimensionless natural frequencies of Rayleigh beam with effects of rotational inertia and variable axial forces are calculated. The results are presented in Table 1.

From Table 1 we find that our results are in excellent agreement with those in (Li *et al.* 2013) by Legendre polynomials.

8.1.2 Effects of rotational inertia, tip mass and axial loading

In this section, the first three dimensionless natural frequencies of Rayleigh beam with a lumped mass at free end under effects of axial forces are calculated. The results are presented in Table 2 and are compared with those obtained in (Li *et al.* 2013).

Table 1 The first three dimensionless natural frequencies of Rayleigh beam with effects of rotational inertia and variable axial loading

		r=0		r=0.1		r=0.2		r=0.3	
Axial force		present	Li *et al.* 2013	present	Li *et al.* 2013	present	Li *et al.* 2013	present	Li *et al.* 2013
$\alpha(\xi)=3(1-\xi)$	Ω_1	2.7645	2.73	2.7018	2.70	2.5355	2.53	2.3138	2.31
	Ω_2	21.437	21.45	18.629	18.63	14.206	14.2	11.056	11.06
	Ω_3	61.107	61.13	46.055	46.1	31.079	31.08	22.850	22.84
$\alpha(\xi)=-4+3(1-\xi)$	Ω_1	4.9734	4.99	4.8776	4.88	4.6172	4.6172	4.2558	4.25
	Ω_2	24.237	24.24	21.095	21.1	16.03	16	12.375	12.38
	Ω_3	63.586	63.58	47.86	47.86	32.187	32.18	23.62	23.62

Table 2 The first three dimensionless natural frequencies of Rayleigh beam with a lumped mass at free end under effects of axial forces

Axial force		β=0.5		β=1.5		β=3	
		present	Li *et al.* 2013	present	Li *et al.* 2013	present	Li *et al.* 2013
$\alpha(\xi)=3(1-\xi)$	Ω_1	1.58701	1.58701	1.03567	1.03567	0.759029	0.75903
	Ω_2	16.4605	16.46046	15.5679	15.56787	15.2939	15.29385
	Ω_3	51.1569	51.14854	50.0547	50.04631	49.7438	49.73534

Table 3 The first dimensionless natural frequency of Rayleigh beam with effects of rotatory inertia, tip mass and axial loading

Axial force		r=0.1 present	r=0.1 Li *et al.* 2013	r=0.2 present	r=0.2 Li *et al.* 2013	r=0.3 present	r=0.3 Li *et al.* 2013
$\alpha(\xi)=-3(1-\xi)$	β=0.5	2.34792	2.4	2.29374	2.3	2.21094	2.2
	β=1	1.81794	1.8	1.79226	1.79	1.75172	1.75

Table 4 The first five natural frequencies of a non-prismatic cantilever beam in axial deformation (rad/sec)

	ω_1	ω_2	ω_3	ω_4	ω_5
Presented approach	1538	3277	5138	7049	8984
SAP-2000	1537	3275	5131	7033	8950

The first dimensionless natural frequency of Rayleigh beam with effects of rotatory inertia, tip mass and axial loading has been calculated and results have been presented in Table 3.

The results presented in Table 3 show excellent agreement with those obtained in (Li *et al.* 2013). By assumption of a constant axial force as: $\alpha(\xi)\equiv$constant$=\alpha$ and calculation of a non-trivial solution for the system of Eq. (67) with R=1, the following equation is obtained for determining dimensionless fundamental frequency

$$\begin{aligned}&\left[7r^4+r^2\right]\Omega^6-\left[7560r^4+2964r^2+40-14\alpha r^2-\alpha\right]\Omega^4\\&+\left[262080r^2+48960-15120\alpha r^2-2964\alpha+7\alpha^2\right]\Omega^2+262080\alpha-7560\alpha^2-604800=0\end{aligned}\tag{70}$$

The first root of Eq. (70) is the first dimensionless fundamental frequency of the beam. Eq. (70) can be used for quick and approximate determination of the fundamental frequency of Rayleigh beam with effects of rotatory inertia, and constant axial loading.

8.2 Free vibration in axial ceformation

8.2.1 Non-prismatic cantilever beam

In this section, the first five natural frequencies of a non-prismatic cantilever beam in axial deformation are obtained. The beam is assumed to have a circular cross section with a linearly varying diameter as: $d(\xi)=2-1.5\xi$ ($0\leq\xi\leq1$). The density of the beam (ρ), elastic modulus (E) and beam length (L) are adopted as: $20.3943\,\frac{ton}{m^3}$, $2\times10^8\,\frac{KN}{m^2}$ and 5 m, respectively. The results are shown in Table 4 and are compared with those obtained using SAP-2000 software.

8.2.2 Non-prismatic cantilever beam with end discrete spring

The natural frequencies of a non-prismatic cantilever beam in axial deformation with end discrete spring are calculated in this example. The beam has the same material and geometric properties as the case 8.2.1. The stiffness of the discrete spring is 10^7 KN/m. The results are presented in Table 5.

Table 5 The first five axial natural frequencies of a non-prismatic cantilever beam in axial deformation with end discrete spring (rad/sec)

	ω_1	ω_2	ω_3	ω_4	ω_5
Presented approach	1625.77	3371	5216	7112	9035
SAP-2000	1625.8	3369	5209	7095	9002

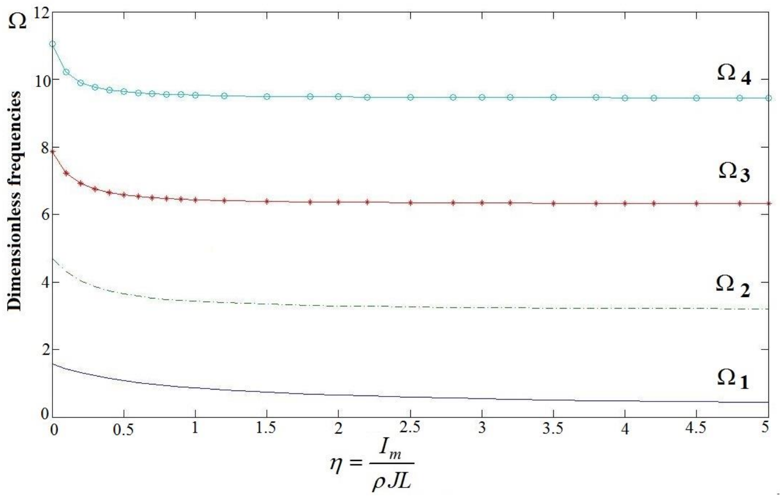

Fig. 4 the first four torsional dimensionless frequencies of a uniform bar with an attached mass moment of inertia

8.3 Torsional vibration

In this example, the first four torsional dimensionless frequencies of a uniform bar are calculated. Fig. 4 presents torsional dimensionless frequencies as a function of η.

From Fig. 4 we find that the results obtained are in excellent agreement with those in (Kelly 2007). The non-trivial solution of the system of Eq. (67) with R=1 results in the following equation for determining torsional dimensionless fundamental frequency

$$[\eta]\Omega^6 - [72\eta + 9]\Omega^4 + [720\eta + 312]\Omega^2 - 720 = 0 \tag{71}$$

The first root of the Eq. (71) is the torsional dimensionless fundamental frequency. The Eq. (71) can be used for quick and approximate calculation of the torsional fundamental frequency of the beam.

8.4 Flexural vibration with distributed elastic support

In this section, the first five natural frequencies of flexural beam with distributed elastic support are calculated. The beam is assumed to has a square cross section with a linearly varying width and height as: $d(\xi)=h(\xi)=2-1.5\xi$ ($0\leq\xi\leq1$). The stiffness of the distributed elastic support is $K = 250000\frac{KN}{m^2}$. The other material and geometric properties of the beam is the same as case

Table 5 The first five natural frequencies of a non-prismatic cantilever beam with distributed elastic support (rad/sec)

	ω_1	ω_2	ω_3	ω_4	ω_5
Presented approach	445.13	1341.8	2984	5405	8612
SAP-2000	445.38	1343.54	2991	5421	8647

Table 6 The first five natural frequencies of a uniform cantilever beam with distributed elastic support (rad/sec)

	ω_1	ω_2	ω_3	ω_4	ω_5
Presented approach	230.4	455.7	1137.26	2197	3620
Clough and Penzien 1975	230.37	455.77	1137.32	2197	3620
SAP-2000	230.38	455.57	1136.1	2194	3613

Table 7 The first five natural frequencies of a non-prismatic flexural cantilever beam without effects of damping (rad/sec)

ω_1	ω_2	ω_3	ω_4	ω_5
420.87	1334.7	2981.9	5403.5	8611.1

Table 8 The first five roots of the frequency equation of a non-prismatic flexural cantilever beam with effects of viscose damping

α_1	α_2	α_3	α_4	α_5
$-168.4\pm385.9\,i$	$-534.5\pm1224.8\,i$	$-1195.3\pm2738.9\,i$	$-2168.1\pm4967.8\,i$	$-3458.5\pm7924.5\,i$

8.2.1. The first five natural frequencies of the beam have been calculated and have been presented in Table 5.

The first five natural frequencies of a uniform beam with $d=h=0.5$ m and $L=5$ m have been calculated and have been presented in Table 6.

8.5 Flexural vibration including viscose damping

In this example, the presented approach is used for free vibration analysis of a non-prismatic cantilever beam with effects of viscose damping. The beam of the case 8.4 is considered in this example. By setting $C(\xi)=0$ in the relations obtained, the first five natural frequencies of the beam without effects of damping are calculated. The results are presented in Table 7.

By assumption of a constant damping ratio $\xi=40\%$ for all modes, the first five roots of the frequency equation have been calculated and have been presented in Table 8.

The results of Table 8 shows that the roots of the frequency equation (α_n $n=1,2,...$) are the complex numbers and are equal to $\alpha_n=\left(-\zeta\omega_n\right)\pm\left(\omega_n\sqrt{1-\zeta^2}\right)\,\mathrm{i}$ or $\alpha_n=-\frac{a_0}{2}\pm\omega_D\,\mathrm{i}$ where $\omega_D=\omega_n\sqrt{1-\zeta^2}$ and a_0 are damped natural frequency of the beam and mass proportional damping coefficient, respectively. And also, the complex norm of α_n is $\left|\alpha_n\right|=\sqrt{\left(-\zeta\omega_n\right)^2+\left(\omega_n\sqrt{1-\zeta^2}\right)^2}=\omega_n$.

9. Conclusions

In this paper, the natural frequencies of the lateral, axial and torsional vibration of the uniform/non-uniform beams have been calculated. The proposed method is based on the conversion of the governing differential equation into its weak form integral equation. The mode shape function has been approximated by a power series, which allows the weak form integral equation to be transformed into a system of linear algebraic equations. The natural frequencies are determined by calculation of a non-trivial solution for system of equations. The various vibration problems such as: Rayleigh beam under variable axial forces, axial vibration of a bar with and without end discrete spring, torsional vibration of a bar with an attached mass moment of inertia, flexural vibration of the beam with laterally distributed elastic springs and also flexural vibration of the beam with effects of viscose damping have been investigated. In the numerical examples presented in the paper, the natural frequencies of the beams were calculated and were compared with those given by other published references and available finite elements software. It was shown that the method proposed in the paper was reliable, efficient, and sufficiently accurate when compared with other references and numerical software.

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