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The dispersion of the flexural waves in a compound hollow cylinder under imperfect contact between layers

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Abstract. The influence of the interface imperfect bonding on the flexural wave dispersion in the bi-layered hollow circular cylinder is studied with utilizing three-dimensional linear theory of elastodynamics. The shear-spring type model is used for describing the imperfect bonding on the interface between the layers and the degree of the imperfectness is estimated through the dimensionless shear-spring parameters which enter the mentioned model. The method for finding the analytical expressions for the sought values and dispersion equation are discussed and detailed. Numerical results on the lowest first and second modes are presented and analyzed. These results are obtained for various values of the shear-spring parameters. According to these results, in particular, it is established that as a results of the imperfection of the bonding between the layers the new branches of the dispersion related the first fundamental mode arise and the character of the dispersion curve related to the second mode becomes more complicated.

Keywords: layered structures; debonding; analytical modelling; flexural waves; dispersion

1. Introduction

It is known that the non-destructive evaluation of various type imperfectness of the structure of the composite material is based on the measurement of the wave propagation velocity in those. To make correct conclusions from these measurement procedures it is necessary to have at hand theoretical results related to the influence of the structural imperfectness of composites (for instance, the bonding between the layers in the layered composites) on the propagation velocity (or on the dispersion) of the waves through which the non-destructive evolution is made. Therefore, the subject of the present paper which regards the study of the influence of the shear-spring type imperfectness of the bonding between the layers on the flexural wave dispersion in the bi-layered hollow circular cylinder has a great significance not only in the theoretical, but also in the application sense and is of great interest to various fields of modern natural science. Because the results of the theoretical investigations carried out in the present paper together with the corresponding experimental data can be employed for the non-destructive determination of the degree of imperfection of the contact between the layers of the cylinder.

Moreover, the mentioned type results also are applied in geophysical and geotechnical

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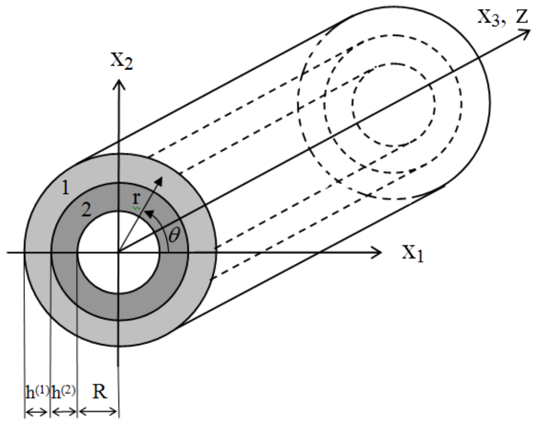

Fig. 1 The geometry of the bi-layered hollow cylinder

engineering for determination of the soil stiffness profile.

We consider here a brief review of the investigations which were made within the scope of the imperfect interface models summarized in paper Martin (1992). Simplification of model (Martin 1992) to the shear-spring type imperfection of the contact conditions was made in a paper Jones and Whitter (1967). According to the subject of the present paper, we consider the investigations related to the compound cylinders. A review of the investigations related to the layered materials in the plane-strain state can be found in papers (Pecorari 2001, Leungvichcharoen and Wijeyewickrema 2003, Kadioglu and Ataoglu 2010, Zhou *et al*. 2012, Küçükarslan 2009, Kumara and Singh 2009, Liu *et al.* 2010, Huang and Liu 2010, Pang and Liu 2011). Moreover, a review of recent investigations related to the wave dispersion in compound cylinders was made in paper Akbarov (2013). Thus, we begin the review with paper Berger *et al*. (2000), in which the torsional wave propagation in a bi-material, imperfectly-compounded circular cylinder is studied and the imperfection of the contact condition is presented according to the model developed in Jones and Whitter (1967). In paper Kepceler (2010) investigations of a similar type were carried out for the initially stressed bi-material compound circular cylinder. Study of the shear- spring type imperfection on the longitudinal bi-material pre-strained compound solid and compound spring type imperfection on the hollow circular cylinders was made in papers Akbarov and Ipek (2010) and Akbarov and Ipek (2012). This seems to be all that has been done in the area of wave dispersion in compound circular cylinders.

In the present paper, the investigations carried out in papers (Akbarov and Ipek 2010, Akbarov and Ipek 2012) are developed for flexural wave propagation in the bi-material compound hollow circular cylinder (Akbarov and Ipek 2015, Akbarov 2015). In other words, in the present paper the influence of the shear-spring type imperfectness of the bonding between the layers of the cylinder on the dispersion curves of the flexural wave propagation in this compound cylinder is studied. Investigations are carried out by utilizing the three-dimensional linear theory of elastodynamics.

2. Formulation of the problem

We consider the hollow (Fig. 1) compound cylinder and assume that the radius of the inner circle of the cross section of the inner hollow cylinder is R .

The thickness of the outer inner cylinders we denote through $h^{(1)}$ and $h^{(2)}$ respectively.

We determine the position of the points of the cylinders in the cylindrical system of coordinates $Or\theta z$ (Fig. 1). The values related to the inner and outer hollow cylinders will be denoted by the upper indices (2) and (1), respectively. Within this framework, let us investigate the flexural wave propagation along the Oz axis in the cylinders using the coordinates r, θ and z in the framework of the three-dimensional linear theory of elastodynamics. Thus, we write the basic relations of the linear theory of elastodynamics for the case under consideration.

The equations of motion

$$\frac{\partial\sigma_{rr}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta r}^{(k)}}{\partial\theta}+\frac{\partial\sigma_{zr}^{(k)}}{\partial z}+\frac{1}{r}(\sigma_{rr}^{(k)}-\sigma_{\theta\theta}^{(k)})=\rho^{(k)}\frac{\partial^2 u_r^{(k)}}{\partial t^2},$$

$$\frac{\partial\sigma_{r\theta}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta\theta}^{(k)}}{\partial\theta}+\frac{\partial\sigma_{z\theta}^{(k)}}{\partial z}+\frac{2}{r}\sigma_{r\theta}^{(k)}=\rho^{(k)}\frac{\partial^2 u_\theta^{(k)}}{\partial t^2},$$

$$\frac{\partial\sigma_{rz}^{(k)}}{\partial r}+\frac{1}{r}\frac{\partial\sigma_{\theta z}^{(k)}}{\partial\theta}+\frac{\partial\sigma_{zz}^{(k)}}{\partial z}+\frac{1}{r}\sigma_{rz}^{(k)}=\rho^{(k)}\frac{\partial^2 u_z^{(k)}}{\partial t^2} \tag{1}$$

The elastic relations

$$\sigma_{rr}^{(k)}=(\lambda^{(k)}+2\mu^{(k)})\frac{\partial u_r^{(k)}}{\partial r}+\lambda^{(k)}\frac{1}{r}(\frac{\partial u_\theta^{(k)}}{\partial\theta}+u_r^{(k)})+\lambda^{(k)}\frac{\partial u_z^{(k)}}{\partial z},$$

$$\sigma_{\theta\theta}^{(k)}=\lambda^{(k)}\frac{\partial u_r^{(k)}}{\partial r}+(\lambda^{(k)}+2\mu^{(k)})\frac{1}{r}(\frac{\partial u_\theta^{(k)}}{\partial\theta}+u_r^{(k)})+\lambda^{(k)}\frac{\partial u_z^{(k)}}{\partial z},$$

$$\sigma_{zz}^{(k)}=\lambda^{(k)}\frac{\partial u_r^{(k)}}{\partial r}+\lambda^{(k)}\frac{1}{r}(\frac{\partial u_\theta^{(k)}}{\partial\theta}+u_r^{(k)})+(\lambda^{(k)}+2\mu^{(k)})\frac{\partial u_z^{(k)}}{\partial z},$$

$$\sigma_{r\theta}^{(k)}=\mu^{(k)}\frac{\partial u_\theta^{(k)}}{\partial r}+\mu^{(k)}(\frac{1}{r}\frac{\partial u_r^{(k)}}{\partial\theta}-\frac{1}{r}u_\theta^{(k)}),$$

$$\sigma_{z\theta}^{(k)}=\mu^{(k)}\frac{\partial u_\theta^{(k)}}{\partial z}+\mu^{(k)}\frac{\partial u_z^{(k)}}{r\partial\theta},\sigma_{zr}^{(k)}=\mu^{(k)}\frac{\partial u_r^{(k)}}{\partial z}+\mu^{(k)}\frac{\partial u_z^{(k)}}{\partial r} \tag{2}$$

In Eq. (2) conventional notation is used.

The boundary conditions on the outer surface of the outer hollow cylinder and inner surface of the inner hollow cylinders are

$$\sigma_{rr}^{(1)}\Big|_{r=R+h^{(1)}+h^{(2)}}=0,\ \sigma_{r\theta}^{(1)}\Big|_{r=R+h^{(1)}+h^{(2)}}=0,\ \sigma_{rz}^{(1)}\Big|_{r=R+h^{(1)}+h^{(2)}}=0,$$

$$\sigma_{rr}^{(2)}\Big|_{r=R}=0,\sigma_{r\theta}^{(2)}\Big|_{r=R}=0,\sigma_{rz}^{(2)}\Big|_{r=R}=0 \tag{3}$$

Now we formulate the contact conditions on the interface surface between the inner and outer hollow cylinders. We assume that the contact conditions with respect to the forces and radial displacements are continuous and can be written as follows

$$\sigma_{rr}^{(1)}\Big|_{r=R+h^{(2)}} = \sigma_{rr}^{(2)}\Big|_{r=R+h^{(2)}}, \quad \sigma_{r\theta}^{(1)}\Big|_{r=R+h^{(2)}} = \sigma_{r\theta}^{(2)}\Big|_{r=R+h^{(2)}},$$

$$\sigma_{rz}^{(1)}\Big|_{r=R+h^{(2)}} = \sigma_{rz}^{(2)}\Big|_{r=R+h^{(2)}}, \quad u_{r}^{(1)}\Big|_{r=R+h^{(2)}} = u_{r}^{(2)}\Big|_{r=R+h^{(2)}} \tag{4}$$

At the same time, we assume that the shear-spring type imperfection occurs in the contact conditions related to the circumferential and axial displacements and, according to (Jones and Whitter 1967, Berger *et al*. 2000, Akbarov and Ipek 2010, Akbarov and Ipek 2012) these conditions are formulated by the following equations

$$u_{\theta}^{(1)}\Big|_{r=R+h^{(2)}} - u_{\theta}^{(2)}\Big|_{r=R+h^{(2)}} = \frac{FR}{\mu^{(1)}}\sigma_{r\theta}^{(1)}\Big|_{r=R+h^{(2)}}$$

$$u_{z}^{(1)}\Big|_{r=R+h^{(2)}} - u_{z}^{(2)}\Big|_{r=R+h^{(2)}} = \frac{F_1 R}{\mu^{(1)}}\sigma_{rz}^{(1)}\Big|_{r=R+h^{(2)}} \tag{5}$$

Here, the dimensionless parameters F and F_1 in Eq. (5) characterize the degree of the imperfection and the change range of these parameters is: $0 \le F \le \infty$ and $0 \le F_1 \le \infty$. Note that the case where $F_1=F=0$ corresponds to complete contact, but the case where $F_1=F=\infty$ corresponds to full slipping contact conditions. At the same time, the case where $F_1=0$ and $F_1=\infty$ under $0<F<\infty$ can be considered, which can be called complete and full slipping contacts with respect to the axial displacements, respectively, or the cases where $F=0$ and $F=\infty$ under $0<F_1<\infty$ can be called complete and full slipping contacts, with respect to the circumferential displacements, respectively. The aim of the present investigation is to study the influence of the shear spring type parameters F_1 and F on the character of the dispersion of the flexural waves in the compound solid cylinder under consideration.

This completes formulation of the problem and consideration of the governing field equations.

3. Solution procedure and obtaining the dispersion equation

For solution of the eigenvalue problems, Eqs. (1)-(5), we use the representation proposed in Guz (2004)

$$u_r^{(k)} = \frac{1}{r}\frac{\partial}{\partial\theta}\Psi^{(k)} - \frac{\partial^2}{\partial r\partial z}\mathrm{X}^{(k)}, \quad u_\theta^{(k)} = -\frac{\partial}{\partial r}\Psi^{(k)} - \frac{1}{r}\frac{\partial^2}{\partial\theta\partial z}\mathrm{X}^{(k)},$$

$$u_z^{(k)} = (\lambda^{(k)}+\mu^{(k)})^{-1}\left((\lambda^{(k)}+2\mu^{(k)})\Delta_1 + \mu^{(k)}\frac{\partial^2}{\partial z^2} - \rho^{(k)}\frac{\partial^2}{\partial t^2}\right)\mathrm{X}^{(k)},$$

$$\Delta_1 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial\theta^2} \tag{6}$$

Here the functions $\Psi^{(k)}$ and $\mathrm{X}^{(k)}$ are the solutions of the equations

$$\left(\Delta_1+\frac{\partial^2}{\partial z^2}-\frac{\rho^{(k)}}{\mu^{(k)}}\frac{\partial^2}{\partial t^2}\right)\Psi^{(k)}=0,\left[\left(\Delta_1+\frac{\partial^2}{\partial z^2}\right)\left(\Delta_1+\frac{\partial^2}{\partial z^2}\right)+\right.$$

$$\left.-\rho^{(k)}\frac{\lambda^{(k)}+3\mu^{(k)}}{\mu^{(k)}(\lambda^{(k)}+2\mu^{(k)})}(\Delta_1+\frac{\partial^2}{\partial z^2})\right)\frac{\partial^2}{\partial t^2}+\frac{(\rho^{(k)})^2}{\mu^{(k)}(\lambda^{(k)}+2\mu^{(k)})}\frac{\partial^4}{\partial t^4}\right]\mathrm{X}^{(k)}=0 \tag{7}$$

For the flexural waves we represent the functions Ψ and X as follows

$$\Psi=\Psi_n(r)\sin n\theta\sin(kz-\omega t),\mathrm{X}=\mathrm{X}_n(r)\cos n\theta\cos(kz-\omega t) \tag{8}$$

Substituting Eq. (8) into Eq. (7) we obtain

$$\left(\Delta_{1n}+\zeta_1^2\right)\Psi_n=0\ \left(\Delta_{1n}+\zeta_2^2\right)\left(\Delta_{1n}+\zeta_3^2\right)\mathrm{X}_n=0$$

$$\Delta_{1n}=\frac{d^2}{dr^2}+\frac{d}{rdr}-\frac{n^2}{r^2} \tag{9}$$

Where

$$\zeta_1^2=k^2\left(\frac{\rho^{(k)}c^2}{\mu^{(k)}}-1\right),c=\frac{k}{\omega} \tag{10}$$

But ζ_2^2 and ζ_3^2 are determined as solutions of the equation

$$\mu^{(k)}(\zeta^{(k)})^4-k^2(\zeta^{(k)})^2\left[\rho^{(k)}c^2-(\lambda^{(k)}+2\mu^{(k)})+\frac{\mu^{(k)}}{\lambda^{(k)}+2\mu^{(k)}}\left(\rho^{(k)}c^2-\mu^{(k)}\right)+\right.$$

$$\left.\frac{(\lambda^{(k)}+\mu^{(k)})^2}{\lambda^{(k)}+2\mu^{(k)}}\right]+k^4\left(\frac{\rho^{(k)}c^2}{\lambda^{(k)}+2\mu^{(k)}}-1\right)\left(\rho^{(k)}c^2-\mu^{(k)}\right)=0 \tag{11}$$

In Eqs. (10) and (11), c is the phase velocity of the flexural waves. Thus, we find the solution of the equations in Eq. (9) as follows
For the inner cylinder

$$\Psi_n^{(2)}=A_1^{(2)}E_n(\zeta_1^{(2)}kr)+B_1^{(2)}D_n(\zeta_1^{(2)}kr),\mathrm{X}_n^{(2)}=A_2^{(2)}E_n(\zeta_2^{(2)}kr)+A_3^{(2)}E_n(\zeta_3^{(2)}kr)+$$

$$A_2^{(2)}D_n(\zeta_2^{(2)}kr)+A_3^{(2)}D_n(\zeta_3^{(2)}kr) \tag{12}$$

For the outer hollow cylinder

$$\Psi_n^{(1)}=A_1^{(1)}E_n(\zeta_1^{(1)}kr)+B_1^{(1)}D_n(\zeta_1^{(1)}kr),\ \mathrm{X}_n^{(1)}=A_2^{(1)}E_n(\zeta_2^{(1)}kr)+A_3^{(1)}E_n(\zeta_3^{(1)}kr)+$$

$$A_2^{(1)}D_n(\zeta_2^{(1)}kr)+A_3^{(1)}D_n(\zeta_3^{(1)}kr) \tag{13}$$

Where

Table 1 The values of elastic constants of selected materials

Materials	Density ($\rho\times10^{-3}$)	Young's Module $E\times10^{-4}$	Pois.'s ratio (ν)	Velocity of wave of dilatation $c_2\times10^{-3}$
Steel (*St*)	7.795 kg/m^3	19.6 MPa	0.27	3.152 m/s
Aluminum (*Al*)	2.77 kg/m^3	7.28 MPa	0.30	3179 m/s

$$E_n(\zeta_j^{(m)}kr) = J_n(\zeta_j^{(m)}kr),\ D_n(\zeta_j^{(m)}kr) = Y_n(\zeta_j^{(m)}kr) \text{ if } (\zeta_j^{(m)})^2 > 0, m = 1,2,$$

$$E_n(\zeta_j^{(m)}kr) = I_n(\zeta_j^{(m)}kr),\ D_n(\zeta_j^{(m)}kr) = K_n(\zeta_j^{(m)}kr) \text{ if } (\zeta_j^{(m)})^2 < 0, j = 1,2,3 \quad (14)$$

In Eq. (14), $J_n(x)$ and $Y_n(x)$ are Bessel functions of the first and second kind of the n-th order, $I_n(x)$ and $K_n(x)$ are Bessel functions of a purely imaginary argument of the n-th order and Macdonald functions of the n-th order, respectively. Thus, using relations Eqs. (6), (12)-(14) and (2) we obtain the dispersion equation

$$\det\left\|\beta_{ij}\right\| = 0,\ i; j = 1,2,...,12 \quad (15)$$

For the compound solid cylinder from the boundary Eq. (3) and contact Eqs. (4) and (5) conditions, we do not give here the explicit expressions of β_{ij}, because they can easily be determined from the corresponding expressions given in paper Akbarov (2013).

4. Numerical results and discussions

Numerical results are obtained for the following pair of materials: the material of the inner and outer cylinders is steel (shortly denoted as *St*) and aluminum (shortly denoted as *Al*) respectively, and this pair of materials will be denoted as *St*+*Al*.

All mechanical characteristics of these materials are given in Table 1 Note that the data given in the tables are selected according to (Guz 2004, Guz *et al*. 2000). Assume that n=1 in Eqs. (8)-(14) and, $h^{(1)}/R$=0.1 and $h^{(2)}/R$=0.3. Within these assumptions we analyze numerical results obtained by numerical solution of the dispersion equation Eq. (15). Note that this numerical solution is carried out by employing the algorithm and PC programs which were used in the previous papers by the authors such as (Akbarov 2013, Akbarov and Ipek 2010, Akbarov and Ipek 2012, Akbarov and Ipek 2015) and others. We recall that in the paper Akbarov (2013) the dispersion of the flexural wave dispersion in the finite pre-strained solid and hollow cylinders made of highly elastic material was studied and programs which were used under this studying were tested by the known classical results obtained, for example, in a paper Abramson (1957). Therefore, the PC programs used in the investigations carried out in the paper Akbarov (2013) after corresponding development and change are employed for the numerical solution of the dispersion equation Eq. (15). Consequently, the algorithm and PC programs used in the present numerical investigations have been already tested, although we will also consider below some fragments on the mentioned testing. Thus, consider the numerical results which are obtained in the following three cases:

- in Case I it is supposed that F=0 and $F_1\geq0$;
- in Case II it is supposed that $F\geq0$ and F_1=0;

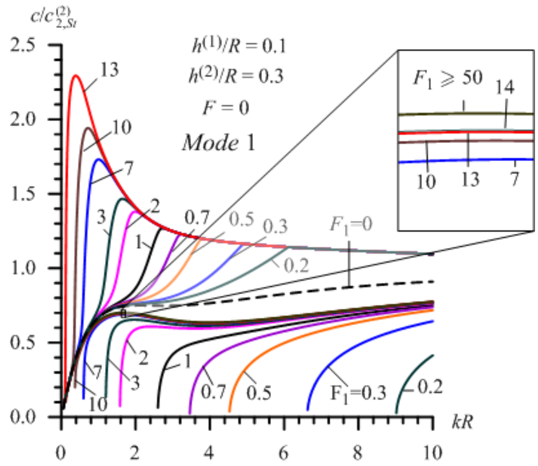

Fig. 2 Dispersion curves of the first mode obtained in Case I

- in Case III it is supposed that $F \geq 0$ and $F_1 \geq 0$.

Where F and F_1 are the shear-spring parameters which enter the contact conditions in Eq. (5). Dispersion curves related to the first mode are given in Figs. 2, 3 and 4, for the Cases I, II and III respectively. First, we analyze the results related to Case I and note that the in the foregoing figures and in the figures which will be considered below the dispersion curve related to the complete contact case, i.e., to the case where $F_1 = F = 0$ is depicted with the dashed line. According to Fig. 2, and other results which are not given here, it can be concluded that there exists such value of the parameter F_1 (denote it by F^*_1) before which (i.e., in the cases where $F_1 < F^*_1$) two branches of the first mode appear. The dispersion curves which are above (below) the dispersion curves related to the complete contact case we call the first (the second) branch. However, in the cases where $F_1 \geq F^*_1$, the dispersion cure related to the first mode has a single branch and this branch approach to the dispersion curve related to the full slipping contact with respect to the axial displacements, i.e., with respect to the $u_z^{(1)}$ and $u_z^{(2)}$. For the cases under consideration it can be concluded that $F^*_1 \approx 14$ for the pair of the materials *St+Al*. It should be noted that, as follows from the Fig. 2 the second branches appear after a certain value of the dimensionless wavenumber kR (denote it by $(kR)_I^{sa}$) for the pair of materials *St+Al* and we call it the cut off value of the dimensionless wavenumber in Case I.

According to the numerical results illustrated in Fig. 2 we can write the following relation for the cut off values of the dimensionless wavenumber

$$(kR)_I^{sa} \to \infty \quad \text{as} \quad F_1 \to 0 \quad \text{and} \quad (kR)_I^{sa} \to 0 \quad \text{as} \quad F_1 \to F_1^* \tag{16}$$

Through c_{fb}^{Isa} and c_{sb}^{Isa} we denote the wave propagation velocity related to the first and second branches of the dispersion curves respectively, but through c_{sa} we denote the wave propagation velocity in the complete contact case between the cylinders for the pair of materials *St+Al*. As noted above, in the case where $F_1 \geq F^*_1$, the flexural wave in the first mode has a single branch and the wave propagation velocity related to this case we denote through c_*^{Isa} for the pair of materials *St+Al*.

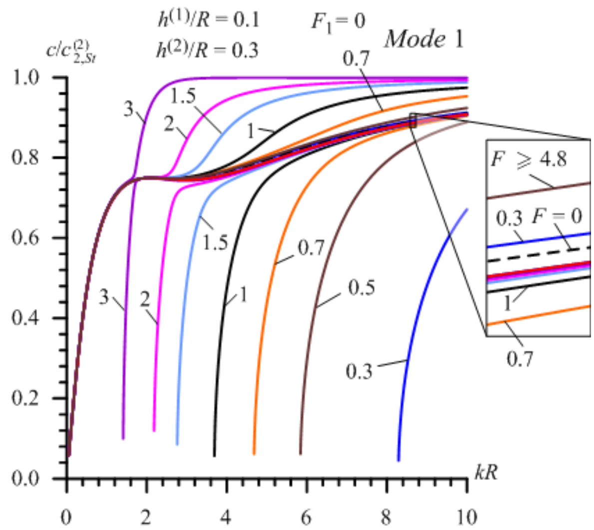

Fig. 3 Dispersion curves of the first mode obtained in Case II

Thus, according to the foregoing notation and according to the results given in Fig. 2 we can write following relations

$$c_{fb}^{Isa} \geq c_{sa}, c_{*}^{Isa} \leq c_{sa} \text{ for each } kR \in (0,\infty) \text{ and } c_{sb}^{Isa} \leq c_{*}^{Isa} \text{ for each } kR \in ((kR)_I^{sa},\infty) \quad (17)$$

Numerical results show that there exist the following limit relations

$$c_{fb}^{Isa}; c_{sb}^{Isa}; c_{*}^{Isa}; c_{sa} \to \min\left\{c_{R,St}; c_{R,Al}\right\} \text{ as } kR \to \infty,$$

$$c_{fb}^{Isa}; c_{*}^{Isa}; c_{sa} \to 0 \text{ as } kR \to 0, \; c_{sb}^{Isa} \to 0 \text{ as } kR \to (kR)_I^{sa} \quad (18)$$

Where $c_{R,St}$ ($c_{R,Al}$) is the Rayleigh wave velocity of the Steel (Aluminum). According to Eringen and Suhubi (1975), $c_{R,St}$=2908.2 m/s and $c_{R,Al}$=2948.2 m/s. Consequently, we can write that $\min\{c_{R,St}; c_{R,Al}\}=c_{R,St}$ and $\max\{c_{R,St}; c_{R,Al}\}=c_{R,Al}$.

Moreover, numerical results show that the dispersion curves obtained in the cases where $F_1 \geq F^*_1$ approach a certain limit which corresponds to the dispersion curve constructed in the full slipping case with respect to the axial displacements.

Now we analyze the dispersion curves related to Case II and constructed also for the first mode. These curves are illustrated in Fig. 3 for the pair of materials *St+Al*. The analyzes of these results show that in Case II the influence of the interface imperfect bonding of the layers of the cylinder is a similar in the qualitative sense with that observed in Case I. Consequently, there exists such value of the parameter F (denote it by F^*) before which the dispersion curves related to the first mode has two branches: the first of them are above, but the second ones below the dispersion curves related to the complete contact case between the layers of the cylinder.

We use the notation F_{sa}^* for the value of the F^* obtained for the pair of materials *St+Al*. It is established that $F_{sa}^* \approx 4.8$. In the cases where $F \geq F^*$ the first mode of the flexural wave in Case II has a single branch and wave propagation velocity related to these cases we denote through c_{*}^{IIsa}

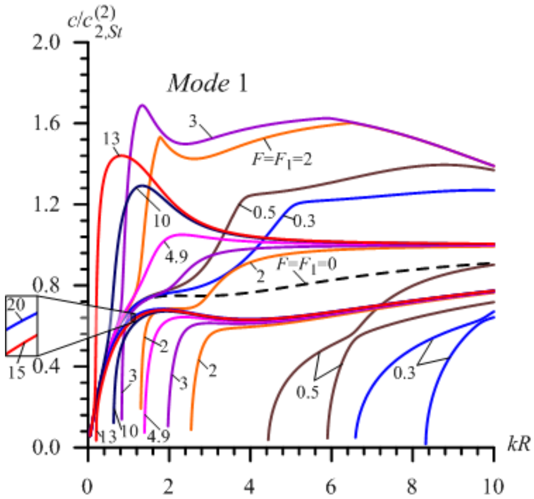

Fig. 4 Dispersion curves of the first mode obtained in Case III

for the pair of materials *St+Al*. Through c_{fb}^{IIsa} and c_{sb}^{IIsa} we denote the wave propagation velocity related to the first and second branches of the first mode for the pair of the materials *St+Al*.

It follows from the Fig. 3 in Case II, as in Case I, the second branch has cut off values for the dimensionless wavenumber *kR*, i.e., the graphs related to the second branch appears after a certain value of *kR* (denote it by $(kR)_{II}^{sa}$) for the pair of materials *St+Al*. According to the numerical results shown in Fig. 3 we can conclude that

$$(kR)_{II}^{sa} \to \infty \ \text{ as } \ F \to 0 \ , \ (kR)_{II}^{sa} \to 0 \ \text{ as } \ F \to F_{sa}^{*} \tag{19}$$

Thus, using the notation described above, the following relations are established from the results illustrated in Fig. 3

$$c_{fb}^{IIsa} > c_{sa}\,, \ c_{*}^{IIsa} < c_{sa} \text{ for each } kR \in (0,\infty)\,, c_{sb}^{IIsa} < c_{*}^{IIsa} \ \text{ for each } \ kR \in ((kR)_{II}^{sa},\infty) \tag{20}$$

At the same time, we obtain the following estimation for the limit cases

$$c_{fb}^{IIsa};\, c_{sb}^{IIsa};\, c_{*}^{IIsa} \to \min\left\{c_{R,St}; c_{R,Al}\right\} \ \text{ as } \ kR \to \infty\,,$$

$$c_{fb}^{IIsa};\, c_{*}^{IIsa} \to 0 \quad \text{as} \ \ kR \to 0\,, \ c_{sb}^{IIsa} \to 0 \ \text{ as } \ kR \to (kR)_{II}^{sa} \tag{21}$$

Now we consider the results obtained in Case III which are illustrated in Fig. 4 for the pair of materials *St+Al*. Analyses show that the number of the branches of the first mode in Case III, as in the previous cases, depends on the value of the parameter $F(=F_1)$. So that, before a certain value of the parameter $F(=F_1)$ (denote it by F_{01}^{*}) the first mode has three branches. We use the notation F_{01}^{*sa} for the F_{01}^{*} for the pair of materials *St+Al*. It is established that $F_{01}^{*sa} \approx 4.8$. Moreover, it follows from the results that after F_{01}^{*} there exists a certain value of the parameter $F(=F_1)$ (denote it by F_{02}^{*}) before which, i.e., in the cases where $F_{01}^{*} \le F(=F_1) < F_{02}^{*}$ the first mode has two branches. The value of F^{*}_{02} related to the pair of materials *St*+*Al* we denote by F^{*sa}_{02}. It follows from the analyses of the numerical results that $F^{*sa}_{02} \approx 14$. Finally, in the cases where $F(=F_1) \geq F^{*}_{02}$ the dispersion curves of the first mode has a single branch.

The numerical results also show that in the cases where $F(=F_1) < F^{*}_{02}$ the curves related to the first branch is above, but the curves related to the second and third branches are below the dispersion curve related to the complete contact case. It should be noted that the graphs related to the second and third branches appear after a certain value of the dimensionless wavenumber *kR* and, as above we call these values of *kR* the cut off values of that. The cut off values of the *kR* for the second and third modes we denote as $(kR)^{sa}_{sIII}$ and $(kR)^{sa}_{tIII}$ for the pair *St*+*Al*. The relations given below follow from the numerical results.

$$(kR)^{sa}_{sIII};\ (kR)^{sa}_{tIII} \to \infty \ \text{ as } \ F(=F_1) \to 0,$$

$$(kR)^{sa}_{tIII} \to 0 \ \text{ as } \ F(=F_1) \to F^{*sa}_{01},\ (kR)^{sa}_{sIII} \to 0 \ \text{ as } \ F(=F_1) \to F^{*sa}_{02} \tag{22}$$

Through c^{IIIsa}_{fb}, c^{IIIsa}_{sb} and c^{IIIsa}_{tb} we denote the wave propagation velocity related to the first, second and third branches of the first mode for the pair of materials *St*+*Al*. At the same time, we use the notation c^{IIIsa}_{*} for the wave propagation velocity related to the case where $F(=F_1) \geq F^{*sa}_{02}$. Using the foregoing notation we can write the following relations for the wave propagation velocities.

$$c^{IIIsa}_{fb} > c_{sa},\ c^{IIIsa}_{*} < c_{sa} \ \text{ for each } kR \in (0,\infty);$$

$$c^{IIIsa}_{fb} > c^{IIIsa}_{sb} > c^{IIIsa}_{tb} \ \text{ for each } \ kR \in ((kR)^{sa}_{tIII},\infty);$$

$$c^{IIIsa}_{fb} > c^{IIIsa}_{sb} \ \text{ for each } \ kR \in ((kR)^{sa}_{sIII},\infty) \tag{23}$$

Moreover, the numerical results illustrated in Fig. 4 allows us to write the following estimations for the limit cases in Case III

$$c^{IIIsa}_{fb};\ c^{IIIsa}_{sb};c^{IIIsa}_{tb};\ c^{IIIsa}_{*} \to \min\left\{c_{R,St};c_{R,Al}\right\} \ \text{ as } \ kR \to \infty,$$

$$c^{IIIsa}_{fb};c^{IIIsa}_{*} \to 0 \ \text{ as } \ kR \to 0,\ c^{IIIsa}_{sb} \to 0 \ \text{ as } \ kR \to (kR)^{sa}_{sIII},$$

$$c^{IIIsa}_{tb} \to 0 \ \text{ as } \ kR \to (kR)^{sa}_{tIII} \tag{24}$$

This completes the analysis of the results related to the first mode. Now we consider the results which regard the second mode. Note that the dispersion curves obtained in Cases I, II and III for the pair of materials *St*+*Al* are given in Figs. 5, 6 and 7 respectively. The results show that the second mode, as in the complete contact case, has a single branch. However, this branch before a certain value of *kR* (denote this value of *kR* through $(kR)^{sa}_{*I}$ and $(kR)^{sa}_{*II}$) for Case I and Case II for the pair of materials *St*+*Al* coincide almost with the dispersion curve obtained in the complete contact case. The relations given below take place for the noted values of the dimensionless wavenumber.

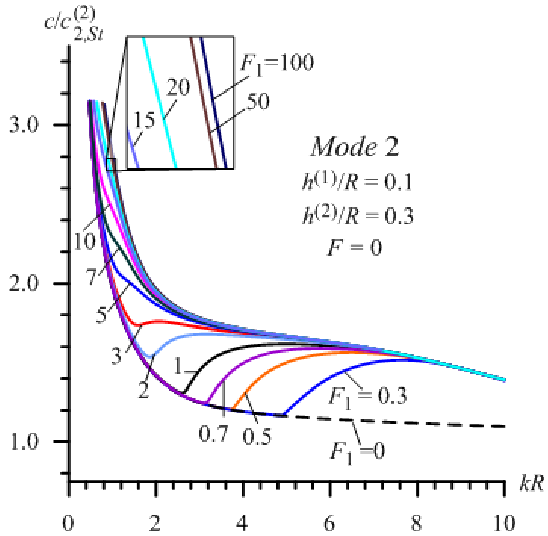

Fig. 5 Dispersion curves of the second mode obtained in Case I

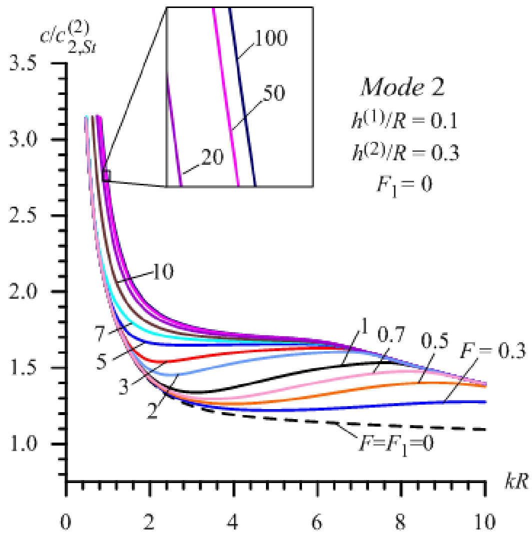

Fig. 6 Dispersion curves of the second mode obtained in Case II

$$(kR)_{*I}^{sa} \to \infty \text{ as } F_1 \to 0 \text{ and } (kR)_{*II}^{sa} \to \infty \text{ as } F \to 0;$$

$$(kR)_{*I}^{sa} \to 0 \text{ as } F_1 \to \infty \text{ and } (kR)_{*II}^{sa} \to 0 \text{ as } F \to \infty \tag{25}$$

However, after of the value of kR indicated in Eq. (25) the dispersion curves obtained in Case I and in Case II are separated from the dispersion curve regarded the complete contact case and increase with kR. The obtained results allow us to write also the relations

$$c_{2m}^{sa} \le c_{2m}^{Isa} \le c_{*2m}^{Isa}, c_{2m}^{sa} \le c_{2m}^{IIsa} \le c_{*2m}^{IIsa} \tag{26}$$

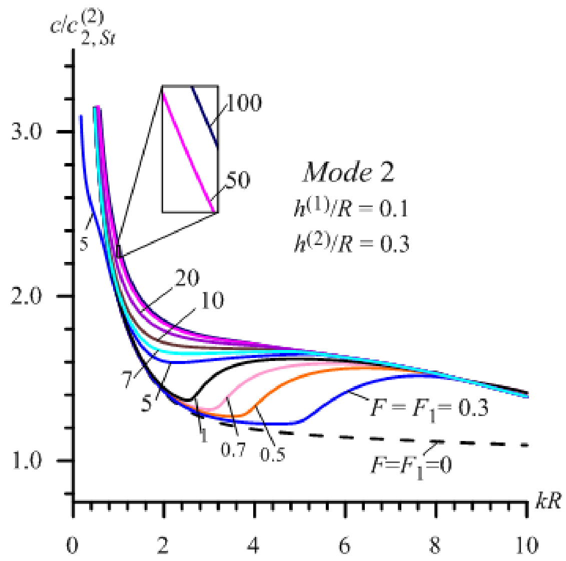

Fig. 7 Dispersion curves of the second mode obtained in Case III

where c^{sa}_{2m}, c^{Isa}_{2m}, c^{IIsa}_{2m}, c^{Isa}_{*2m} and c^{IIsa}_{*2m} are the wave propagation velocity in the second mode in the complete contact case, in Case I, in Case II, in the full slipping case with respect to axial displacements and in the full slipping case with respect to circumferential displacements respectively for the pair of materials *St+Al*.

With this we restrict to consideration of the dispersion curves obtained for the second mode in Case I and in Case II. It follows from the foregoing analysis that the influence of the imperfect bonding between the layers of the cylinder has a certain rules determined by the expressions Eqs. (25) and (26). However, according to results given in Fig. 7 in Case III the mentioned influence has more complicated character. For instance, the relations which are similar to the Eq. (25) or (26) do not take place in Case III. However, the wave propagation velocity approach a certain limit one with $F(=F_1)$ and this limit one corresponds the full slipping case between the layers with respect to the circumferential and axial displacements simultaneously. Moreover, Fig. 7 shows that in contrast to the first mode for which the wave propagation velocity obtained under full slipping case is less than corresponding ones obtained under complete contact case, in the second mode the wave propagation velocity related to the full slipping case may be greater or less than that obtained for other possible cases. At the same time, according to Fig. 7 we can conclude that in Case III, as well as in the other two previous cases, the flexural wave propagation velocity of the second mode for the pair of materials under considerations approach the $c_{R,Al}$ as $kR \rightarrow \infty$.

5. Conclusions

Thus, in the present paper within the scope of the piecewise homogeneous body model by utilizing the three-dimensional linear theory of elastodynamics the influence of the imperfectness of the bonding between the layers of the bi-layered hollow cylinder on the flexural wave dispersion in this cylinder has been studied. The shear-spring type model is used for describing of the imperfectness which characterizes the discontinuity of the axial and circumferential displacements of the constituents on the interface surface. For estimation of the degree of the imperfectness the dimensionless, so called shear spring parameters are introduced. The method for finding the analytical expressions for the sought values have been discussed and detailed. Numerical results on the lowest first and second modes are presented and analyzed in the three cases determined in the beginning of the previous section for the pair of materials *St+Al*. According to these analyses it can be made the following main conclusions:

• As a result of the considered type imperfectness the new branches appear for the first mode.

• The number of the mentioned branches depend on the values of the shear spring parameters.

• The rules of the wave propagation velocity related to the branches are described through the expressions Eqs. (16)-(24).

• The results obtained for the first mode can not be limited with the corresponding ones obtained in the complete and full slipping contact cases between the constituents of the cylinder.

• The dispersion curves obtained for the second mode has only a single branch.

• In Case I and in Case II the rules of the wave propagation velocity related to the second branch are described by the expressions Eqs. (25) and (26).

• In Case I and in Case II the results for the second mode are limited , but in Case III do not be limited with the corresponding ones obtained under complete and full slipping contact cases between the layers of the cylinder.

Although, the discussed results are obtained for the concrete selected pair of materials, these results have also a general meaning.

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