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Non-linear transverse vibrations of tensioned nanobeams using nonlocal beam theory

Süleyman M. Bağdatlı*

Faculty of Engineering, Department of Mechanical Engineering, Celal Bayar University, 45140, Yunusemre, Manisa, Turkey

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Abstract. In this study, nonlinear transverse vibrations of tensioned Euler-Bernoulli nanobeams are studied. The nonlinear equations of motion including stretching of the neutral axis and axial tension are derived using nonlocal beam theory. Forcing and damping effects are included in the equations. Equation of motion is made dimensionless via dimensionless parameters. A perturbation technique, the multiple scale methods is employed for solving the nonlinear problem. Approximate solutions are applied for the equations of motion. Natural frequencies of the nanobeams for the linear problem are found from the first equation of the perturbation series. From nonlinear term of the perturbation series appear as corrections to the linear problem. The effects of the various axial tension parameters and different nonlocal parameters as well as effects of different boundary conditions on the vibrations are determined. Nonlinear frequencies are estimated; amplitude-phase modulation figures are presented for simple-simple and clamped-clamped cases.

Keywords: nanobeam; transverse vibration; axial tension; perturbation method; nonlocal elasticity

1. Introduction

Nano-sized structures have gained considerable interest in scientific research studies in the last two decades due to their superior electrical and mechanical properties. Due to these superior properties, nanostructures are used by scientists in various areas such as sensor technologies, composites and electromechanical systems.

Nonlocal continuum theory proposed by Eringen (1983, 2002) has been used to model the nanostructures. The nonlocal continuum theory was initially applied to nanotechnology by Peddison *et al*. (2003). After this study, many scientific researchers have used the nonlocal models in the nanostructure analysis.

Nonlocal continuum mechanics is used on the column buckling of multiwalled carbon nanotubes (MWCNTs) (Sudak 2003). Nonlocal Euler Bernoulli and Timoshenko beam theories were applied to study the static deformation of micro and nano rods and tubes (Wang and Liew 2007). The nonlocal elasticity theory of Eringen was developed to study the bending, buckling and free vibration of nanobeams (Aydogdu 2009, Reddy 2007, Thai 2012). Wave propagation of single walled carbon nanotubes (SWCNTs) based on the nonlocal beam models was investigated to

*Corresponding author, Assistant Professor, E-mail: murat.bagdatli@cbu.edu.tr

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obtain the influences of nonlocal effects on the wave properties (Lu *et al*. 2007). Buckling of SWCNTs and MWNTs due to the axial compressive loads were studied by molecular mechanics simulation (Sears Batra 2006). The axial vibration of single walled carbon nanotube embedded in an elastic medium is studied using nonlocal elasticity theory (Main and Jones 2007a). Exact analytical solutions were formulated by Main and Jones for free vibrations of tensioned beams with an intermediate viscous damper (Main and Jones 2007b) and tensioned beams with a viscous damper attached transversely near a support (Main and Jones 2007c). Tensioned pipes conveying fluid and carrying a concentrated mass (Öz and Boyaci 2000, Öz 2002) and axially traveling tensioned beams in contact with a stationary mass (Öz 2003) were considered to investigate the natural frequencies. Nonlinear transverse vibrations of a tensioned beam resting on multiple supports (Bağdatlı *et al*. 2011) and transverse vibrations of an axially accelerating beam resting (Bağdatlı *et al*. 2011) on simple supports were investigated using the Euler- Bernoulli beam model. Tensioned beam on an elastic foundation were considered to find the classical critical speed and deflection response (Adams 1995). Nonlinear vibrations of an axially moving mid-supported and multi-supported string have been investigated. There are non-ideal supports allowing minimal deflections between ideal supports at both ends of the beam (Bağdatli and Uslu 2015) and at both ends of the string (Yurddaş *et al*. 2013, Yurddaş *et al*. 2014). A cantilever beam attached to an axially moving base in fluid was studied to investigate the free vibration and stability based on Galerkin approach (Ni *et al*. 2014). Kural *et al*. (2012), studied string–beam transition problem and they found approximately solution by using perturbation methods. Kural *et al*. (2015) were investigated the size effects of beam behavior by adding “Modified Couple Stress Theory” to “Hamilton Principle” and perturbations methods.

Transverse vibration of nanobeam subjected to an initial axial tension based on the nonlocal stress theory is presented, with considering the effects of the dimensionless nanoscale parameter and pre-tension on natural frequencies (Li *et al*. 2011, Lim *et al*. 2009a). Lim *et al*. (2010) analyzed the transverse free vibrations of axially moving nano-beams subjected to axial tension based on nonlocal stress elasticity theory. Also this author and its coauthors (Lim at al. 2009b) presented a new nonlocal stress variational principle approach for the transverse free vibration of an Euler-Bernoulli cantilever nanobeam with an initial axial tension at its free end to obtain the relationship between natural frequency and nanoscale. Vibrational properties of an axially moving SWCNT with simply supported ends (Kiani 2013) and of an axially loaded non-prismatic SWCNT embedded in two parameter elastic medium (Mustapha and Zhong 2010) were studied using nonlocal Rayleigh beam model. The nonlinear primary resonance of nanobeam with the axial initial load were investigated based on the nonlocal continuum theory, with the influences of small scale effect, axial initial load, mode number, Winkler foundation modulus and the length to the diameter (Wang and Li 2014). Bağdatli (2015) was studied vibration of nanobeam using nonlocal beam theory for different boundary conditions. A new methodology were applied to detect the free vibration response of different nano-Timoshenko beams with different boundary conditions, material exponents and nonlocality parameters on the fundamental frequencies of nanobeam (Eltaher *et al*. 2014). Free and force axial vibrations of damped nonlocal rods were investigated (Adhikari *et al*. 2013). Vibration analysis of coupled nanobeam system under initial compressive pre-stressed condition were presented using Eringen nonlocal Elasticity model to detect the pre-load effects on the nonlocal frequencies (Murmu and Adhikari 2012). Coupled mechanical and electronic behaviors of SWCNT under applied electric field and tensile loading were investigated by the use of quantum mechanics (Guo and Guo 2003). A nonlocal Euler-Bernoulli beam model with axial prestress was established based on the nonlocal elasticity theory (Lu 2007).

Up to now, various aspects of vibrations of nanobeam have been addressed; however, vibrations of tensioned nanobeam have not been thoroughly assessed. Here in, nonlinear transverse vibrations of a tensioned nanobeam under two boundary conditions are studied using nonlocal Euler-Bernoulli beam theory. The nanobeam is stretched during vibration due to immovable supports. Transverse forcing and damping terms are also added in the problem. Exact natural frequencies are calculated for two boundary conditions, different axial tension parameters (v_p) and different nonlocal parameters (γ). The method of multiple scales, a perturbation technique, is used to solve the nonlinear equations approximately. The first terms in the expansions lead to the linear problem. The natural frequencies are calculated exactly and showed for different parameters and conditions. The addition of nonlinear terms then introduces corrections to the linear problem. The amplitude and phase modulation equations are determined from the nonlinear analysis. Free vibrations and forced vibrations with damping are investigated in detail. The effects of stretching on the nanobeam vibrations are considered for nonlocal parameters and sup-port conditions.

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$$\sigma(\hat{x}) = \int_V K(|\hat{x} - \hat{x}'|, \gamma) T(\hat{x}') dV(\hat{x}') \tag{1}$$

$$T(\hat{x}) = C(\hat{x}) : \varepsilon(\hat{x}) \tag{2}$$

where $\hat{x}$ is a reference point within domain V; $\sigma(\hat{x})$ and $\varepsilon(\hat{x})$are the second order tensors representing stress and strain fields, respectively; $T(\hat{x})$ is the classical, macroscopic stress tensor at point $\hat{x}$, $C(\hat{x})$ is the fourth order elasticity tensor, $K(|\hat{x}-\hat{x}'|, \gamma)$ is the nonlocal modulus or attenuation function. Typically, $K(|\hat{x}-\hat{x}'|, \gamma)$ is a function of material constant γ and the Euclidian distance $|\hat{x}-\hat{x}'|$ (Eringen 1983, 2002). The material constant γ defined as e_0a/L depends on the internal characteristic lengths a, external characteristics lengths L and e_0 is a constant appropriate to each material. The parameter e_0a is the nonlocal parameter revealing the small-scale effect on the responses of nanoscale structures.

The solution of nonlocal elasticity problems is very hard to solve mathematically because of the spatial integrals in the nonlocal relations. But, these integropartial equations can be approximately transformed to equivalent differential constitutive equations by using Green's function (Nayfeh and Mook 1979, Peddieson *et al*. 2003). Therefore, the constitutive relation is given as follows

$$\left(1 - (e_0\mathrm{a})^2 \nabla^2\right)\sigma = T \tag{3}$$

where ∇^2 is the Laplacian operator. The nonlocal constitutive relation for a homogeneous isotropic

Euler-Bernoulli beam takes the following form

$$\sigma(\hat{x}) - (e_0 \mathrm{a})^2 \sigma''(\hat{x}) = E\varepsilon(\hat{x}) \tag{4}$$

where E is the elasticity modulus.

3. Equations of motion

Governing equations of the tensioned nanobeam derived Lagrange's equations from Hamilton's Principles. The Lagrange equations of the proposed model is given in Eq. (5), $\hat{w}$ denotes the transverse displacement of the nanobeam section between supports, ρA is the mass, EI is flexural rigidity, L is the length of the nanobeam, e_0a is the nonlocal parameter of nanobeam, $\hat{t}$ is the time, EA is longitudinal rigidity and $\hat{P}$ is the axial tension force on nanobeam.

$$\begin{aligned} £ = \frac{1}{2}\int_0^L \rho A \dot{\hat{w}}^2 d\hat{x} - \frac{1}{2}\int_0^L \left(EI\hat{w}'' + (e_0 a)^2 \left(\frac{EA}{2L}\int_0^L \hat{w}'^2 d\hat{x} \right) \hat{w}'' + (e_0 a)^2 \hat{P}\hat{w}^{iv} - (e_0 a)^2 \rho A \ddot{\hat{w}} \right) \hat{w}'' d\hat{x} \\ - \frac{1}{2}\int_0^L \left(\frac{EA}{2L}\int_0^L \hat{w}'^2 d\hat{x} \right) \hat{w}'^2 - \frac{1}{2}\int_0^L \hat{P}\hat{w}'^2 \end{aligned} \tag{5}$$

where ($\dot{}$) shows respect to $d/d\hat{t}$ and ( )′ shows respect to $d/d\hat{x}$. The kinetic energy of the beam, the elastic energy in bending, the elastic energy in extension due to stretching of the neutral axis and the elastic energy due to axial tension are written in Eq. (5), respectively. The equations of motion and boundary conditions for the boundary condition cases for the nanobeam in dimensional form is obtained by applying Hamilton's principle and performing the necessary algebra as follows

$$\begin{aligned} EI\hat{w}^{iv} + \rho A \ddot{\hat{w}} + (e_0 a)^2 \left(\frac{EA}{2L}\int_0^L \hat{w}'^2 d\hat{x} \right) \hat{w}^{iv} - (e_0 a)^2 \hat{P}\hat{w}^{iv} - (e_0 a)^2 \rho A \ddot{\hat{w}}'' \\ - \left(\frac{EA}{2L}\int_0^L \hat{w}'^2 d\hat{x} \right) \hat{w}'^2 - \hat{P}\hat{w}'^2 = 0 \end{aligned} \tag{6}$$

The boundary conditions can be shown as follows

$$\begin{array}{ll} \underline{\text{Simple-Simple Case:}} & \underline{\text{Clamped-Clamped Case:}} \\ \hat{w}(0)=0, \quad \hat{w}(1)=0, & \hat{w}(0)=0, \quad \hat{w}(1)=0, \\ \hat{w}''(0)=0, \quad \hat{w}''(1)=0 & \hat{w}'(0)=0, \quad \hat{w}'(1)=0 \end{array} \tag{7}$$

The dimensionless quantities are related to the dimensional ones through the following relations

$$x = \frac{\hat{x}}{L}, w = \frac{\hat{w}}{R}, \ t = \frac{\hat{t}}{\sqrt{\rho A L^4 / EI}}, \gamma = \frac{e_0 a}{L}, v_p^2 = \frac{\hat{P}}{EI / L^2} \tag{8}$$

where R is the radius of gyration of the nanobeam cross-section with respect to the neutral axis. Using the Eq. (8) into the Eq. (6) yields

$$w^{iv} + \ddot{w} + \gamma^2 v_p^2 w^{iv} - \gamma^2 \ddot{w}'' - v_p^2 w'' = \frac{1}{2}\left[\int_0^L w'^2 dx\right]\left[w'' - \gamma^2 w^{iv}\right] \tag{9}$$

The non-dimensional form of boundary conditions can be shown as follows

$$\begin{array}{llll} \underline{\text{Simple-Simple Case:}} & & \underline{\text{Clamped-Clamped Case:}} & \\ w(0)=0, & w(1)=0 & w(0)=0, & w(1)=0, \\ w''(0)=0, & w''(1)=0 & w'(0)=0, & w'(1)=0 \end{array} \tag{10}$$

4. Multiple scales method

The multiple scales method will be used to solve the problem (Nayfeh and Mook 1979).

$$w^{iv} + \ddot{w} + \gamma^2 v_p^2 w^{iv} - \gamma^2 \ddot{w}'' - v_p^2 w'' = \frac{1}{2}\left[\int_0^L w'^2 dx\right]\left(w'' - \gamma^2 w^{iv}\right) + \tilde{F}\cos\Omega t - 2\tilde{\mu}\dot{w} \tag{11}$$

The multiple scales method will be applied to the partial differential equation system and boundary conditions directly (Nayfeh and Mook 1979, Nayfeh 1981). There is no quadratic non-linearity, that's why one can write an expansion of the form

$$w(x,t;\varepsilon) = \varepsilon w_0(x,T_0;T_2) + \varepsilon^3 w_1(x,T_0;T_2) \tag{12}$$

where ε is a small parameter that the deflections are small. This procedure models a weak non-linear system. $T_0=t$ and $T_2=\varepsilon^2 t$ are the fast and slow time scales. The forcing and damping terms are ordered as expressed below so that they are counter effect of nonlinearity, $\tilde{\mu} = \varepsilon^3\mu$ and $\tilde{F} = \varepsilon^3 F$ the time derivatives are written in terms of the new time variables $\partial/\partial t=D_0+\varepsilon D_2$, $\partial^2/\partial t^2=D_0^2+2\varepsilon D_0 D_2$, where $D_n=\partial/\partial T_n$. One obtains equations of motion after expansion as follows

Order (ε)

$$w_0^{iv} + \gamma^2 v_p^2 w_0^{iv} + D_0^2 w_0 - \gamma^2 D_0^2 w_0'' - v_p^2 w_0'' = 0 \tag{13}$$

Order (ε^3)

$$\begin{aligned} w_1^{iv} + \gamma^2 v_p^2 w_1^{iv} + D_0^2 w_1 - \gamma^2 D_0^2 w_1'' - v_p^2 w_1'' = -2D_0 D_2 w_0 + 2\gamma^2 D_0 D_2 w_0'' + \frac{1}{2}\left[\int_0^1 w_0''^2 dx\right] w_0'' \\ -\frac{1}{2}\gamma^2\left[\int_0^1 w_0''^2 dx\right] w_0^{iv} + F\cos\Omega t - 2\mu D_0 y_0 \end{aligned} \tag{14}$$

Solution of the first order of expansion gives natural frequency values and a solvability condition is obtained from the third order of expansion. The first order of perturbation is linear given in the Eq. (13); the solution may be represented by

$$w_0(x,T_0,T_2)=\left[A(T_2)e^{i\omega T_0}+cc\right]Y(x) \tag{15}$$

where $A(T_2)$ complex amplitudes and cc stands for their conjugate of the preceding terms. $Y(x)$ estimated the following equations and boundary conditions. Substituting Eq. (15) into Eq. (13), one has

$$Y^{iv}+\gamma^2 v_p^2\, Y^{iv}+\omega^2\gamma^2 Y''-\omega^2 Y-v_p^2 Y''=0 \tag{16}$$

The solution of the equations can be sought by assuming the following shape function and boundary conditions

$$Y(x)=c_1e^{i\beta_1 x}+c_2e^{i\beta_2 x}+c_3e^{i\beta_3 x}+c_4e^{i\beta_4 x} \tag{17}$$

$$\text{S-S Case: } Y(0)=0,\ \ Y''(0)=0,\ \ Y(1)=0,\ \ Y''(1)=0$$

$$\text{C-C Case: } Y(0)=0,\ \ Y'(0)=0,\ \ Y(1)=0,\ \ Y'(1)=0 \tag{18}$$

Numerical values of β_n is calculated by using Eq. (16) and Eq. (17) as follows

$$\beta_n=\pm\sqrt{\frac{-\left(\omega^2\gamma^2-v_p^2\right)\pm\sqrt{\left(\omega^2\gamma^2-v_p^2\right)^2+4\omega^2\left(1+\gamma^2 v_p^2\right)}}{2\left(1+\gamma^2 v_p^2\right)}}\qquad (n=1,2,3,4) \tag{19}$$

When the boundary conditions are applied the frequency equations can be obtained. Since the homogenous problems described by Eq. (13) have a non-trivial solution, the inhomogenous Eq. (14) has a non-secular solution only if the following solvability condition is determined as explained in reference (Nayfeh and Mook 1979).

$$w_1(x,T_2,T_1)=\phi(x,T_2)e^{i\omega T_0}+W(x,T_0,T_2)+cc \tag{20}$$

and substituting Eq. (20) into Eq. (14), we eliminate the terms producing secularities. Here $W(x,T_0,T_2)$ stands for the solution related with non-secular terms. One obtains

$$\begin{aligned}\phi^{\mathrm{iv}}+\gamma^2v_p^2\phi^{\mathrm{iv}}-\omega^2\phi+\gamma^2\omega^2\phi''-v_p^2\phi''&=-2i\omega \mathrm{D}_2AY+2i\omega\gamma^2\mathrm{D}_2AY''\\&+\frac{3}{2}A^2\overline{A}\left(\int_0^1 Y'^2\mathrm{dx}\right)Y''-\frac{3}{2}\gamma^2A^2\overline{A}\left(\int_0^1 Y'^2\mathrm{dx}\right)Y^{iv}\\&+\frac{1}{2}\mathrm{F}e^{\mathrm{i}\sigma \mathrm{T}_1}-4i\mu\omega AY+cc+NST\end{aligned} \tag{21}$$

where cc stands for complex conjugate of preceding terms and NST stands for non-secular terms. We also assume that excitation frequency is close to one of the natural frequencies of the system; that is

$$\Omega=\omega+\varepsilon^2\sigma(T_2) \tag{22}$$

where σ is a detuning parameter of order 1, the solvability condition for Eqs. (22)-(21) are obtained as follows

$$2i\omega\left(D_2A+2\mu A\right)+2i\omega\gamma^2 D_2Ab+\frac{3}{2}A^2\bar{A}\left(b^2+\gamma^2 bd\right)-\frac{1}{2}e^{i\sigma T_1}f=0 \tag{23}$$

where $\int_0^1 Y^2 dx = 1,\quad \int_0^1 Y'^2 dx = b,\quad \int_0^1 Y''^2 dx = d,\quad \int_0^1 FY dx = f$. The complex amplitude A in Eq. (24) can be written in terms of a real amplitude a and a phase θ

$$A=\frac{1}{2}a(T_2)e^{i\theta(T_2)} \tag{24}$$

Then amplitude and phase modulation equations can be obtained as follows

$$\omega a D_2\psi=\omega a\sigma+\omega\gamma^2 ab\sigma-\omega\gamma^2 ab D_2\psi-\frac{3}{16}a^3\left(b^2+\gamma^2 bd\right)+\frac{1}{2}f\cos\psi \tag{25}$$

$$\omega D_2 a\left(1+\gamma^2\right)+2\mu\omega a=\frac{1}{2}f\sin\psi \tag{26}$$

where, $\theta=\sigma T_2-\psi$. The Eqs. (25)-(26) will be solved for steady-state case in the next section and variation of nonlinear amplitude will be discussed.

5. Numerical results

In this section numerical studies for frequencies will be shown for different cases. Firstly, the linear natural frequencies for different support conditions will be calculated. Then, the nonlinear frequencies for free, undamped vibrations will be calculated. For this case, by taking $\mu=f=\sigma=0$, one obtains

$$D_2 a=0 \quad \text{and} \quad a=a_0 \quad \text{(constant)} \tag{27}$$

from Eq. (27). Here a_0 is the steady-state real amplitude. The non-linear frequency is

$$\omega_{n1}=\omega+a_0^2\lambda=\omega+\frac{3a_0^2}{16}\frac{\left(b^2+\gamma^2 bd\right)}{\omega\left(1+\gamma^2 b\right)} \tag{28}$$

where λ is the correction coefficient due to nonlinear terms. At the steady state, $D_2a=0$, $D_2\psi=0$ become zero. The frequency detuning parameter is as follows

$$\sigma=\frac{3a_0^2}{16}\frac{\left(b^2+\gamma^2 bd\right)}{\omega\left(1+\gamma^2 b\right)}\mp\sqrt{\left(\frac{f}{2\omega a\left(1+\gamma^2 b\right)}\right)^2-\mu^2} \tag{29}$$

Table 1 and Table 2 show the linear frequencies and nonlinear correction terms for the first frequencies for simple-simple case and clamped-clamped case in different axial tension parameters (v_p) and different nonlocal (γ) parameters, respectively. The effects of support conditions, stretching parameters and nonlocal parameters are given. Generally, when the γ values increase, the linear frequencies decrease, but the correction terms increase. It can be seen in the tables that nonlocal linear frequencies are smaller than the local linear frequencies for both type of boundary conditions. Local (classical) linear frequencies can be obtained by applying the nonlocal parameter equals γ to zero value. Furthermore, the small-scale effects play an important role in the analysis of nanobeam. When v_p values increase, the linear frequencies increase, but the correction term decreases and same γ values as shown in Tables 1-2. With increasing the v_p values, the system becomes more stiffness than previous. So, this phenomenon increases the frequencies of the system. Supporting condition has also an effect on the stiffness of system. When Tables 1 and 2 are compared according to the supporting conditions, frequencies corresponding to the clamped-clamped boundary condition are higher than the simple-simple boundary conditions.

Table 1 The first five frequencies and correction term due to nonlinear terms for different v_p and γ values for Simple-Simple support condition

				v_p=0				
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	9.8696	8.3569	6.1456	4.6254	3.6488	2.9936	2.048773	1.551272
ω_2	39.4784	24.5823	14.5951	10.1219	7.7030	6.2051	6.267525	3.131692
ω_3	88.8264	41.6285	22.7743	15.4680	11.6787	9.3722	8.365816	4.705771
ω_4	157.9140	58.3803	30.8121	20.7621	15.6308	12.5268	12.55852	6.278218
ω_5	246.7400	74.8398	38.7818	26.0338	19.5731	15.6762	14.65403	7.850006
λ	1.8506	2.1855	2.9719	3.9487	5.0055	6.1011	8.9147	11.7737
				v_p=1				
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	10.3575	8.9279	6.9020	5.5914	4.8149	4.3395	3.75061	3.503719
ω_2	39.9753	25.3726	15.8901	11.9135	9.9406	8.8307	7.538504	7.020393
ω_3	89.3250	42.6821	24.6475	18.1131	15.0073	13.2915	11.31849	10.53426
ω_4	158.4130	59.7175	33.2761	24.2689	20.0558	17.7436	15.09638	14.04741
ω_5	247.2400	76.4705	41.8422	30.4056	25.0967	22.1920	18.8734	17.56026
λ	1.7634	2.0457	2.6462	3.2665	3.7933	4.2088	4.86966	5.21281
				v_p=5				
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	18.5513	17.7926	16.8674	16.3748	16.1262	15.9907	15.84101	15.78438
ω_2	50.4530	39.8905	34.6407	33.0063	32.3465	32.0229	31.69087	31.57163
ω_3	100.5525	62.8776	52.3386	49.5976	48.5495	48.0468	47.53886	47.35827
ω_4	169.9546	85.7677	69.9802	66.1733	64.7469	64.0684	63.38634	63.14474
ω_5	258.9386	108.4874	87.5930	82.7421	80.9420	80.0890	79.23363	78.93114
λ	0.9845	1.0265	1.0828	1.1154	1.1326	1.1422	1.15297	1.15711
				v_p=10				
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	32.9298	32.5084	32.0114	31.7546	31.6271	31.5582	31.48266	31.4542
ω_2	74.2050	67.4695	64.5047	63.6419	63.3023	63.1375	62.96977	62.90985
ω_3	129.5098	103.0319	96.9604	95.5086	94.9686	94.7126	94.45595	94.36519
ω_4	201.8120	138.5627	129.3861	127.3673	126.6321	126.2865	125.9419	125.8204
ω_5	292.4973	173.9972	161.7963	159.2224	158.2944	157.8599	157.4277	157.2757
λ	0.5546	0.5618	0.5706	0.5752	0.5775	0.5787	0.580135	0.58066

Table 2 The first five frequencies and correction term due to nonlinear terms for different v_p and γ values for Clamped-Clamped support condition

v_p=0								
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	22.3733	18.2894	12.9047	9.5092	7.4236	6.0566	4.119695	3.112145
ω_2	61.6728	36.4239	21.1398	14.5656	11.0565	8.8954	5.963916	4.481861
ω_3	120.9034	54.5240	29.9624	20.4589	15.4924	12.4530	8.343044	6.268465
ω_4	199.8594	71.6126	37.8277	25.5073	19.2093	15.3971	10.28442	7.718526
ω_5	298.5555	88.4869	46.1502	31.0922	23.4182	18.7740	12.54335	9.414965
λ	12.3026	12.5277	12.8090	12.9590	13.0342	13.0750	13.1198	13.13672
v_p=1								
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	22.6464	18.8739	14.0491	11.1922	9.5799	8.6193	7.455771	6.976571
ω_2	62.0450	37.3649	22.8894	17.0634	14.2126	12.6189	10.77206	10.03415
ω_3	121.3117	55.7801	32.3596	23.9146	19.8783	17.6391	15.05528	14.02559
ω_4	200.2882	73.1787	40.8153	29.7922	24.6313	21.7975	18.55245	17.26638
ω_5	298.9973	90.3642	49.7660	36.2970	30.0157	26.5691	22.62263	21.05843
λ	12.2877	12.5158	12.8023	12.9555	13.0323	13.0739	13.11951	13.1366
v_p=5								
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	31.25623	31.17285	31.00693	30.62057	29.61107	28.35411	28.34307	28.31456
ω_2	70.3486	55.3389	48.7107	46.7678	46.0002	45.6272	45.24704	45.11111
ω_3	130.7192	80.2328	68.0636	65.2087	64.1739	63.6914	63.21382	63.04665
ω_4	210.3133	103.9103	85.4674	81.0826	79.4455	78.6679	77.88691	77.61043
ω_5	309.4094	150.3101	103.9232	98.6683	96.7555	95.8585	94.96596	94.65218
λ	11.9874	12.3816	12.7703	12.9472	13.0297	13.0730	13.83763	13.839965
v_p=10								
γ	0.1	0.2	0.4	0.6	0.8	1	1.5	2
ω_1	61.6260	61.0054	59.7866	57.0162	50.0189	40.9932	40.27869	40.21717
ω_2	91.2643	90.9340	90.2787	90.0690	89.9852	89.9443	89.90258	89.88765
ω_3	166.7472	157.8793	156.3955	125.5434	125.5115	125.5110	125.5991	125.6250
ω_4	238.8921	239.4884	257.2408	156.0310	155.3685	155.0592	154.7520	154.6441
ω_5	339.8710	346.8556	354.3935	189.8461	189.2122	188.9391	188.6853	188.6007
λ	11.4550	12.3028	12.7636	12.9462	13.0295	13.0729	13.39438	13.41505

Nonlinear frequency versus amplitude curves are plotted in Figs. 1-6 for different γ values, different v_p and different modes. In Figs. 1-2, the variation of nonlinear frequency is plotted for the first mode and v_p=1 when γ=0-0.2-0.4-0.6-0.8-1-1.5-2. From this figures, as nonlocal parameter (γ) increases, the nonlinear frequencies decrease. However, with increasing the nonlocal parameter (γ) influence of amplitude to the nonlinear frequency has been increased. Figs. 3-4, the variation of nonlinear frequency is plotted for different modes when v_p=1 and γ=0.4. From this figures, as modes increases, the nonlinear frequencies increases. It can also be observed that with increasing of the axial tension parameter $v_{p,}$ the influence of amplitude to the nonlinear frequency has been increased. In Figs. 5-6, the variation of nonlinear frequency is plotted for the first modes when

γ=0.4. From this figures, as the effect of axial tension (v_p) increases, the nonlinear frequencies increase. The same phenomena can also be observed with increasing the mode number.

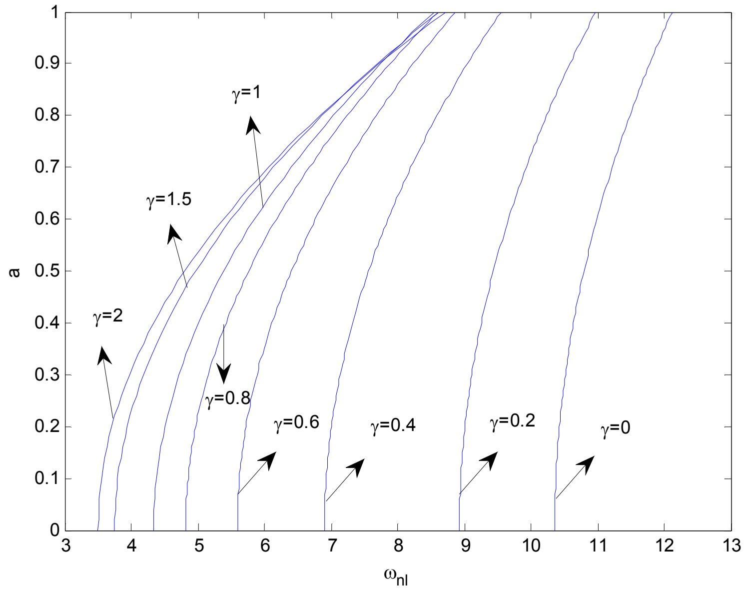

Fig. 1 Nonlinear natural frequency versus amplitude for v_p=1 and different nonlocal parameters (first mode, S-S Case)

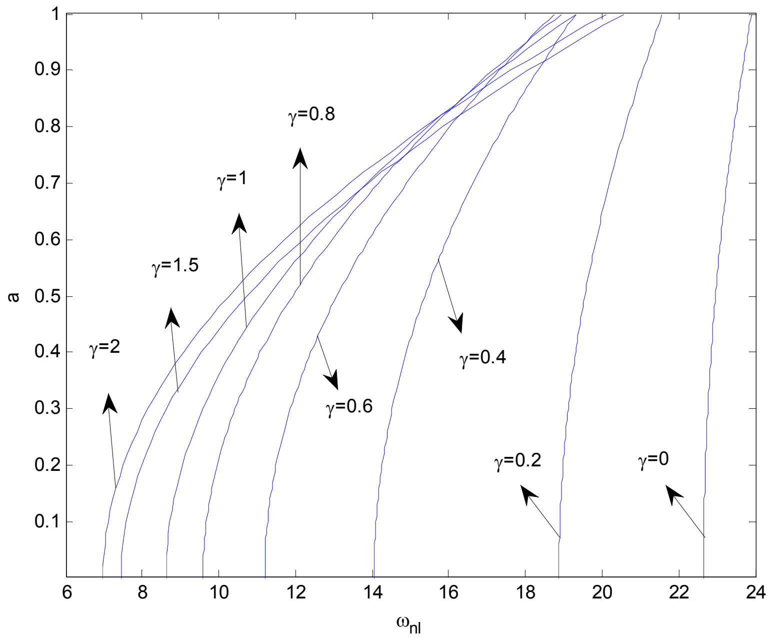

Fig. 2 Nonlinear natural frequency versus amplitude for v_p=1 and different nonlocal parameters (first mode, C-C Case)

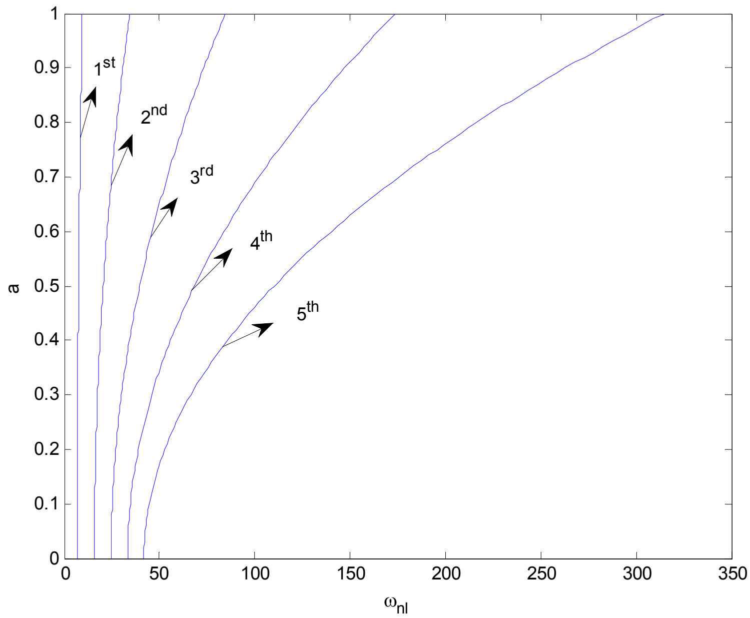

Fig. 3 Nonlinear frequency versus amplitude for different modes (S-S Case, v_p=1 and γ=0.4)

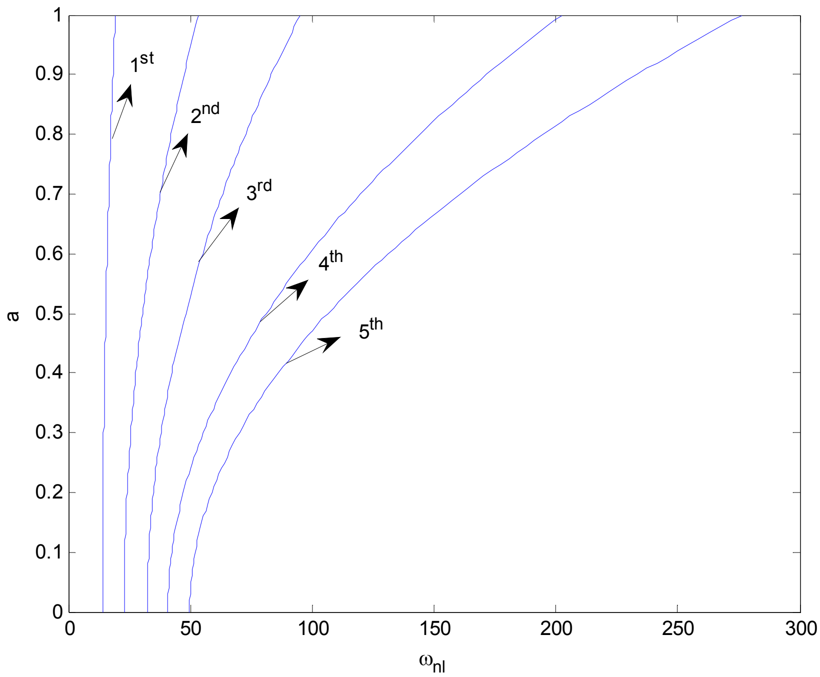

Fig. 4 Nonlinear frequency versus amplitude for different modes (C-C Case, v_p=1 and γ=0.4)

Frequency response curves are presented in Figs. 7-12. The detuning parameter shows the nearness of the external excitation frequency to the natural frequency of the system. Several figures are drawn using Eq. (29) assuming f =1 and damping coefficient μ=0.1. Figs. 7-8 show the influence of nonlocal parameter on the frequency response curves for S-S and C-C case, respectively. It can be seen that for two boundary conditions, the hardening effect is increased by increasing the nonlocal parameter. Hence, the nonlocal parameter is very important for a particular

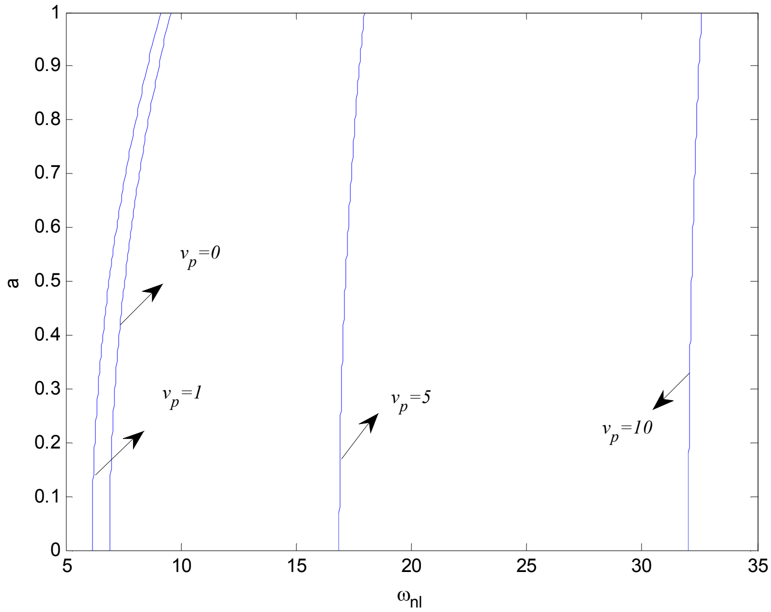

Fig. 5 Nonlinear frequency versus amplitude for γ=0.4 (first mode, S-S Case)

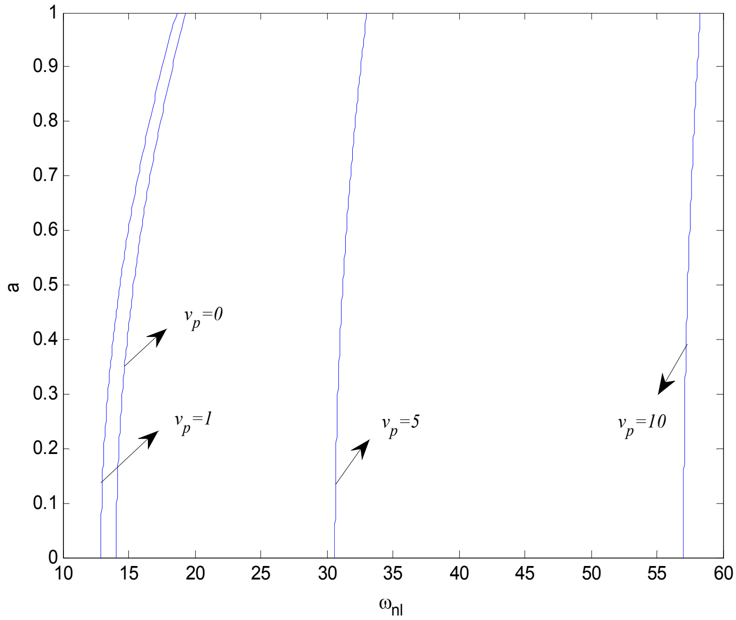

Fig. 6 Nonlinear frequency versus amplitude for γ=0.4 (first mode, C-C Case)

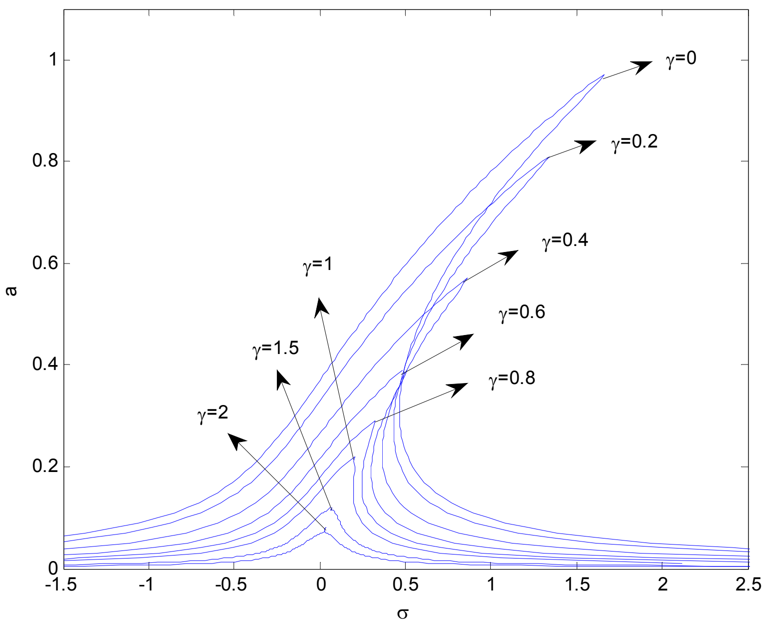

Fig. 7 Forcing frequency-amplitude curves for v_p=1 and different nonlocal parameters (first mode, S-S Case)

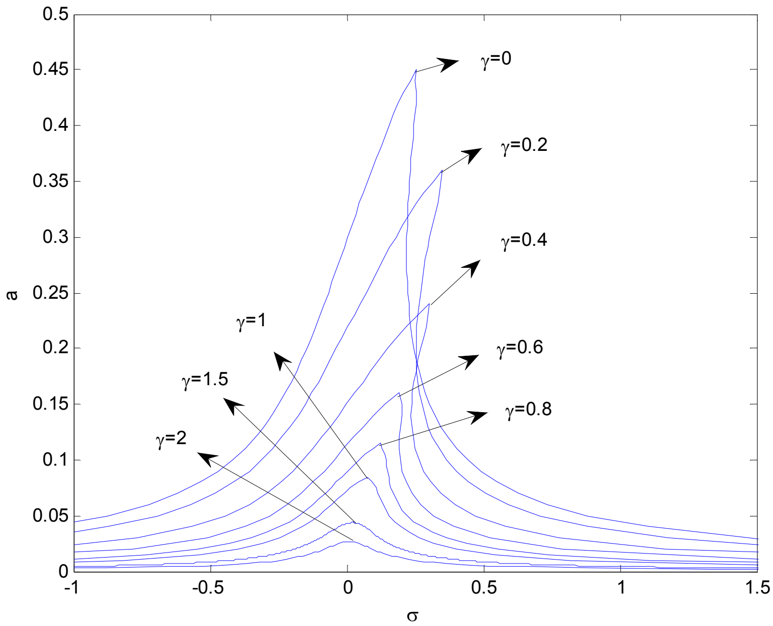

Fig. 8 Forcing frequency-amplitude curves for v_p=1 and different nonlocal parameters(first mode, C-C Case)

system. Figs. 9-10 denotes the mode number effect on the hardening nonlinearity. The first five mode numbers are considered and compared. The axial tension parameter v_p=1 and nonlocal parameter γ=0.4 is taken. It can be observed in the figures that the amplitude is larger and the width is broader for the primary mode (mode1). Figs. 11-12 show the frequency response curve for the simple-simple and clamped-clamped boundary conditions, respectively. It can be observed in the same figures that increasing the axial tension parameter v_p, the amplitude reaches lower value both type of boundary condition. The frequency response bending to the left side is called

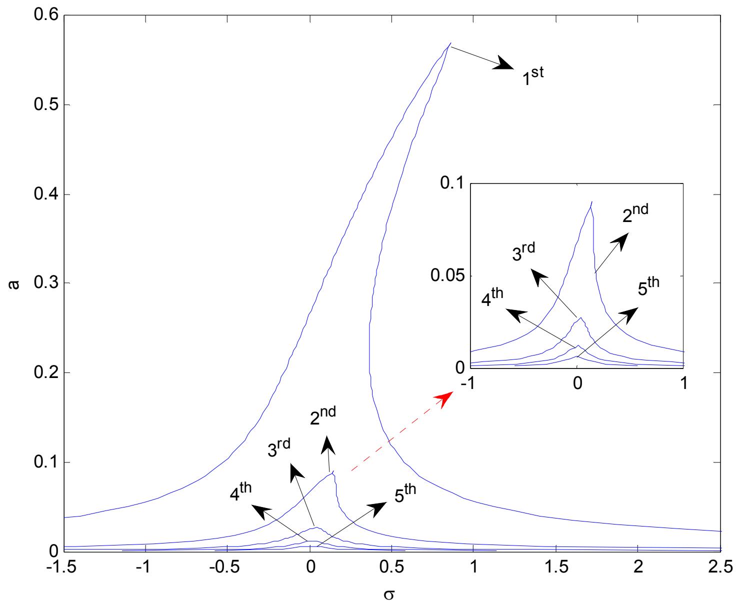

Fig. 9 Forcing frequency-amplitude curves for different modes (S-S Case, v_p=1 and γ=0.4)

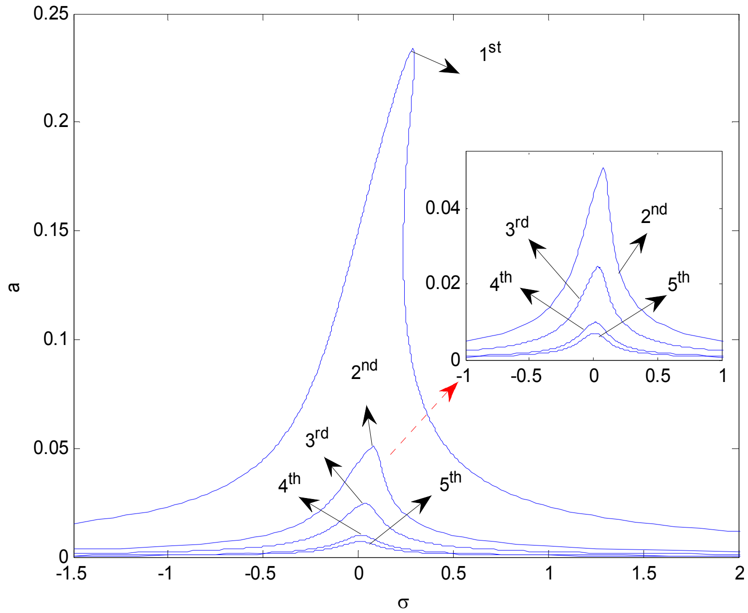

Fig. 10 Forcing frequency-amplitude curves for different modes (C-C Case, v_p=1 and γ=0.4)

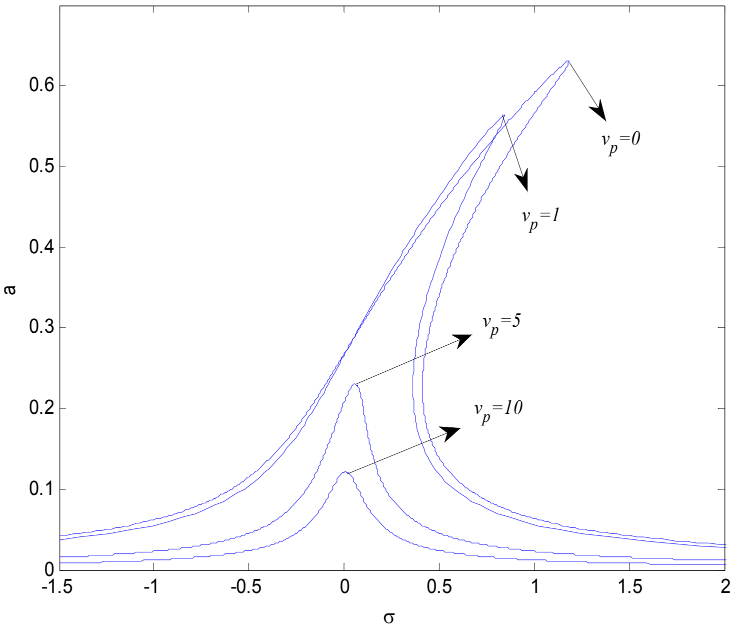

Fig. 11 Forcing frequency-amplitude curves for γ=0.4 (first mode, S-S Case)

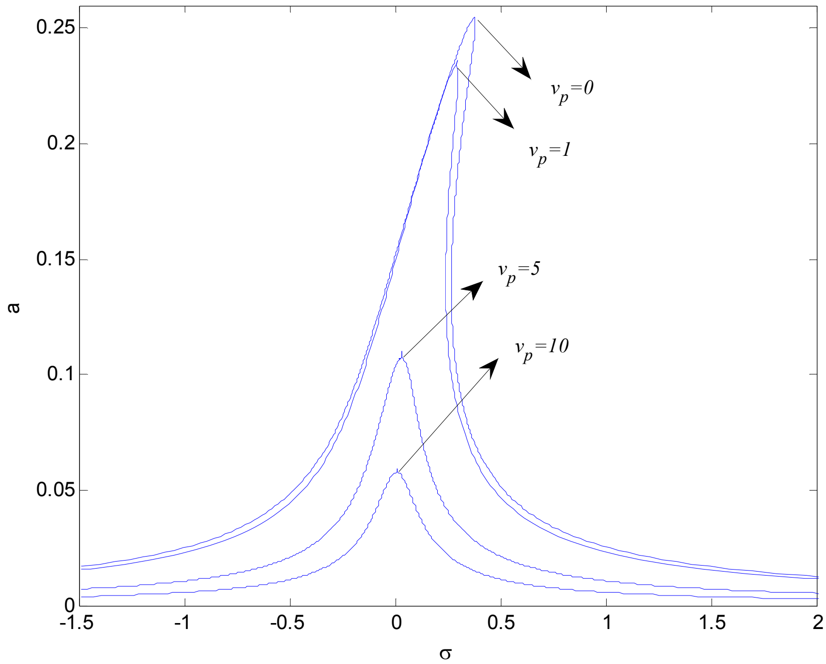

Fig. 12 Forcing frequency-amplitude curves for γ=0.4 (first mode, C-C Case)

the softening nonlinearity, but to the right side is called the hardening nonlinearity. So, the behavior in Figs. 11-12 is of hardening type. Nanobeam hardening nonlinearity can be decreased by increasing the v_p values.

When axial tension decreases, the jump region and maximum amplitude increases. Similarly, the jump region and maximum amplitude increases, when nonlocal parameter decreases. On the other hand, the jump region and maximum amplitude increases, when mode values decreases and vice versa.

6. Conclusions

The vibrations of nanobeam having different boundary conditions are presented as nonlocal Euler-Bernoulli beam type. The equation of motion is derived including axial tension and stretching of the neutral axis. The multiple scales method is used to acquire approximate solutions. For linear problem, exact solutions for natural frequencies and numerical values are investigated. For the non-linear problem, correction terms to linear problem are acquired. Nonlinear terms of the perturbation series appear as corrections to the linear problem. Nonlinear free and forced vibrations are given in detail. The effects of the γ and v_p are determined. As γ increases, natural frequencies decrease and correction terms increase. As v_p increases, natural frequencies increase and the correction terms decrease. Axial tension and stretching of the neutral axis cause a hardening nonlinearity type. The behavior is of hardening type in all figures. Nonlocal parameter (γ) and axial tension parameter (v_p) has an effect on the jump region.

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