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Frequency analysis of eccentric hemispherical shells with variable thickness

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Abstract. A three-dimensional (3-D) method of analysis is presented for determining the free vibration frequencies of eccentric hemi-spherical shells of revolution with variable thickness. Unlike conventional shell theories, which are mathematically two-dimensional (2-D), the present method is based upon the 3-D dynamic equations of elasticity. Displacement components u_r, u_θ, and u_z in the radial, circumferential, and axial directions, respectively, are taken to be periodic in θ and in time, and algebraic polynomials in the r and z directions. Potential and kinetic energies of eccentric hemi-spherical shells with variable thickness are formulated, and the Ritz method is used to solve the eigenvalue problem, thus yielding upper bound values of the frequencies by minimizing the frequencies. As the degree of the polynomials is increased, frequencies converge to the exact values. Convergence to three or four-digit exactitude is demonstrated for the first five frequencies of the shells. Numerical results are presented for a variety of eccentric hemi-spherical shells with variable thickness.

Keywords: vibration; eccentric hemi-spherical shell; variable thickness; shell of revolution

1. Introduction

Many researchers have analyzed by three-dimensional (3-D) approaches the free vibration problems of solid spheres (Poisson 1829, Jaerisch 1880, Lamb 1882, Chree 1889, Sato and Usami 1962, Buchanan and Rich 2002), complete hollow spheres (Sato and Usami 1962, Buchanan and Rich 2002, Shah *et al*. 1969, Cohen and Shah 1972, Grigorenko and Kilina 1990, Chang and Demkowicz 1995, Ding and Chen 1996, Jiang *et al*. 1996, McGee 1997), and incomplete hollow spheres (Buchanan and Rich 2002, Kang and Leissa 2004, Kang 2013, Fazzolari 2014, Sahoo 2014, Su *et al*. 2014, Tornabene *et al*. 2014, Ye *et al*. 2014). A review article by Qatu (2002) summarizes research on vibrations of homogeneous shells, including spherical ones, published between 1989 and 2000. Recently, some researchers (Panda and Singh 2009, Fadaee *et al*. 2013, Ghavanloo and Fazelzadeh 2013) analyzed spherical shells having uniform or variable thickness by 2-D shell theory. Recently, Kang (2012) studied on the vibrations of hemi-spherical shell with eccentricity with uniform shell thickness. However, it is known that there is no published literature on the problem of eccentric hemispherical shells of revolution with variable thickness based on 3-

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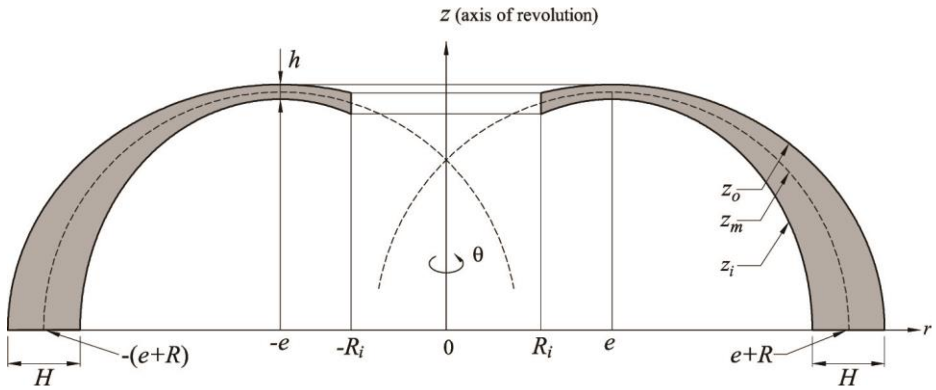


Fig. 1 A cross-section of an eccentric hemispherical shell of revolution with eccentricity (e) having variable thickness and the circular cylindrical coordinate system (r, z, θ)

D analysis. Conventional shell analysis, which is mathematically 2-D, makes simple kinematic assumptions about the variation of displacements through the thickness. Almost all such shell analyses assume, at most, constant plus linear variations. Such assumptions reduce the 3-D theory to a 2-D one, characterized by the middle surface displacements. And such analyses are typically accurate for thin or moderately thick shells of homogeneous and isotropic materials, and at least for the lower frequencies (i.e., long wave lengths of mode shapes). But for thicker shells or higher frequencies, a 3-D analysis becomes necessary for accurate frequencies.

In the present work eccentric hemispherical shells of revolution with variable thickness are analyzed by a 3-D approach. Instead of attempting to solve equations of motion, an energy approach is followed which, as sufficient freedom is given to the three displacement components, yields frequency values as close to the exact ones as desired. To evaluate the energy integrations over the shell volume, displacements and strains are expressed in terms of the circular cylindrical coordinates, instead of related 3-D shell coordinates which are normal and tangent to the shell midsurface. Natural frequencies are obtained for fifteen eccentric hemispherical shells of revolution with variable thickness. Such a shell structure may be used in architectural engineering. Very diverse types of shells can be obtained as the values of the eccentricity.

2. Method of analysis

A representative cross-section of an eccentric hemi-spherical shell of revolution with eccentricity of e and a radius of R of the mid-surface (z_m) of the hemi-spherical shell segment of variable thickness is shown in Fig. 1. The shell thicknesses at z=0 and r=e are H and h, respectively. The eccentric hemi-spherical shells of revolution with variable thickness are obtained by rotating the cross-section for $r\geq0$ in Fig. 1 360° about the z-axis. The eccentricity (e) is the distance from the axis of revolution of the shell (z-axis) to the center of the hemi-spherical shell segment. Thus the mid-surface (z_m) of the shell for $r\geq0$ has the equation of $(r-e)^2+z^2=R^2$. The cylindrical coordinate system (r, z, θ), also shown in the figure, is used in the analysis, where θ is the circumferential angle. The domain (Λ) of the shell is obtained by subtracting the inner portion

$$R_i \leq r \leq e+R-H/2\,,\quad 0\leq z\leq z_i\,,\quad 0\leq\theta\leq 2\pi \tag{1}$$

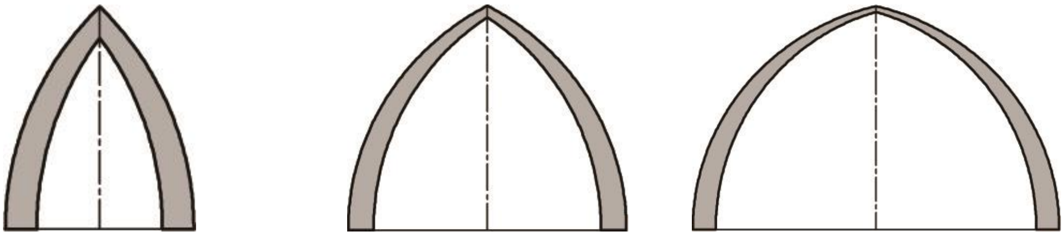


Fig. 2 Cross-sections of closed (R_i=0) hemi-spherical shells of revolution with eccentricity of $-R<e<0$ having variable thickness of H/R=0.1 and h/H=0.2

from the outer portion

$$R_i \le r \le e + R + H/2\,,\quad 0 \le z \le z_o\,,\quad 0 \le \theta \le 2\pi\,, \tag{2}$$

where R_i is the radius of the inner hole of the shell of revolution and $z_{i,o}$ are the coordinates of the elliptical inner and outer surfaces of the cross-section for $r \ge 0$ in Fig. 1, respectively

$$z_{i,o} \equiv \left(R \mp h/2\right)\sqrt{1-\left(\frac{r-e}{R \mp H/2}\right)^2}\,. \tag{3}$$

For mathematical convenience, the radial (r) and axial (z) coordinates are made dimensionless as

$$\psi \equiv r/R\,,\quad \zeta \equiv z/R\,. \tag{4}$$

Thus the domain (Λ) of the shell in terms of the nondimensional circular cylindrical coordinates (ψ, ζ, θ) are given by subtracting the inner portion

$$R_i^* \le \psi \le e^* + 1 - H^*/2\,,\quad 0 \le \zeta \le \zeta_i\,,\quad 0 \le \theta \le 2\pi\,, \tag{5}$$

from the outer portion

$$R_i^* \le \psi \le e^* + 1 + H^*/2\,,\quad 0 \le \zeta \le \zeta_o\,,\quad 0 \le \theta \le 2\pi\,, \tag{6}$$

where

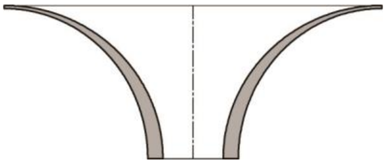
$$\zeta_{i,o} \equiv (1 \mp h^* H^*/2)\sqrt{1-\left(\frac{\psi - e^*}{1 \mp H^*/2}\right)^2}\,, \tag{7}$$

with nondimensional parameters of R_i^*, e^*, H^*, and h^* defined by

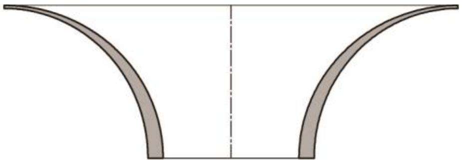
$$R_i^* \equiv R_i/R\,,\quad e^* \equiv e/R\,,\quad H^* \equiv H/R\,,\quad h^* \equiv h/H\,. \tag{8}$$

Utilizing tensor analysis, the three equations of motion in terms of the circular cylindrical coordinate system (r, z, θ) are found to be (Sokolnikoff 1956)

$$\sigma_{rr,r} + \sigma_{rz,z} + \frac{1}{r}(\sigma_{rr} - \sigma_{\theta\theta} + \sigma_{r\theta,\theta}) = \rho \ddot{u}_r\,, \tag{9a}$$



(a) e/R=1.25, R_o/R=1.25



(b) e/R=1.5, R_o/R=1.5

(c) e/R=1.75

Fig. 3 Cross-sections of hemi-spherical shells of revolution with eccentricity of e/R>1 having variable thickness of H/R=0.1 and h/H=0.2

$$\sigma_{rz,r} + \sigma_{zz,z} + \frac{1}{r}(\sigma_{rz} + \sigma_{z\theta,\theta}) = \rho \ddot{u}_z , \tag{9b}$$

$$\sigma_{r\theta,r} + \sigma_{z\theta,z} + \frac{1}{r}(2\sigma_{r\theta} + \sigma_{\theta\theta,\theta}) = \rho \ddot{u}_\theta , \tag{9c}$$

where the σ_{ij} are the normal (i=j) and shear (i≠j) stress components; u_r, u_z, and u_θ are the displacement components in the r, z, and θ directions, respectively; ρ is mass density per unit volume; the commas indicate spatial derivatives; and the dots denote time derivatives.

The well-known relationships between the tensorial stresses (σ_{ij}) and strains (ε_{ij}) of isotropic, linear elasticity are

$$\sigma_{ij} = \lambda \varepsilon \delta_{ij} + 2G\varepsilon_{ij} , \tag{10}$$

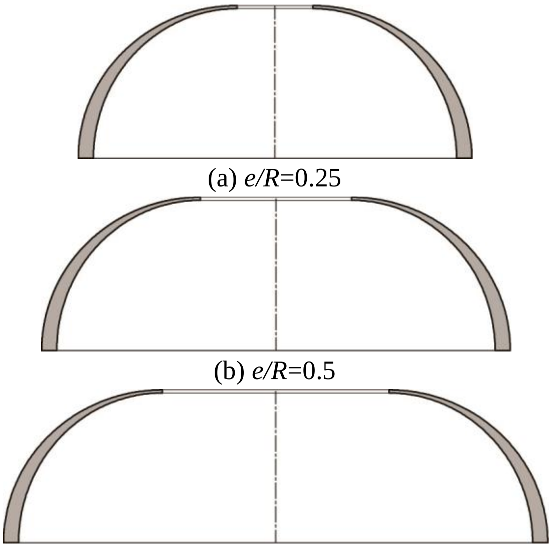
where λ and G are the Lamé parameters, expressed in terms of Young's modulus (E) and Poisson's ratio (ν) for an isotropic solid as

$$\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} , \quad G = \frac{E}{2(1+\nu)} , \tag{11}$$

$\varepsilon \equiv \varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\theta\theta}$ is the trace of the strain tensor, and δ_{ij} is Kronecker's delta.

The 3-D tensorial strains (ε_{ij}) are found to be related to the three displacements u_r, u_z, and u_θ, by (Sokolnikoff 1956)

$$\varepsilon_{rr} = u_{r,r} , \quad \varepsilon_{zz} = u_{z,z} , \quad \varepsilon_{\theta\theta} = \frac{u_r + u_{\theta,\theta}}{r} , \tag{12a}$$



(a) e/R=0.25

(b) e/R=0.5

(c) e/R=0.75

Fig. 4 Cross-sections of open (R_i=e) hemi-spherical shells of revolution with eccentricity of $0<e/R<1$ having variable thickness of H/R=0.1 and h/H=0.2

$$2\varepsilon_{rz} = u_{r,z} + u_{z,r}\,,\quad 2\varepsilon_{r\theta} = u_{\theta,r} + \frac{u_{r,\theta} - u_\theta}{r}\,,\quad 2\varepsilon_{z\theta} = u_{\theta,z} + \frac{u_{z,\theta}}{r}\,. \tag{12b}$$

Substituting Eqs. (10) and (12) into Eqs. (9), one obtains a set of three second-order partial differential equations in u_r, u_z, and u_θ governing free vibrations. However, in the case of eccentric hemi-spherical shells of revolution with variable thickness, exact solutions are intractable because of the variable coefficients that appear in many terms. Alternatively, one may approach the problem from an energy perspective.

Because the strains are related to the displacement components by Eqs. (12), unacceptable strain singularities may be encountered exactly at r=0 due to the term $1/r$ when R_i=0 (no hole). Since a negligibly small hole (R_i≈0) does not affect the frequencies (Kang and Leissa 2004), such singularities may be avoided by replacing R_i^* in Eqs. (5) and (6) with 10^{-3}.

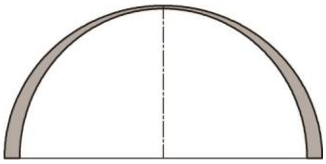
During vibratory deformation of the body, its strain (potential) energy (V) is the integral over the domain (Λ)

$$V = \frac{1}{2}\int_\Lambda (\sigma_{rr}\varepsilon_{rr} + \sigma_{zz}\varepsilon_{zz} + \sigma_{\theta\theta}\varepsilon_{\theta\theta} + 2\sigma_{rz}\varepsilon_{rz} + 2\sigma_{r\theta}\varepsilon_{r\theta} + 2\sigma_{z\theta}\varepsilon_{z\theta})\, r\,dr\,dz\,d\theta\,. \tag{13}$$

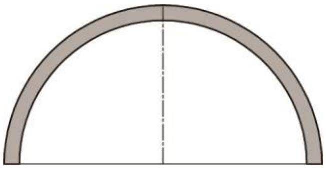
Substituting Eqs. (10) and (12) into Eq. (13) results in the strain energy in terms of the three displacements

$$V = \frac{1}{2}\int_\Lambda [\lambda(\varepsilon_{rr} + \varepsilon_{zz} + \varepsilon_{\theta\theta})^2 + 2G\{\varepsilon_{rr}{}^2 + \varepsilon_{zz}{}^2 + \varepsilon_{\theta\theta}{}^2 + 2(\varepsilon_{rz}{}^2 + \varepsilon_{z\theta}{}^2 + \varepsilon_{r\theta}{}^2)\}]\, r\,dr\,dz\,d\theta\,, \tag{14}$$

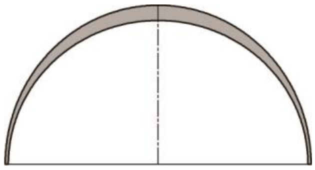
where the tensorial strains ε_{ij} are expressed in terms of the three displacements by Eqs. (12).



(a) H/R=0.1, h/H=0.2



(b) H/R=0.1, h/H=1



(c) H/R=0.02, h/H=5

Fig. 5 Cross-sections of closed (R_i=0) hemi-spherical shells of revolution with eccentricity of e=0 having variable thickness

The kinetic energy (T) is simply

$$T = \frac{1}{2}\int_{\Lambda} \rho(\dot{u}_r^{\,2} + \dot{u}_z^{\,2} + \dot{u}_\theta^{\,2})\, r\,dr\,dz\,d\theta \ . \tag{15}$$

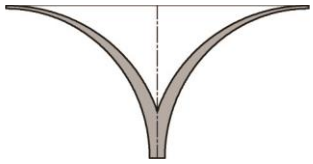
For the free, undamped vibration, the time (t) response of the three displacements is sinusoidal and, moreover, the circular symmetry of the body of revolution allows the displacements to be expressed by

$$u_r(\psi,\zeta,\theta,t) = U_r(\psi,\zeta)\cos n\theta\,\sin(\omega t + \alpha)\,, \tag{16a}$$

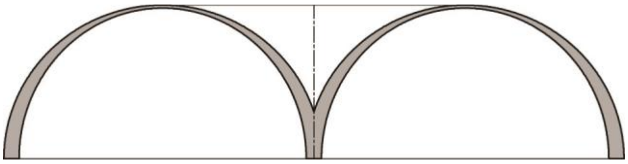
$$u_z(\psi,\zeta,\theta,t) = U_z(\psi,\zeta)\cos n\theta\,\sin(\omega t + \alpha)\,, \tag{16b}$$

$$u_\theta(\psi,\zeta,\theta,t) = U_\theta(\psi,\zeta)\sin n\theta\,\sin(\omega t + \alpha)\,, \tag{16c}$$

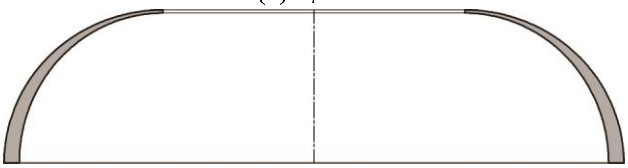
where U_r, U_z, and U_θ are displacement functions of ψ and ζ, ω is a natural frequency, and α is an arbitrary phase angle determined by the initial conditions. The circumferential wave number is taken to be an integer (n=0, 1, 2, …,∞), to ensure periodicity in θ. It may be verified by substituting the displacements into the 3-D equations of motion that the variables separable form of Eq. (16) does apply. Then Eq. (16) account for all free vibration modes except for the torsional ones. These modes arise from an alternative set of solutions which are the same as Eq. (16), except that cos$n\theta$ and sin$n\theta$ are interchanged. For $n\geq1$, this set duplicates the solutions of Eq. (16), with the symmetry axes of the mode shapes being rotated. But for n=0 the alternative set reduces to $u_r=u_z=0$, $u_\theta=U_\theta^*(\psi,\zeta)\sin(\omega t+\alpha)$, which corresponds to the torsional modes. The displacements uncouple by circumferential wavenumber (n), leaving only coupling in r (or ψ) and z (or ζ).



(a) R_o/R=1



(b) R_i/R=0



(c) R_i/R=1

Fig. 6 Cross-sections of hemi-spherical shells of revolution with eccentricity of e/R=1 having variable thickness of H/R=0.1 and h/H=0.2

The Ritz method uses the maximum potential (strain) energy (V_{max}) and the maximum kinetic energy (T_{max}) functionals in a cycle of vibratory motion. The functionals for the shell are obtained by setting $\sin^2(\omega t+\alpha)$ and $\cos^2(\omega t+\alpha)$ equal to unity in Eqs. (14) and (15) after the displacements (16) are substituted, and by using the nondimensional coordinates ψ and ζ, as follows

$$V_{\max}=\frac{RG}{2}\left[\int_{R_i^*}^{e^*+1+H^*/2}\int_0^{\zeta_o} I_V\ \psi\, d\zeta\, d\psi-\int_{R_i^*}^{e^*+1-H^*/2}\int_0^{\zeta_i} I_V\ \psi\, d\zeta\, d\psi\right], \tag{17}$$

$$T_{\max}=\frac{R^3\rho\omega^2}{2}\left[\int_{R_i^*}^{e^*+1+H^*/2}\int_0^{\zeta_o} I_T\ \psi\, d\zeta\, d\psi-\int_{R_i^*}^{e^*+1-H^*/2}\int_0^{\zeta_i} I_T\ \psi\, d\zeta\, d\psi\right], \tag{18}$$

where

$$I_V\equiv\left[\frac{\lambda}{G}(\kappa_1+\kappa_2+\kappa_3)^2+2(\kappa_1{}^2+\kappa_2{}^2+\kappa_3{}^2)+\kappa_4{}^2\right]\Gamma_1+(\kappa_5{}^2+\kappa_6{}^2)\Gamma_2, \tag{19}$$

$$I_T\equiv(U_r{}^2+U_z{}^2)\Gamma_1+U_\theta{}^2\Gamma_2, \tag{20}$$

and

$$\kappa_1\equiv\frac{U_r+nU_\theta}{\psi},\quad \kappa_2\equiv U_{r,\psi},\quad \kappa_3\equiv U_{z,\zeta}, \tag{21a}$$

$$\kappa_4\equiv U_{z,\psi}+U_{r,\zeta},\quad \kappa_5\equiv\frac{nU_z}{\psi}-U_{\theta,\zeta},\quad \kappa_6\equiv\frac{nU_r+U_\theta}{\psi}-U_{\theta,\psi}, \tag{21b}$$

Table 1 Convergence of frequencies $\omega R\sqrt{\rho/G}$ of a free, closed (R_i=0) hemi-spherical shell of revolution with eccentricity (e/R=−0.75) having variable thickness (H/R=0.1, h/H=0.2) for the five lowest torsional modes ($n=0^T$) for v=0.3.

TZ[a]	TR[b]	DET[c]	1	2	3	4	5
3	2	6	6.894	14.10	36.64	46.66	89.41
	4	12	6.807	12.54	20.70	32.84	34.86
	6	18	6.798	12.48	19.22	27.80	33.24
	8	24	6.790	12.45	18.71	26.87	33.15
	10	30	6.787	12.43	18.31	26.20	32.92
4	2	8	6.755	11.99	22.19	36.63	40.47
	4	16	6.704	11.39	17.45	25.67	33.10
	6	24	6.703	11.34	16.93	23.15	32.07
	8	32	6.703	11.32	16.86	22.63	30.00
	10	40	6.703	11.31	16.80	22.32	29.46
5	2	10	6.739	11.45	17.37	31.87	36.56
	4	20	6.703	11.23	15.81	22.25	31.07
	6	30	6.703	11.22	15.69	20.97	27.63
	8	40	6.703	11.22	15.63	20.76	26.64
	10	50	6.703	11.22	15.62	20.68	26.32
6	2	12	6.737	11.38	16.20	23.13	36.50
	4	24	6.703	11.21	15.62	20.26	26.97
	6	36	6.703	11.21	15.56	20.03	25.19
	8	48	6.703	11.21	15.56	19.90	24.74
	9	54	6.703	11.21	15.56	19.89	24.65
	10	60	6.703	11.21	15.56	19.88	24.61
7	2	14	6.736	11.35	15.98	21.21	29.30
	4	28	6.703	11.21	15.55	20.00	24.79
	6	42	6.703	11.21	15.54	19.84	24.40
	8	56	6.703	11.21	15.54	19.82	24.16
	9	63	6.703	11.21	15.54	19.81	24.14
	10	70	6.703	11.21	15.54	19.81	24.12

[a]***TZ***=Total numbers of polynomial terms used in the z (or ζ) direction.
[b]***TR***=Total numbers of polynomial terms used in the r (or ψ) direction.
[c]***DET***=Frequency determinant order.

and Γ_1 and Γ_2 are constants, defined by

$$\Gamma_1 \equiv \int_0^{2\pi} \cos^2 n\theta \, d\theta = \begin{cases} 2\pi & \text{if } n=0 \\ \pi & \text{if } n \ge 1 \end{cases} , \quad \Gamma_2 \equiv \int_0^{2\pi} \sin^2 n\theta \, d\theta = \begin{cases} 0 & \text{if } n=0 \\ \pi & \text{if } n \ge 1 \end{cases} . \tag{22}$$

From Eqs. (11) it is seen that the nondimensional constant λ/G in Eq. (19) involves only ν as follows

$$\frac{\lambda}{G} = \frac{2\nu}{1-2\nu} . \tag{23}$$

The displacement functions U_r, U_z and U_θ in Eqs. (16) are further assumed as algebraic polynomials

$$U_r(\psi,\zeta) = \eta_r(\psi,\zeta) \sum_{i=0}^{I} \sum_{j=0}^{J} A_{ij}\, \psi^i \zeta^j \tag{24a}$$

$$U_z(\psi,\zeta) = \eta_z(\psi,\zeta) \sum_{k=0}^{K} \sum_{l=0}^{L} B_{kl}\, \psi^k \zeta^l \tag{24b}$$

Table 2 Convergence of frequencies $\omega R\sqrt{\rho/G}$ of a free, hemi-spherical shell of revolution with eccentricity (e/R=1.25) having variable thickness (H/R=0.1, h/H=0.2) for the five lowest axisymmetric modes (n=0^A) for R_o/R=1.25 and ν=0.3

TZ[a]	*TR*[b]	*DET*[c]	1	2	3	4	5
4	2	16	0.9262	2.672	4.090	5.314	6.644
	4	32	0.4669	1.369	2.624	3.794	4.625
	6	48	0.4124	1.137	2.039	3.145	3.888
	8	64	0.4024	1.067	1.797	2.642	3.620
	10	80	0.4007	1.053	1.741	2.508	3.368
5	2	20	0.8997	2.515	4.003	4.734	6.331
	4	40	0.4482	1.272	2.367	3.619	4.180
	6	60	0.4077	1.103	1.932	2.972	3.818
	8	80	0.4015	1.060	1.767	2.567	3.455
	10	100	0.4005	1.052	1.732	2.478	3.297
	11	110	0.4004	1.051	1.727	2.462	3.255
	12	120	0.4004	1.051	1.726	2.455	3.236
6	2	24	0.8432	2.417	3.951	4.637	6.251
	4	48	0.4383	1.216	2.173	3.378	3.995
	6	72	0.4052	1.083	1.861	2.825	3.742
	8	96	0.4011	1.057	1.753	2.532	3.378
	10	120	0.4005	1.051	1.729	2.467	3.264
	11	132	0.4004	1.051	1.726	2.457	3.241
	12	144	0.4004	1.051	1.725	2.453	3.229
7	2	28	0.8079	2.250	3.834	4.327	6.011
	4	56	0.4300	1.195	2.069	3.154	3.903
	6	84	0.4039	1.073	1.811	2.709	3.656
	8	112	0.4008	1.054	1.743	2.509	3.339
	10	140	0.4004	1.051	1.727	2.461	3.249
	11	154	0.4004	1.051	1.725	2.455	3.233
8	2	32	0.8020	2.193	3.735	4.110	5.576
	4	64	0.4212	1.171	2.035	3.003	3.825
	6	96	0.4029	1.067	1.782	2.614	3.549
	8	128	0.4006	1.053	1.737	2.492	3.312
	9	144	0.4005	1.052	1.729	2.467	3.264
	10	160	0.4004	1.051	1.726	2.457	3.239

[a] ***TZ***=Total numbers of polynomial terms used in the z (or ζ) direction.
[b] ***TR***=Total numbers of polynomial terms used in the r (or ψ) direction.
[c] ***DET***=Frequency determinant order.

Table 3 Convergence of frequencies $\omega R\sqrt{\rho/G}$ of a free, open (R_i/R=0.75) hemi-spherical shell of revolution with eccentricity (e/R=0.25) having variable thickness (H/R=0.1, h/H=0.2) for the five lowest bending modes (n=2) for ν=0.3

TZ[a]	*TR*[b]	*DET*[c]	1	2	3	4	5
4	6	72	0.1097	0.6551	1.051	1.292	1.452
	8	96	0.1095	0.6464	1.046	1.288	1.428
	10	120	0.1092	0.6432	1.044	1.287	1.419
	12	144	0.1091	0.6426	1.044	1.286	1.417
	13	156	0.1091	0.6425	1.043	1.286	1.417
5	6	90	0.1092	0.6531	1.050	1.291	1.439
	8	120	0.1090	0.6450	1.046	1.288	1.422
	10	150	0.1089	0.6428	1.044	1.286	1.417
	11	165	0.1089	0.6426	1.044	1.286	1.416
	12	180	0.1089	0.6425	1.043	1.286	1.416
	13	195	0.1089	0.6424	1.043	1.286	1.416
6	6	108	0.1089	0.6514	1.049	1.290	1.433
	8	144	0.1089	0.6442	1.045	1.287	1.421
	10	180	0.1088	0.6427	1.044	1.286	1.417
	11	198	0.1088	0.6425	1.043	1.286	1.416
	12	216	0.1088	0.6424	1.043	1.286	1.416
7	4	84	0.1089	0.6605	1.061	1.298	1.490
	6	126	0.1088	0.6498	1.048	1.289	1.429
	8	168	0.1088	0.6436	1.044	1.287	1.419
	9	189	0.1088	0.6429	1.044	1.286	1.417
	10	210	0.1088	0.6426	1.043	1.286	1.416
	11	231	0.1088	0.6424	1.043	1.286	1.416
8	4	96	0.1088	0.6573	1.056	1.295	1.468
	6	144	0.1088	0.6482	1.047	1.289	1.427
	7	168	0.1088	0.6446	1.045	1.287	1.421
	8	192	0.1088	0.6432	1.044	1.287	1.418
9	4	108	0.1088	0.6556	1.053	1.293	1.453
	5	135	0.1088	0.6512	1.048	1.289	1.431
	6	162	0.1088	0.6467	1.046	1.288	1.424
	7	189	0.1087	0.6439	1.045	1.287	1.419
10	3	90	0.1090	0.6814	1.071	1.303	1.516
	4	120	0.1088	0.6543	1.051	1.292	1.445
	5	150	0.1087	0.6497	1.048	1.289	1.428
	6	180	0.1087	0.6455	1.046	1.288	1.422
11	3	99	0.1090	0.6771	1.064	1.300	1.497
	4	132	0.1087	0.6529	1.050	1.291	1.440
	5	165	0.1087	0.6481	1.047	1.288	1.425
12	3	108	0.1090	0.6731	1.060	1.298	1.481
	4	144	0.1087	0.6513	1.049	1.290	1.437
	5	180	0.1087	0.6468	1.046	1.288	1.423

[a]***TZ***=Total numbers of polynomial terms used in the z (or ζ) direction.
[b]***TR***=Total numbers of polynomial terms used in the r (or ψ) direction.
[c]***DET***=Frequency determinant order.

$$U_\theta(\psi,\zeta) = \eta_\theta(\psi,\zeta) \sum_{m=0}^{M} \sum_{n=0}^{N} C_{mn} \psi^m \zeta^n \tag{24c}$$

and similarly for U_θ^*, where *i*, *j*, *k*, *l*, *m*, and *n* are integers; *I*, *J*, *K*, *L*, *M*, and *N* are the highest degrees taken in the polynomial terms; A_{ij}, B_{kl} and C_{mn} are arbitrary coefficients to be determined, and the $\eta(\psi, \zeta)$ are functions depending upon the geometric boundary conditions to be enforced. For example:

1. completely free: $\eta_r = \eta_z = \eta_\theta = 1$,
2. the bottom edge fixed: $\eta_r = \eta_z = \eta_\theta = \zeta$,

The functions of η shown above, impose only the necessary geometric constraints related to displacement boundary conditions. Together with the algebraic polynomials in Eq. (24), they form function sets which are mathematically complete (Kantorovich and Krylov 1958). Thus, the function sets are capable of representing any 3-D motion of the shell with increasing accuracy as the indices *I*, *J*, …, *N* are increased. In the limit, as sufficient terms are taken, all internal kinematic constraints vanish, and the functions (24) will approach the exact solution as closely as desired.

The eigenvalue problem is formulated by minimizing the free vibration frequencies with respect to the arbitrary coefficients A_{ij}, B_{kl} and C_{mn}, thereby minimizing the effects of the internal constraints present, when the upper limits (*I*, *J*, …, *N*) become large. This corresponds to the equations

$$\frac{\partial}{\partial A_{ij}}(V_{\max} - \omega^2 T^*_{\max}) = 0, \quad (i = 0,1,2,\ldots,I;\ j = 0,1,2,\ldots,J)\,,$$

$$\frac{\partial}{\partial B_{kl}}(V_{\max} - \omega^2 T^*_{\max}) = 0, \quad (k = 0,1,2,\ldots,K;\ l = 0,1,2,\ldots,L)\,,$$

$$\frac{\partial}{\partial C_{mn}}(V_{\max} - \omega^2 T^*_{\max}) = 0, \quad (m = 0,1,2,\ldots,M;\ n = 0,1,2,\ldots,N)\,, \tag{25}$$

where $T_{\max} = \omega^2 T^*_{\max}$. Eq. (25) yield a set of $(I+1)(J+1)+(K+1)(L+1)+(M+1)(N+1)$ linear, homogeneous, algebraic equations in the unknowns A_{ij}, B_{kl} and C_{mn}. The equations can be written in the form

$$(\mathbf{K} - \Omega\,\mathbf{M})\,\mathbf{x} = \mathbf{0}\,, \tag{26}$$

where **K** and **M** are stiffness and mass matrices resulting from the maximum strain energy ($V_{\max}$) and the maximum kinetic energy ($T_{\max}$), respectively, and Ω is an eigenvalue of the vibrating system, expressed as the square of non-dimensional frequency, $\Omega \equiv \omega^2 R^2 \rho / G$, and the vector **x** takes the form

$$\mathbf{x} = (A_{00}, A_{01}, \ldots, A_{IJ};\ B_{00}, B_{01}, \ldots, B_{KL};\ C_{00}, C_{01}, \ldots, C_{MN})^T\,. \tag{27}$$

For a nontrivial solution, the determinant of the coefficient matrix is set equal to zero, which yields the frequencies (eigenvalues); that is to say $|\mathbf{K} - \Omega \mathbf{M}| = 0$. These frequencies are upper bounds on the exact values. The mode shape (eigenfunction) corresponding to each frequency is obtained, in the usual manner, by substituting each Ω back into the set of algebraic equations, and solving for the ratios of coefficients.

Table 4 Frequencies $\omega R\sqrt{\rho/G}$ of free hemispherical shells of revolution with eccentricity ($-1<e/R<0$, $e/R>1$) having variable thickness (H/R=0.1, h/H=0.2) for ν=0.3

n	s	$-1<e/R<0$ (R_i=0)			$e/R>1$		
		e/R=−0.75	e/R=−0.5	e/R=−0.25	e/R=1.25 R_o/R=1.25	e/R=1.5 R_o/R=1.5	e/R=1.75
		Fig. 2(a)	Fig. 2(b)	Fig. 2(c)	Fig. 3(a)	Fig. 3(b)	Fig. 3(c)
0^T	1	6.703	4.852	4.014	2.443	2.136	1.033
	2	11.21	8.023	6.527	4.089	3.986	2.321
	3	15.54	11.08	8.945	6.055	5.987	3.059
	4	19.81	14.10	11.34	8.051	7.996	4.107
	5	24.12	17.12	13.73	10.05	10.01	5.060
0^A	1	6.219	3.140	2.079	0.4004(4)	0.3588(5)	0.05545(2)
	2	7.028	3.597	2.344	1.051	0.9200	0.5440
	3	9.465	4.857	3.010	1.725	1.476	0.5701
	4	10.44	6.026	3.786	2.453	2.028	0.6763
	5	12.92	6.579	4.346	3.229	2.544	0.8779
1	1	5.624(4)	2.952(5)	2.001(5)	0.4283(5)	0.3807	0.06728(3)
	2	6.170(5)	3.918	2.533	1.086	0.9566	0.4459
	3	8.183	4.437	3.105	1.776	1.533	0.5565
	4	9.440	5.359	3.257	2.206	1.932	0.6949
	5	11.14	6.194	4.194	2.541	2.119	0.8480
2	1	2.130(1)	0.5766(1)	0.2701(1)	0.3030(1)	0.2513(1)	0.02829(1)
	2	4.787(2)	2.275(3)	1.603(4)	0.4988	0.2989(2)	0.2023
	3	8.331	3.981	2.475	0.8465	0.5207	0.2409
	4	8.827	4.131	2.582	1.438	0.9621	0.4204
	5	12.34	5.847	3.461	2.003	1.548	0.5545
3	1	5.268(3)	1.533(2)	0.7321(2)	0.3347(2)	0.3002(3)	0.08038(4)
	2	8.219	3.299	2.063	0.7424	0.6277	0.4825
	3	11.62	5.173	3.009	1.224	0.9126	0.4929
	4	12.56	5.768	3.700	1.968	1.348	0.6534
	5	16.29	7.222	4.098	2.833	1.769	0.6652
4	1	8.895	2.760(4)	1.332(3)	0.3880(3)	0.3416(4)	0.1489(5)
	2	12.02	4.617	2.618	0.9379	0.7985	0.6333
	3	15.17	6.625	3.688	1.542	1.228	0.6657
	4	16.58	7.588	4.866	2.406	1.815	0.8301
	5	20.39	8.884	5.022	3.427	2.410	0.9478
5	1	12.75	4.195	2.049	0.4580	0.3936	0.2294
	2	15.93	6.172	3.344	1.135	0.9545	0.7487
	3	18.82	8.299	4.542	1.851	1.479	0.7892
	4	20.68	9.408	5.884	2.826	2.183	0.9662
	5	24.48	10.71	6.218	3.978	2.976	1.168
6	1	16.68	5.790	2.872	0.5418	0.4553	0.3201
	2	19.84	7.887	4.212	1.337	1.111	0.8613
	3	22.53	10.11	5.523	2.159	1.719	0.9020
	4	24.80	11.23	6.974	3.243	2.519	1.087
	5	28.31	12.65	7.439	4.521	3.437	1.327

3. Convergence study

To guarantee the accuracy of frequencies obtained by the procedure described above, it is necessary to conduct some convergence studies to determine the number of terms required in the power series of Eq. (24). A convergence study is based upon the fact that, if the displacements are expressed as power series, all the frequencies obtained by the Ritz method should converge to their exact values in an upper bound manner. If the results do not converge properly, or converge too slowly, it would be likely that the assumed displacement functions chosen are poor ones, or be missing some functions from a minimal complete set of polynomials.

Tables 1-3 are such a study for completely free hemi-spherical shells of revolution with eccentricity of e/R=−0.75 (Table 1), 1.25 (Table 2), and 0.75 (Table 3) having variable thickness (H/R=0.1, h/H=0.2), depicted as the configurations in Fig. 2(a), Fig. 3(a), and Fig. 4(c), respectively. The radius R_o in Table 2 and Fig. 3(a) stands for the maximum outer one of the shell. The tables list the first five nondimensional frequencies $\omega R\sqrt{\rho/G}$ for ν=0.3, for torsional (n=0^T), axisymmetric (n=0^A), and bending (n=2) modes, respectively. The bending modes (n=2) have two circumferential waves in their mode shapes.

The range of ψ in Eqs. (5) and (6), $R_i^* \le \psi \le e^* + 1 \mp H^*/2$, must be replaced by $e^* - 1 \pm H^*/2 \le \psi \le R_o/R$ for the shell configurations of (a) and (b) in Fig. 3 and $e^* - 1 \pm H^*/2 \le \psi \le e^* + 1 \mp H^*/2$ for the shell configuration of (c) in Fig. 3.

To make the study of convergence less complicated, equal numbers of polynomial terms were taken in both the r (or ψ) coordinate (i.e., I=K=M) and z (or ζ) coordinate (i.e., J=L=N), although some computational optimization could be obtained for some configurations and some mode shapes by using unequal numbers of polynomial terms.

The symbols ***TZ*** and ***TR*** in the table indicate the total numbers of polynomial terms used through the axial (z or ζ) and the radial (r or ψ) directions, respectively. Note that the frequency determinant order ***DET*** is related to ***TZ*** and ***TR*** as follows

$$\boldsymbol{DET}=\begin{cases} TZ\times TR & \text{for torsional modes } (n=0),\\ 2\times TZ\times TR & \text{for axisymmetric modes } (n=0),\\ 3\times TZ\times TR & \text{for general modes } (n\ge 1).\end{cases} \tag{28}$$

Tables 1-3 shows the monotonic convergence of all five frequencies as ***TZ*** (=J+1, L+1, and N+1 in Eq. (24)) are increased, as well as ***TR*** (=I+1, K+1, and M+1 in Eqs. (24)). One sees in Table 3, for example, that the first nondimensional frequency $\omega R\sqrt{\rho/G}$ for n=2 converges to four digits (0.1087) when (***TZ***,***TR***)=(11,4) terms are used, which results in ***DET***=3×(11×4)=132.

It is interesting to note in Tables 1-3 that the modes for n=2 require much larger size of ***DET*** compared with the torsional (n=0^T) and the axisymmetric modes (n=0^A). This is primarily because only the circumferential displacement components (u_θ) are involved in the torsional modes, and the radial (u_r) and axial (u_z) displacement components are involved in the axisymmetrical modes, whereas all three components enter into the modes having n≥1, as seen in Eq. (28).

Frequencies in underlined, bold-faced type in Tables 1-3 are the most accurate values (to four significant figures) achieved with the smallest determinant sizes.

4. Numerical results and discussion

Table 5 Frequencies $\omega R\sqrt{\rho/G}$ of free hemispherical shells of revolution with eccentricity ($0\leq e/R<1$) having variable thickness for ν=0.3

n	s	R_i/e=1, H/R=0.1, h/H=0.2			e/R=0 (R_i=0)		
		e/R=0.25	e/R=0.5	e/R=0.75	H/R=0.1 h/H=0.2	H/R=0.1 h/H=1	H/R=0.02 h/H=5
		Fig. 4(a)	Fig. 4(b)	Fig. 4(c)	Fig. 5(a)	Fig. 5(b)	Fig. 5(c)
0^T	1	3.172	2.942	2.791	3.515	3.156	3.046
	2	4.954	4.619	4.455	5.600	5.280	5.177
	3	6.728	6.419	6.294	7.596	7.333	7.256
	4	8.576	8.315	8.218	9.580	9.360	9.304
	5	10.47	10.25	10.17	11.57	11.38	11.34
0^A	1	0.8124(5)	0.5759	0.4688	1.541	1.445(5)	1.353
	2	1.207	0.9877	0.8439	1.667	1.647	1.583
	3	1.287	1.075	0.9436	1.845	2.165	1.726
	4	1.585	1.421	1.228	2.176	3.117	2.109
	5	2.120	1.558	1.335	2.773	3.330	2.745
1	1	0.8534	0.5751(5)	0.4627	1.498	1.452	1.376
	2	1.177	0.9625	0.8233	1.712	1.844	1.609
	3	1.293	1.076	0.9481	1.976	2.365	1.881
	4	1.635	1.446	1.351	2.387	2.598	2.386
	5	1.939	1.632	1.405	2.461	3.726	2.482
2	1	0.1087(1)	0.07922(1)	0.06064(1)	0.1609(1)	0.1872(1)	0.1083(1)
	2	0.6422(4)	0.2965(3)	0.1864(3)	1.313(5)	1.617	1.440
	3	1.043	0.7121	0.5424	1.716	1.952	1.705
	4	1.286	0.9929	0.8246	1.819	2.185	2.080
	5	1.416	1.138	0.9922	2.228	3.132	2.494
3	1	0.2996(2)	0.2199(2)	0.1698(2)	0.4401(2)	0.4783(2)	0.1817(2)
	2	1.243	0.5873	0.3697(5)	1.606	1.867	1.531
	3	1.353	1.059	0.8182	2.047	2.630	1.836
	4	1.633	1.244	1.065	2.582	2.963	2.319
	5	2.026	1.579	1.267	2.674	3.751	3.066
4	1	0.5471(3)	0.4024(4)	0.3114(4)	0.8022(3)	0.8528(3)	0.2737(3)
	2	1.545	0.8723	0.5389	1.877	2.216	1.634
	3	1.703	1.300	1.050	2.400	3.174	2.000
	4	1.958	1.429	1.196	3.059	3.888	2.595
	5	2.422	1.833	1.605	3.633	4.439	3.458
5	1	0.8405	0.6182	0.4785	1.233(4)	1.298(4)	0.3894(4)
	2	1.753	1.180	0.7204	2.237	2.670	1.747
	3	2.021	1.488	1.256	2.861	3.796	2.194
	4	2.378	1.644	1.348	3.621	4.793	2.904
	5	2.875	2.055	1.751	4.519	5.184	3.880
6	1	1.175	0.8636	0.6681	1.728	1.809	0.5267(5)
	2	2.004	1.509	0.9173	2.697	3.217	1.875
	3	2.348	1.664	1.431	3.417	4.484	2.419
	4	2.800	1.880	1.519	4.263	5.692	3.242
	5	3.373	2.311	1.923	5.249	5.979	4.333

Table 6 Frequencies $\omega R\sqrt{\rho/G}$ of free hemispherical shells of revolution with eccentricity (e/R=1) having variable thickness (h/H=0.2) for H/R=0.1 (v=0.3)

n	s	R_o/R=1	R_i/R=0	R_i/R=1	n	s	R_o/R=1	R_i/R=0	R_i/R=1
		Fig. 6(a)	Fig. 6(b)	Fig. 6(c)			Fig. 6(a)	Fig. 6(b)	Fig. 6(c)
0^T	1	3.850	2.063	2.690	3	1	0.3795(2)	0.1361(3)	0.1359(3)
	2	5.953	2.997	4.363		2	0.9042	0.7672	0.2721(5)
	3	7.211	4.144	6.229		3	1.660	0.8081	0.6635
	4	8.790	5.129	8.169		4	2.772	1.038	0.9481
	5	10.70	6.080	10.13		5	4.098	1.060	1.046
0^A	1	0.4712(4)	0.09804(2)	0.4053	4	1	0.4595(3)	0.2501(5)	0.2499(4)
	2	1.273	0.7453	0.7405		2	1.176	0.9374	0.3894
	3	2.142	0.7739	0.8566		3	2.080	0.9911	0.8578
	4	3.117	0.9014	1.017		4	3.336	1.216	1.057
	5	4.202	1.154	1.284		5	4.850	1.417	1.476
1	1	0.5246(5)	0.1369(4)	0.3986	5	1	0.5640	0.3844	0.3842
	2	1.309	0.6403	0.7240		2	1.452	1.094	0.5124
	3	1.971	0.7473	0.8630		3	2.502	1.157	1.043
	4	2.588	0.9224	1.228		4	3.901	1.387	1.180
	5	2.718	1.176	1.297		5	5.586	1.586	1.586
2	1	0.3290(1)	0.04863(1)	0.04810(1)	6	1	0.6907	0.5365	0.5361
	2	0.6138	0.4841	0.1347(2)		2	1.740	1.245	0.6442
	3	1.240	0.5989	0.4454		3	2.933	1.314	1.214
	4	2.191	0.7290	0.7126		4	4.474	1.534	1.317
	5	3.269	0.8904	0.8971		5	6.329	1.706	1.712

Tables 4-6 present the nondimensional frequencies $\omega R\sqrt{\rho/G}$ of completely free hemispherical shells of revolution with various range of eccentricity of −0.75≤e/R≤1.75. Each table is for the shells of variable thickness of H/R=0.1 and h/H=0.2, except for the shells in the 5th and 6th columns of Table 5. Poisson's ratio (v) was taken to be 0.3. The shell configurations for Tables 4-6 are depicted in Figs. 2-3, Figs 4-5, and Fig. 6, respectively. Forty frequencies are given for each configuration, which arise from eight circumferential wave numbers (n=0^T, 0^A, 1, 2, 3, 4, 5, 6) and the first five modes (s=1, 2, 3, 4, 5) for each value of n, where the superscripts T and A indicate torsional and axisymmetric modes, respectively. The numbers in parentheses identify the first five frequencies for each configuration. For example, in the case of e/R=−0.75 in Table 4, the first five frequencies are modes for (n,s)=(2,1), (2,2), (3,1), (1,1), and (1,2) in this order. The zero frequencies of rigid body modes are omitted from the tables.

The ranges of ψ in Eqs. (5) and (6), $R_i^* \le \psi \le e^* + 1 \mp H^*/2$, must be replace by $H^*/2 \le \psi \le R_o/R$ and $0 \le \psi \le R_o/R$ for the shell in Fig. 6(a) and by $H^*/2 \le \psi \le e^* + 1 - H^*/2$ and $0 \le \psi \le e^* + 1 + H^*/2$ for the shell in Fig. 6(b), respectively.

It is interesting to note in Tables 4-6 that, irrespective of shell configurations, the fundamental (lowest) frequencies and the second ones are for modes having two (n=2) and three (n=3) circumferential waves in their modes, respectively, and the torsional frequencies (n=0^T) are all for higher modes. It is also seen that the axisymmetric modes (n=0^A) are important for positive eccentricity ($e > 0$). That is, they are among the lowest frequencies of the shells.

5. Conclusions

Extensive and accurate frequency data determined by the 3-D Ritz analysis have been presented for eccentric hemi-spherical shells of revolution with variable thickness. Such a shell structure may be used in architectural engineering. Very diverse types of shells can be obtained as the values of the eccentricity (e) as shown in Figs. 2-6. The analysis uses the 3-D equations of the theory of elasticity in their general forms for isotropic materials. They are only limited to small strains. No other constraints are placed upon the displacements. This is in stark contrast with the classical 2-D thin shell theories, which make very limiting assumptions about the displacement variation through the shell thickness.

The method is capable of determining frequencies as close to the exact ones as desired. Therefore, the data in Tables 4-6 may be regarded as benchmark results against which 3-D results obtained by other methods, such as finite elements and finite differences, may be compared to determine the accuracy of the latter. Moreover, the frequency determinants required by the present method are at least an order of magnitude smaller than those needed by finite element analyses of comparable accuracy. This was demonstrated extensively in a paper by McGee and Leissa (1991). The Ritz method guarantees upper bound convergence of the frequencies in terms of functions sets that are mathematically complete, such as algebraic polynomials. Some finite element methods can also accomplish this, but at much greater costs, and others cannot.

The method presented could also be extended to circumferentially open ($0 \le \theta \le \theta_0$) hemi-spherical shells with eccentricity having variable thickness, instead of circumferentially closed ($0 \le \theta \le 2\pi$) ones of revolution considered in the present work. However, the periodicity in θ would not be present. It would be necessary then to replace the double sums of algebraic polynomials in Eq. (24) by triple sums, with polynomials in θ being included.

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