Structural Engineering and Mechanics, *Vol. 54, No. 6 (2015) 1061-1073*
DOI: http://dx.doi.org/10.12989/sem.2015.54.6.1061

Nonlinear vibration analysis of a nonlocal sinusoidal shear deformation carbon nanotube using differential quadrature method

Hasan Rahimi Pour*[1], Hossein Vossough[2], Mohammad Mehdi Heydari[2]
Gholamhossein Beygipoor[3] and Alireza Azimzadeh[4]

[1]Young Researchers and Elite Club, Khomeyni Shahr Branch, Islamic Azad University, Khomeyni Shahr, Iran
[2]Department of Civil Engineering, Jasb Branch, Islamic Azad University, Jasb, Iran
[3]Young Researchers and Elite Club, Bandarabbas Branch, Islamic Azad University, Bandarabbas, Iran
[4]Department of Mechanical Engineering, Kashan Branch, Islamic Azad University, Kashan, Iran

(Received October 14, 2014, Revised January 15, 2015, Accepted February 21, 2015)

Abstract. This paper presents a nonlocal sinusoidal shear deformation beam theory (SDBT) for the nonlinear vibration of single walled carbon nanotubes (CNTs). The present model is capable of capturing both small scale effect and transverse shear deformation effects of CNTs, and does not require shear correction factors. The surrounding elastic medium is simulated based on Pasternak foundation. Based on the nonlocal differential constitutive relations of Eringen, the equations of motion of the CNTs are derived using Hamilton's principle. Differential quadrature method (DQM) for the natural frequency is presented for different boundary conditions, and the obtained results are compared with those predicted by the nonlocal Timoshenko beam theory (TBT). The effects of nonlocal parameter, boundary condition, aspect ratio on the frequency of CNTs are considered. The compar¬ison firmly establishes that the present beam theory can accurately predict the vibration responses of CNTs.

Keywords: nonlinear vibration; sinusoidal shear deformation theory; DQM; Eringen theory

1. Introduction

Nanostructures are widely used in micro- and nano-scale devices and systems such as biosensors, atomic force microscopes, micro-electro-mechanical systems (MEMS) and nano-electro-mechanical systems (NEMS) due to their superior mechanical, chemical, and electronic properties (Libhushan Takashima, Beak and Kim 2003). In such applications, small scale effects are often observed. These effects can be captured using size-dependent continuum mechanics such as strain gradient theory (Nix and Gao 1998), modified couple stress theory (Ma and Reddy 2008), and nonlocal elasticity theory (Eringen 1972). Among these theories, the nonlocal elasticity theory initiated by Eringen is the most commonly used theory. Unlike the local theories which assume that the stress at a point is a function of strain at that point, the nonlocal elasticity theory assumes that the stress at a point is a function of strains at all points in the continuum.

*Corresponding author, Ph.D. Student, E-mail: h.rahimipour@pogc.ir

http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

Based on the nonlocal constitutive relation of Eringen, a number of papers have been published attempting to develop nonlocal beam models for predicting the responses of carbon nanotubes. The nonlocal Euler-Bernoulli beam theory (EBT) and Timoshenko beam theory (TBT) first proposed by Peddieson *et al*. (2003), Wang (2005), respectively, were adopted by many researchers to investigate bending (Civalek and Demir 2011, Wang *et al*. 2008, Wang and Liew 2007), buckling (Murmu and Pradhan 2009, Wang *et al*. 2006, Wang *et al*. 2006), and vibration (Wang and Varadan 2006, Wang *et al*. 2007, Zhang *et al*. 2005) responses of carbon nanotubes.

A complete development of EBT and TBT was presented by Reddy and Pang (2008) who provided the ana-lytical solutions for the deflection, buckling load, and natural frequency of nanobeams with various boundary conditions. It should be noted that the EBT is only applicable for slender beams where the shear deformation effect is negligible. However, it underestimates deflection and overestimates buckling load as well as natural frequency for short beams. The TBT accounts for the shear deformation effect for short beams by assuming a constant shear strain through the height of the beam. Therefore, a shear correction factor is required to compensate for the difference between the actual stress state and the constant stress state. To avoid the use of shear correction factor, higher-order shear deformation theories were developed based on the assumption of the higher-order variation of axial displacement through the height of the beam, notable among them are the third-order theory of Reddy (2007), generalized theory of Aydogdu (2009), refined theory of Thai (2012), and sinusoidal shear deformation theory of Touratier (1991).

The sinusoidal shear deformation theory of Touratier (1991) is based on the assumption that the transverse shear stress vanishes on the top and bottom surfaces of the beam and is nonzero elsewhere. Thus there is no need to use shear correction factors as in the case of TBT. This theory is also employed to predict the response of laminate plate (Zenkor 2004) and functionally graded sandwich plates (Zenkour 2005a). The aim of this paper is to propose a nonlocal SDBT which accounts for both small scale and shear deformation effects of CNTs. The small scale effect is taken into account by using the nonlocal constitutive relations of Eringen, while the shear deformation effect is captured using the SDBT (Touratier 1991). The nonlocal equations of motion are derived using Hamilton's principle. DQM for natural frequency of CNT with different boundary conditions is presented, and the obtained results are compared with those predicted by the TBT to verify the accuracy of the present solution. The effects of nonlocal parameter, boundary condition, aspect ratio are discussed in detail.

2. Equations of motion of the sinoisoidal beam theory

Consider a beam length L and rectangular cross section $b\times h$, with b being the width and h being the height. The x, y and z coordinates are taken along the length width, and height of the beam, respectively.

According to the sinusoidal theory, the displacement field is chosen based on the assumption that the transverse shear stress vanishes on the top and bottom surfaces of the beam and is nonzero elsewhere. The displacement field is given as (touratier)

$$u_1(x,z,t) = u(x,t) - z\frac{dw}{dx} + f\varphi$$

$$u_2(x,z,t) = 0$$

$$u_3(x,z,t) = w(x,t) \tag{1}$$

where f=(h/p)sin($\pi z/h$) u and w are the axial and transverse displacements, respectively, of a point on the mid-plane of the beam and φ is the rotation of the cross section about the y-axis. The only nonzero strains are

$$\begin{aligned} \varepsilon_x &= \frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2 - z\frac{d^2w}{dx^2} + f\,\frac{d\varphi}{dx} \\ \gamma_{xz} &= \cos\left(\frac{\pi z}{h}\right)\varphi \end{aligned} \tag{2}$$

It can be observed from Eq. (2) that the transverse shear strain γ_{xz} is zero at the top ($z=h/2$) and bottom ($z=-h/2$) surfaces of the beam thus satisfying the traction free conditions for σ_{xz}.

3. Nonlocal theory

Unlike the local theory, the nonlocal theory assumes that the stress at a point depends not only on the strain at that point but also on strains at all other points of the body. According to Eringen (1983), the nonlocal stress tensor r at point x is expressed as

$$\sigma - \mu\nabla^2\sigma = \tau \tag{3}$$

where s is classical stress tensor at a point x related to the strain by the Hooke's law; $\mu=(e_0a)^2$ is the nonlocal parameter which incorporates the small scale effect, a is the internal characteristic length and e_0 is a constant appropriate to each material. The nonlocal parameter depends on the boundary conditions, chirality, mode shapes, number of walls, and type of motion (Arash and Wang 2012). So far, there is no rigorous study made on estimating the value of the nonlocal parameter. It is suggested that the value of nonlocal parameter can be determined by experiment or by conducting a comparison of dispersion curves from the nonlocal continuum mechanics and molecular dynamics simulation (Arash and Ansari 2010, Wang 2005). In general, a conservative estimate of the nonlocal parameter is e_0a<2 nm for a single wall carbon nanotube (Wang and Wang 2007).

For an isotropic material in a one-dimensional case, the nonlocal constitutive relation in Eq. (3) takes the following forms

$$\sigma_x - \mu\frac{d^2\sigma_x}{dx^2} = E\varepsilon_x \tag{4}$$

$$\sigma_{xz} - \mu\frac{d^2\sigma_{xz}}{dx^2} = G\gamma_{xz} \tag{5}$$

where E and G are the elastic and shear modulus of the CNT, respectively.

4. Energy method

Equations of motion are derived using Hamilton's principle. The principle can be stated in analytical from as (Reddy)

$$0 = \int_0^t (\delta u + \delta v - \delta k)dt \tag{6}$$

where δu is the variation of the strain energy δv is variation of external works; and δk is the variation of the kinetic energy.

The variation of the strain energy of the CNT can be stated as

$$\delta u = \int_0^L \int_a (\sigma_x \delta\varepsilon_x + \sigma_{xz}\delta\gamma_{xz})dAdx = \int_0^L \left[N\left(\frac{d\delta u}{dx} + \frac{dw}{dx}\frac{d\delta w}{dx}\right) - M\frac{d^2\delta w}{dx^2} + P\frac{d\delta\varphi}{dx} + Q\delta\varphi \right]dx \tag{7}$$

where N, M, P and q are the stress resultants defined as

$$\begin{aligned} (N, M, P) &= \int_a (1.z.f)\sigma_x dA, \\ Q &= \int_a \cos\left(\frac{\pi z}{h}\right)\sigma_{xz} dA \end{aligned} \tag{8}$$

The variation of the external works of the CNT can be stated as

$$\delta v = \int_0^l q\delta w dx \tag{9}$$

where q is the Pasternak medium force which may be expressed as

$$q = -K_w w + K_g \frac{d^2 w}{dx^2} \tag{10}$$

where K_w and K_g are respectively spring constant of Winkler type and shear constant of Pasternak type.

The variation of the kinetic energy is obtained as

$$\begin{aligned} \delta k &= \int_0^L \int_a \rho(\dot{u}_1\delta\dot{u}_1 + \dot{u}_3\delta\dot{u}_3)dadz \\ &= \left[\int_0^L m_0(\dot{u}\delta\dot{u}_1 + \dot{w}\delta\dot{w}) + m_2\frac{d\dot{w}}{dx}\frac{d\delta\dot{w}}{dx} - \frac{24m_2}{\pi^3}\left(\frac{d\dot{w}}{dx}\delta\dot{\varphi} + \dot{\varphi}\frac{d\delta\dot{w}}{dx}\right) + \frac{6m_2}{\pi_2}\dot{\varphi}\delta\dot{\varphi} \right]dx \end{aligned} \tag{11}$$

where dot-superscript convention indicates the differentiation with respect to the time variable t; ρ is the mass density; and (m_0, m_2) are mass inertias defined as

$$(m_0, m_2) = \int_a (1, z^2)\rho da \tag{12}$$

Substituting the expressions for, δu, δv and δk from Eqs. (7), (6) and (11) into Eq. (6) and integrating by parts, and collecting the coefficients of, δu, δw and, $\delta\varphi$ the following equations of motion of the CNT are obtained

$$\delta u : \frac{dN}{dx} = m_0 \ddot{u} \tag{13}$$

$$\delta\varphi : \frac{dP}{dx} - Q = \frac{6m_2}{\pi^2}\ddot{\varphi} - \frac{24m_2}{\pi^3}\frac{d\ddot{w}}{dx} \tag{14}$$

$$\delta w : \frac{d^2M}{dx^2} + q = \frac{24m_2}{\pi^3}\frac{d\ddot{\varphi}}{dx} + m_0\ddot{w} - m_2\frac{d^2\ddot{w}}{dx^2} \tag{15}$$

By substituting Eq. (2) into Eqs. (4) and (5) and the subsequent results into Eq. (8), the stress resultants are obtained as

$$N - \mu\frac{d^2N}{dx^2} = EA\left(\frac{du}{dx} + \frac{1}{2}\left(\frac{dw}{dx}\right)^2\right) \tag{16}$$

$$M - \mu\frac{d^2M}{dx^2} = -EI\frac{d^2w}{dx^2} + \frac{24EI}{\pi^3}\frac{d\varphi}{dx} \tag{17}$$

$$P - \mu\frac{d^2p}{dx^2} = -\frac{24EI}{\pi^3}\frac{d^2w}{dx^2} + \frac{6EI}{\pi^2}\frac{d\varphi}{dx} \tag{18}$$

$$Q - \mu\frac{d^2Q}{dx^2} = \frac{GA}{2}\varphi \tag{19}$$

where

$$(A, I) = \int_a (1, z^2)\, dA \tag{20}$$

The nonlocal equations of motion of the proposed CNT theory can be expressed in terms of displacements (u, w, φ) by substituting stress resultants in Eqs. (13)-(15) as

$$EA\left(\frac{d^2u}{dx^2} + \frac{d^2w}{dx^2}\frac{dw}{dx}\right) = m_0\left(\ddot{u} - \mu\frac{d^2\ddot{u}}{dx^2}\right) \tag{21}$$

$$-\frac{24EI}{\pi^3}\frac{d^3w}{dx^3} + \frac{6EI}{\pi^2}\frac{d^2\varphi}{dx^2} - \frac{GA}{2}\varphi = \frac{6m_2}{\pi^2}\left(\ddot{\varphi} - \mu\frac{d^2\ddot{\varphi}}{dx^2}\right) - \frac{24m_2}{\pi^3}\left(\frac{d\ddot{w}}{dx} - \mu\frac{d^3\ddot{w}}{dx^3}\right) \tag{22}$$

$$\begin{aligned} &-EI\frac{d^4w}{dx^4} + \frac{24EI}{\pi^3}\frac{d^3\varphi}{dx^3} + q - \mu\frac{d^2q}{dx^2} - N_x\left(\frac{d^2w}{dx^2} - \mu\frac{d^4w}{dx^4}\right) = \frac{24m_2}{\pi^3}\left(\frac{d\ddot{\varphi}}{dx} - \mu\frac{d^3\ddot{\varphi}}{dx^3}\right) + \\ &+ m_0\left(\ddot{w} - \mu\frac{d^2\ddot{w}}{dx^2}\right) - m_2\left(\frac{d^2\ddot{w}}{dx^2} - \mu\frac{d^4\ddot{w}}{dx^4}\right) \end{aligned} \tag{23}$$

The equations of motion of local CNT theory can be recovered from Eqs. (21)-(23) by setting the nonlocal parameter μ equal to zero.

5. GDQM

In this method, the differential equations are changed into a first order algebraic equation by employing appropriate weighting coefficients. Because weighting coefficients do not relate to any special problem and only depend on the grid spacing. In other words, the partial derivatives of a function (say φ, w here) are approximated with respect to specific variables (say x), at a discontinuous point in a defined domain ($0<x<L$) as a set of linear weighting coefficients and the amount represented by the function itself at that point and other points throughout the domain. The approximation of the n^{th} derivative function with respect to x may be expressed in general form as

$$f_x^{(n)}(x_i)=\sum_{k=1}^{N_x} A^{(n)}{}_{ik}\, f(x_k), \tag{24}$$

where N_x, denotes the number of points in x direction, $f(x)$ is the function and A_{ik} is the weighting coefficients defined as

$$A^{(1)}{}_{ij}=\frac{M(x_i)}{(x_i-x_j)M(x_j)}, \tag{25}$$

where M is Lagrangian operators defined as

$$M(x_i)=\prod_{j=1}^{N_x}(x_i-x_j),\ \ i\neq j \tag{26}$$

The weighting coefficients for the second, third and fourth derivatives are determined via matrix multiplication

$$A^{(2)}{}_{ij}=\sum_{k=1}^{N_x} A^{(1)}{}_{ik}A^{(1)}{}_{kj},\ A^{(3)}{}_{ij}=\sum_{k=1}^{N_x} A^{(2)}{}_{ik}A^{(1)}{}_{kj},\ A^{(4)}{}_{ij}=\sum_{k=1}^{N_x} A^{(3)}{}_{ik}A^{(1)}{}_{kj},\ \ i,j=1,2,...,N_x, \tag{27}$$

Using the following rule, the distribution of grid points in domain is calculated as

$$x_i=\frac{L}{2}[1-\cos(\frac{\pi i}{N_x})], \tag{28}$$

By introducing

$$(u(x,t),\varphi(x,t),w(x,t))=(u(x),\varphi(x),w(x))e^{i\omega t}, \tag{29}$$

where ω is frequency of system. Substituting Eq. (24) into the governing equations turns it into a set of algebraic equations expressed as

$$EA\left[\sum_{k=1}^{N_x} A^{(2)}{}_{ik}u(x_k)+\sum_{k=1}^{N_x} A^{(2)}{}_{ik}w(x_k)\sum_{k=1}^{N_x} A^{(1)}{}_{ik}w(x_k)\right]=-m_0\omega^2(u(x_k)-\mu\sum_{k=1}^{N_x} A^{(2)}{}_{ik}u(x_k)) \tag{30}$$

$$\begin{aligned}&-\frac{24EI}{\pi^3}\sum_{k=1}^{N_x} A^{(3)}{}_{ik}w(x_k)+\frac{6EI}{\pi^2}\sum_{k=1}^{N_x} A^{(2)}{}_{ik}\varphi(x_k)-\frac{GA}{2}\varphi(x_k)=\\&-\frac{6m_2}{\pi^2}\omega^2\left[\varphi(x_k)-\mu\sum_{k=1}^{N_x} A^{(2)}{}_{ik}\varphi(x_k)\right]+\frac{24m_2}{\pi^3}\omega^2\left[\sum_{k=1}^{N_x} A^{(1)}{}_{ik}w(x_k)-\mu\sum_{k=1}^{N_x} A^{(3)}{}_{ik}w(x_k)\right],\end{aligned} \tag{31}$$

$$
\begin{aligned}
&-EI\sum_{k=1}^{N_x}A^{(4)}{}_{ik}w(x_k)+\frac{24EI}{\pi^3}\sum_{k=1}^{N_x}A^{(3)}{}_{ik}\varphi(x_k)-K_w w(x_k)+K_g\sum_{k=1}^{N_x}A^{(2)}{}_{ik}w(x_k)\\
&-\mu\left[-K_w\sum_{k=1}^{N_x}A^{(2)}{}_{ik}w(x_k)+K_g\sum_{k=1}^{N_x}A^{(4)}{}_{ik}w(x_k)\right]=-\frac{24m_2}{\pi^3}\omega^2\left[\sum_{k=1}^{N_x}A^{(1)}{}_{ik}\varphi(x_k)-\mu\sum_{k=1}^{N_x}A^{(3)}{}_{ik}\varphi(x_k)\right]+\\
&-m_0\omega^2\left[w(x_k)-\mu\sum_{k=1}^{N_x}A^{(2)}{}_{ik}w(x_k)\right]+m_2\omega^2\left[\sum_{k=1}^{N_x}A^{(2)}{}_{ik}w(x_k)-\mu\sum_{k=1}^{N_x}A^{(4)}{}_{ik}w(x_k)\right],
\end{aligned}
\tag{32}
$$

Finally, the governing equations (i.e., Eqs. (30)-(32)) in matrix form can be expressed as

$$
\left\{\underbrace{\begin{bmatrix}[K_{1u}] & [0] & [0]\\ [0] & [K_{2w}] & [K_{2\varphi}]\\ [0] & [K_{3w}] & [K_{3\varphi}]\end{bmatrix}}_{[K_L]}+\underbrace{\begin{bmatrix}[0] & [KN_{1w}] & [0]\\ [0] & [0] & [0]\\ [0] & [0] & [0]\end{bmatrix}}_{[K_{NL}]}+\omega^2\begin{bmatrix}[M_{1u}] & [0] & [0]\\ [0] & [M_{2w}] & [M_{2\varphi}]\\ [0] & [M_{3w}] & [M_{3\varphi}]\end{bmatrix}\right\}\begin{bmatrix}u\\ w\\ \varphi\end{bmatrix}=\begin{bmatrix}0\\ 0\\ 0\end{bmatrix},
\tag{33}
$$

where $[K]$ and $[M]$ are respectively, stiffness and mass matrixes which can be defined as

$$K_{1u}=EAA^{(2)},\tag{34}$$

$$KN_{1w}=EAA^{(2)}wA^{(1)},\tag{35}$$

$$M_{1u}=m_0\left(II-\mu A^{(2)}\right),\tag{36}$$

$$K_{2w}=-\frac{24EI}{\pi^3}A^{(2)},\tag{37}$$

$$K_{2\varphi}=\frac{6EI}{\pi^2}A^{(2)}-\frac{GA}{2}II,\tag{38}$$

$$M_{2w}=-\frac{24m_2}{\pi^3}\left(A^{(1)}-\mu A^{(3)}\right),\tag{39}$$

$$M_{2\varphi}=\frac{6m_2}{\pi^2}\left(II-\mu A^{(2)}\right),\tag{40}$$

$$K_{3w}=-EIA^{(4)}-K_wII+K_gA^{(2)}-\mu\left(-K_wA^{(2)}+K_gA^{(4)}\right),\tag{41}$$

$$K_{3\varphi}=\frac{24EI}{\pi^3}A^{(3)},\tag{42}$$

$$M_{3w}=m_0\left(II-\mu A^{(2)}\right)-m_2\left(A^{(2)}-\mu A^{(4)}\right),\tag{43}$$

$$M_{3\varphi}=\frac{24m_2}{\pi^3}\left(A^{(1)}-\mu A^{(3)}\right),\tag{44}$$

Table 1 First three non-dimensional frequency ϖ of simply supported nano-beams

L/h	μ (nm^2)	ϖ_1		ϖ_2		ϖ_3	
		TBT (Thai 2012)	Present	TBT (Thai 2012)	Present	TBT (Thai 2012)	Present
5	0	9.2740	9.2752	32.1665	32.1948	61.4581	61.6192
	1	8.8477	8.8488	27.2364	27.2604	44.7247	44.8420
	2	8.4752	8.4763	24.0453	24.0664	36.8831	36.9798
	3	8.1461	8.1472	21.7642	21.7833	32.1036	32.1878
	4	7.8528	7.8536	20.0293	20.0470	28.8023	28.8778
10	0	9.7075	9.7077	37.0962	37.1009	78.1547	78.1855
	1	9.2612	9.2614	31.4105	31.4146	56.8753	56.8977
	2	8.8713	8.8710	27.7303	27.7339	46.9034	46.9219
	3	8.5269	8.5271	25.0996	25.1079	40.8254	40.8415
	4	8.2196	8.2198	23.0989	23.1019	36.6272	36.6416
20	0	9.8281	9.8282	38.8299	38.8308	85.6619	85.6671
	1	9.3763	9.3764	32.8786	32.8793	62.3385	62.3422
	2	8.9816	8.9816	29.0263	29.0270	51.4087	51.4118
	3	8.6328	8.6329	26.2727	26.2733	44.7469	44.7496
	4	8.3218	8.3218	24.1785	24.1790	40.1454	40.1478
1000	0	9.8679	9.8679	39.4517	39.4517	88.6914	88.6915
	1	9.4143	9.4143	33.4051	33.4051	64.5431	64.5432
	2	9.0180	9.0180	29.4911	29.4912	53.2268	53.2269
	3	8.6678	8.6678	26.6934	26.6934	46.3294	46.3295
	4	8.3555	8.3555	24.5657	24.5657	41.5652	41.5653

The above nonlinear equation can now be solved using a direct iterative process as follows:

First, nonlinearity is ignored by taking to solve Eq. (33). This yields the linear frequency and displacements. The displacements are then scaled up.

Using linear deflection, nonlinear coefficient could be evaluated. The problem is then solved by substituting nonlinear coefficient into Eq. (33). This would give the nonlinear frequency and displacements.

The new nonlinear deflection is scaled up again and the above procedure is repeated iteratively until the difference between displacements values from the two subsequent iterations becomes less than 0.01%.

6. Results and discussion

In order to show the effects of nonlocal parameter, surrounding elastic medium, aspect ratio and boundary condition, the frequency reduction percent is defined as follows

$$FRP = \left(\frac{\Omega_{local} - \Omega_{nonlocal}}{\Omega_{local}} \right) \times 100$$

In the absence of similar publications in the literature covering the same scope of the problem, one can not directly validate the results found here. However, the present work could be partially validated based on a simplified analysis suggested by Thai (2012) neglecting nonlinear terms in motion equations and elastic foundation. Table 1 shows the non-dimensional fundamental frequency (i.e., $\varpi = \omega L^2 \sqrt{m_0 / EI}$) of a simply supported nano-beam. The obtained results are compared with those reported by Thai (2012) based on nonlocal TBT. It can be seen that the results of present theory are in excellent agreement with those predicted by TBT for all values of small scale coefficient and length-to-depth ratio even for short beams at the higher vibration modes where the effects of transverse shear deformation and rotary inertia are significant. It is worth noting that the TBT requires a shear correction factor to satisfy the free transverse shear stress conditions on the top and bottom surfaces of the beam, whereas the present theory satisfies the free transverse shear stress conditions on the top and bottom surfaces of the beam without using any shear correction factors.

Fig. 1 illustrates the FRP versus the nonlocal parameter for four cases including:

Case1: linear vibration analysis of TBT

Case 2: Nonlinear vibration analysis of TBT

Case3: linear vibration analysis of SDBT

Case 4: Nonlinear vibration analysis of SDBT

As can be seen, the FRP increases with increasing μ. It means that with increasing μ, the frequency of the nano-beam becomes lower. This is due to the fact that the increase of nonlocal parameter decreases the interaction force between nano-beam atoms, and that leads to a softer structure. It is also concluded that the FRP (or frequency) of SDBT is lower (or higher) than TBT. Hence, application of SDBT in modeling of nanostructures based vibration analysis is better than TBT. Furthermore, the FRP of SDBT and TBT in nonlinear vibration response is higher than

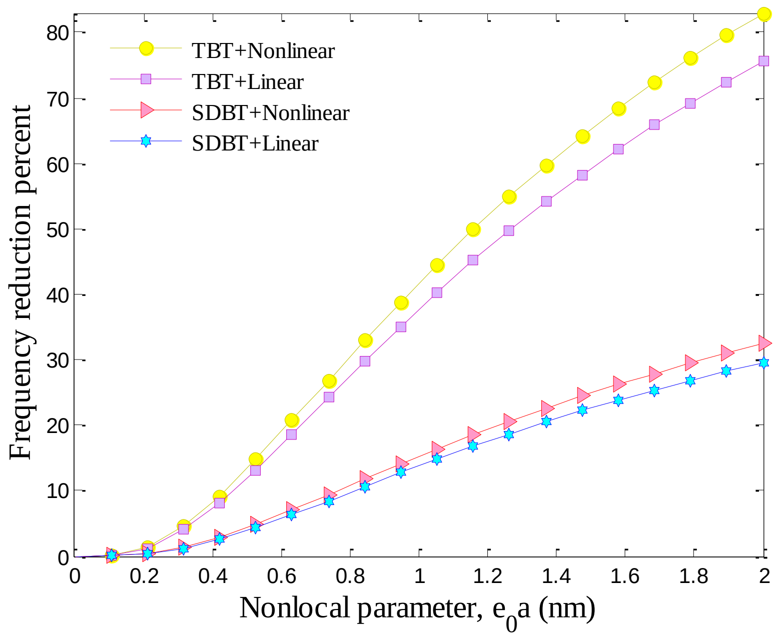

Fig. 1 FRP versus nonlocal parameter for TBT and SDBT

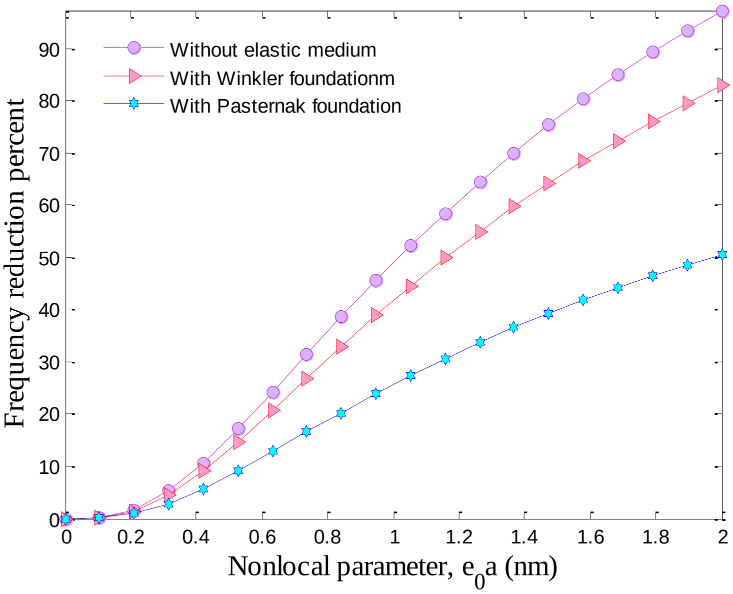

Fig. 2 The effect of elastic medium on the FRP versus the nonlocal parameter

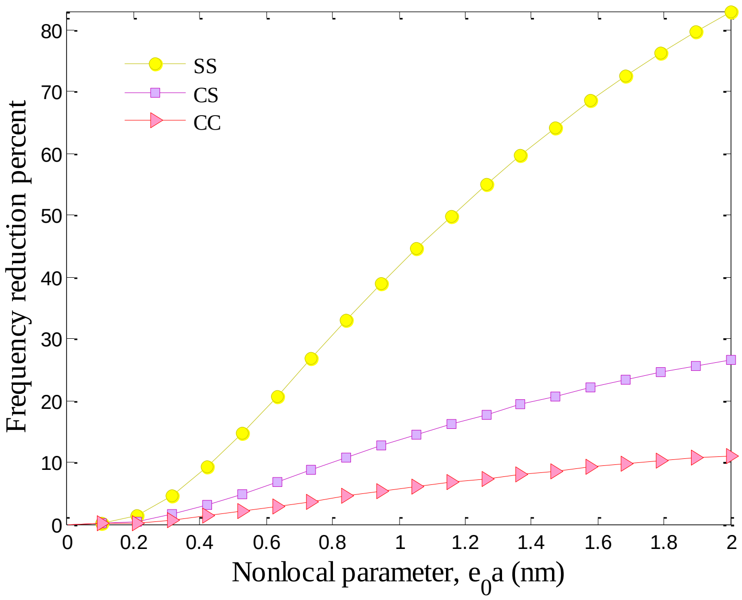

Fig. 3 The effects of boundary conditions on the FRP versus nonlocal parameter

linear one. It is due the fact that in nonlinear analyzing the accuracy of the obtained results is higher than linear one.

The effect of elastic medium on the FRP versus the nonlocal parameter of nano-beam, is shown in Fig. 2. Three different cases of elastic medium are considered. Case 1, Case2 and Case 3 depict the (i) without elastic medium (ii) with Winkler medium (iii) with Pasternak medium, respectively.

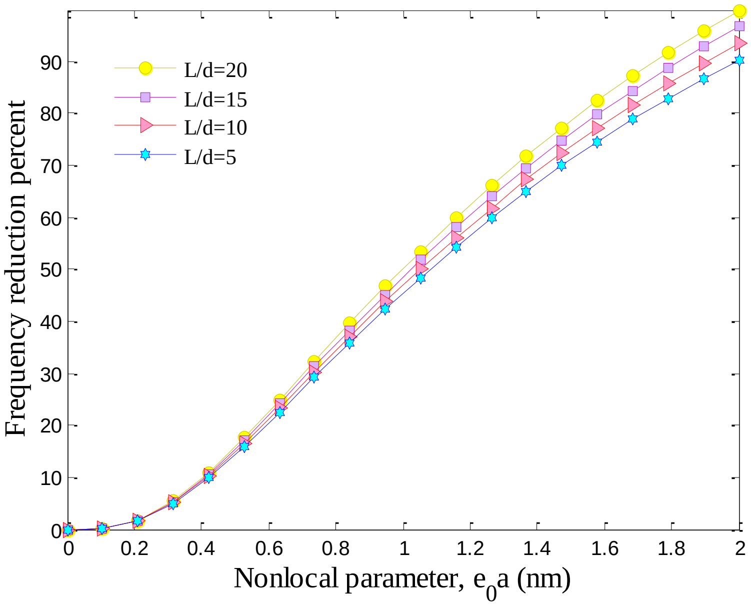

Fig. 4 The effect of aspect ration on the FRP of system versus nonlocal parameter

As can be seen, the FRP (or frequency) increases (or decreases) with increasing nonlocal parameter. It can be observed that the FRP for cases 1 and 3 is maximum and minimum, respectively. In the other words, the frequency of the system for the case of nano-beam embedded in elastic medium is higher than other cases. It is because considering elastic medium increases the stiffness of the system. It is also obvious that the FRP (or frequency) of the case 3 is lower (or higher) than case 2. It is due to the fact that in Winkler medium, a proportional interaction between pressure and deflection of nano-beam is assumed, which is carried out in the form of discrete and independent vertical springs. Whereas, Pasternak medium considers not only the normal stresses but also the transverse shear deformation and continuity among the spring elements.

Fig. 3 demonstrates the influence of boundary conditions on the FRP with respect to the nonlocal parameter. Three boundary conditions namely as clamped-clamped (CC), clamped-simply (CS) and simply-simply (SS) are considered. It could be said however, that FRP (or frequency) of nano-beam for CC boundary condition is minimum (or maximum). This is perhaps because in CC boundary condition, the stiffness of structure is higher than other cases. In addition, the effect of boundary conditions on the FRP becomes more prominent at higher nonlocal parameter.

Fig. 4 illustrates the effect of aspect ration (i.e., length to diameter of SWCNT) on the FRP of system versus nonlocal parameter. As can be seen, increasing aspect ratio, decreases frequency of SWCNT. It is due to the fact that, increasing aspect ration leads to soffer structure.

7. Conclusions

A nonlocal sinusoidal beam theory is developed for the nonlinear vibration of CNTs embedded in Pasternak medium. The present model is capable of capturing both small scale and shear deformation effects of CNTs, and does not require shear correction factors. Based on the nonlocal differential constitutive relations of Eringen, the equations of motion of the CNT are derived using Hamilton's principle. DQM for natural frequency is presented for different boundary conditions, and the obtained results are compared well with those predicted by the TBT. It is concluded that the FRP (or frequency) of SDBT is lower (or higher) than TBT. The FRP of SDBT and TBT in nonlinear vibration response is higher than linear one. It is also obvious that the FRP (or frequency) of the Pasternak medium is lower (or higher) than Winkler medium. Furthermore, increasing aspect ratio, decreases frequency of CNT. It could be said however, that FRP (or frequency) of nano-beam for CC boundary condition is minimum (or maximum). In addition, the effect of boundary conditions on the FRP becomes more prominent at higher nonlocal parameter. Meanwhile, the FRP (or frequency) increases (or decreases) with increasing nonlocal parameter.

Acknowledgements

I should thank Pars Oil and Gas Co. for having sponsored this project.

References

Arash, B. and Ansari, R. (2010), "Evaluation of nonlocal parameter in the vibrations of single-walled carbon nanotubes with initial strain", *Physica E: Low Dimen. Syst. Nanostruct*., **42**, 2058-2064.

Arash, B. and Wang, Q. (2012), "A review on the application of nonlocal elastic models in modeling of carbon nanotubes and graphenes", *Comput. Mater. Sci*., **51**, 303-313.

Thai, H.T. and Vo, T.P. (2012), "A nonlocal sinusoidal shear deformation beam theory with application to bending, buckling, and vibration of nanobeams", *Int. J. Eng. Sci*., **54**, 58-66.

Aydogdu, M. (2009). "A general nonlocal beam theory: Its application to nanobeam bending, buckling and vibration", *Physica E: Low Dimen. Syst. Nanostruct*., **41**, 1651-1655.

Civalek, O. and Demir, C. (2011), "Bending analysis of microtubules using nonlocal Euler-Bernoulli beam theory", *Appl. Math. Model*., **35**, 2053-2067.

Eringen, A.C. (1972), "Nonlocal polar elastic continua", *Int. J. Eng. Sci*., **10**, 1-16.

Eringen, A C. (1983), "On differential equations of nonlocal elasticity and solutions of screw dislocation and surface waves", *J. Appl. Phys*., **54**, 4703-4710.

Li, X., Bhushan, B., Takashima, K., Baek, C.W. and Kim, Y.K. (2003), "Mechanical characterization of micro/nanoscale structures for MEMS/NEMS applications using nanoindentation techniques", *Ultramicroscopy*, **97**, 481-494.

Ma, H.M., Gao, X.L. and Reddy, J.N. (2008), "A microstructure-dependent Timoshenko beam model based on a modified couple stress theory", *J. Mech. Phys. Solid*., **56**, 3379-3391.

Murmu, T. and Pradhan, S.C. (2009), "Buckling analysis of a single-walled carbon nanotube embedded in an elastic medium based on nonlocal elasticity and Timoshenko beam theory and using DQM", *Physica E: Low Dimen. Syst. Nanostruct*., **41**, 1232-1239.

Nix, W.D. and Gao, H. (1998), "Indentation size effects in crystalline materials: A law for strain gradient plasticity", J. Mech. Phys. Solid., **46**, 411-425.

Peddieson, J., Buchanan, G.R. and McNitt, R.P. (2003), "Application of nonlocal continuum models to nanotechnology", *Int. J. Eng. Sci*., **41**, 305-312.

Reddy, J.N. (2002), *Energy Principles and Variational Methods in Applied Mechanics*, John Wiley & Sons Inc.

Reddy, J.N. (2007), "Nonlocal theories for bending, buckling and vibration of beams", *Int. J. Eng. Sci*., **45**, 288-307.

Reddy, J.N. and Pang, S.D. (2008), "Nonlocal continuum theories of beams for the analysis of carbon nanotubes", *J. Appl. Phys*., **103**, 023511.

Thai, H.T. (2012), "A nonlocal beam theory for bending, buckling, and vibration of nanobeams", *Int. J. Eng. Sci*., **52**, 56-64.

Touratier, M. (1991), "An efficient standard plate theory", *Int. J. Eng. Sci*., **29**, 901-916.

Wang, Q. (2005), Wave propagation in carbon nanotubes via nonlocal continuum mechanics", *J. Appl. Phys.*, **98**, 124301.

Wang, C.M., Kitipornchai, S., Lim, C.W. and Eisenberger, M. (2008), "Beam bending solutions based on nonlocal Timoshenko beam theory", *J. Eng. Mech*., **134**, 475-481.

Wang, Q. and Liew, K.M. (2007), "Application of nonlocal continuum mechanics to static analysis of micro- and nano-structures", *Phys. Lett. A*, **363**, 236-242.

Wang, Q. and Varadan, V.K. (2006), "Vibration of carbon nanotubes studied using nonlocal continuum mechanics", *Smart Mater. Struct*., **15**, 659-666.

Wang, Q., Varadan, V.K. and Quek, S.T. (2006), "Small scale effect on elastic buckling of carbon nanotubes with nonlocal continuum models", *Phys. Lett. A*, **357**, 130-135.

Wang, Q. and Wang, C. (2007), "The constitutive relation and small scale parameter of nonlocal continuum mechanics for modelling carbon nanotubes", *Nanotechnology*, **18**, 075702.

Wang, C.M., Zhang, Y.Y. and He, X.Q. (2007), "Vibration of nonlocal Timoshenko beams", *Nanotechnology*, **18**, 105401.

Wang, C.M., Zhang, Y.Y., Ramesh, S.S. and Kitipornchai, S. (2006), "Buckling analysis of micro- and nano-rods/tubes based on nonlocal Timoshenko beam theory", *J. Phys. D: Appl. Phys*., **39**, 3904-3909.

Yakobson, B.I., Brabec, C.J. and Bernholc, J. (1996), "Nanomechanics of carbon tubes: Instabilities beyond linear response", *Phys. Rev. Lett*., **76**, 2511-2514.

Zenkour, A.M. (2004), "Analytical solution for bending of cross-ply laminated plates under thermo-mechanical loading", *Compos. Struct*., **65**, 367-379.

Zenkour, A.M. (2005a), "A comprehensive analysis of functionally graded sandwich plates: Part 1-Deflection and stresses", *Int. J. Solid. Struct*., **42**, 5224-5242.

Zenkour, A.M. (2005b), "A comprehensive analysis of functionally graded sandwich plates: Part 2-Buckling and free vibration", *Int. J. Solid. Struct*., **42**, 5243-5258.

Zenkour, A.M. (2006), "Generalized shear deformation theory for bending analysis of functionally graded plates", *Appl. Math. Model*., **30**, 67-84.

Zhang, Y.Q., Liu, G.R. and Xie, X.Y. (2005), "Free transverse vibrations of double-walled carbon nanotubes using a theory of nonlocal elasticity", *Phys. Rev. B*, **71**, 195404

CC