Structural Engineering and Mechanics, *Vol. 54, No. 4 (2015) 793-807*
DOI: http://dx.doi.org/10.12989/sem.2015.54.4.793

Direct kinematic method for exactly constructing influence lines of forces of statically indeterminate structures

Dixiong Yang*[1,2], Guohai Chen[1] and Zongliang Du[1]

[1]*Department of Engineering Mechanics, State Key Laboratory of Structural Analysis for Industrial Equipment, Dalian University of Technology, Dalian 116023, China*
[2]*State Key Laboratory Base of Eco-hydraulic Engineering in Arid Area, Xi'an University of Technology, Xi'an 710048, China*

(Received August 31, 2014, Revised January 14, 2015, Accepted April 4, 2015)

Abstract. Constructing the influence lines of forces of statically indeterminate structures is a traditional issue in structural engineering and mechanics. However, the existing kinematic method for establishing these force influence lines is an indirect or mixed approach by combining the force method with the theorem of reciprocal displacements, which is yet inconsistent with the kinematic method for statically determinate structure. This paper proposes the direct kinematic method in conjunction with the load-displacement differential relation for exactly constructing influence lines of reaction and internal forces of indeterminate structures. Firstly, through applying the principle of virtual displacement, the formula for influence lines of reaction and internal forces of indeterminate structure via direct kinematic method is derived based on the released structure. Then, a computational approach with a clear concept and unified procedure as well as wide applicability based on the load-displacement differential relation of beam is suggested to achieve conveniently the closed-form expression of force influence lines, and exactly draw them. Finally, three representative examples for constructing force influence lines of statically indeterminate beams and frame illustrate the superiority of the proposed method.

Keywords: statically indeterminate structure; influence lines of reaction and internal forces; closed-form solution; direct kinematic method; principle of virtual displacement

1. Introduction

Influence line of reaction and internal force of structure is an imperative tool for the design and state identification of bridge and building engineering subjected to moving load (Strauss *et al*. 2012, Zhu *et al.* 2014, Zhao *et al*. 2015). Nowadays, there are two kinds of approach, such as the static method and kinematic method, for constructing the influence lines of forces of engineering structures (Timoshenko and Young 1965, Buckley 1997, Hibbeler 2002, Buckley 2003, Li 2010, Long and Bao 2012). The kinematic method is derived from the Muller-Breslau principle, usually also called as qualitative method (Kassimali 1999, Hibbeler 2002, Ghali *et al*. 2003). For statically determinate structure, the influence line of force constructed by kinematic

*Corresponding author, Professor, E-mail: yangdx@dlut.edu.cn

http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

corresponding constraint of original structure and introducing the relevant unit displacement. Nevertheless, for statically indeterminate structure especially with multiple degrees of indeterminacy, it seems that the static method based on finite element analysis is tedious (Strauss *et al*. 2011, Strauss *et al*. 2012). Further, it cannot outline the shape of influence line before detailed calculation. As a consequence, the kinematic method is recommended, which can quickly determine the shape of influence line of statically indeterminate structure.

Actually, the kinematic method suggested in some popular references (Kassimali 1999, Hibbeler 2002, Bhavikatti 2005, Li 2010, Long and Bao 2012) is firstly to adopt both the force method (obviously, which belongs to the static method) and the Maxwell's theorem of reciprocal displacements (i.e., energy method), and then transform the problem of establishing the force influence line of statically indeterminate structure under a vertical unit moving load into that of constructing deflected curve of released structure under a unit displacement. However, it is an indirect or mixed kinematic method, instead of a direct or pure kinematic method as same as that for statically determinate structure. This point may make the structural engineers, especially the beginners confused. Some researchers had noted this problem and made certain modifications. For instance, Liu and Wang (2001) pointed out the above mentioned limitation, and established the equation of force influence line of the released structure by using the theorem of reciprocal works. Unfortunately, they failed to show the convenient computational process for quantitative influence lines of indeterminate structure. Chen (2002) derived the formula of influence line of reaction and internal force for indeterminate structure using the theorem of reciprocal displacement and reaction, in which the influence line corresponds to the deflected curve of original structure caused by unit joint displacement. Nonetheless, it is still not a direct kinematic method.

On the other hand, the formula of force influence line in the framework of kinematic method by using the Betti's theorem of reciprocal works was deduced in a few works (Timoshenko and Young 1965, Thompson and Haywood 1986, Leet and Uang 2002, Ghali *et al*. 2003) for drawing the shape of force influence line of indeterminate structure. Nevertheless, the detailed computational procedure for drawing quantitative influence line is approximate and cumbersome, either through calculating point by point the displacement of successive points of deflection curve with the unit load moving across the released structure at certain distance, or by applying the specific or exclusive approach (Buckley 1997, Kassimali 1999, Hibbeler 2002, Leet and Uang 2002, Ghali *et al*. 2003, Bhavikatti 2005, Long and Bao 2012). As a result, although drawing influence lines of forces of statically indeterminate structure is a traditional and important topic in structural engineering (Kurrer 2008), to develop a clear, simple and exact method for this task is still a pending issue.

In this paper, the formula based on direct kinematic method to construct influence lines of forces of statically indeterminate structures is derived by using the principle of virtual displacement. It is shown that the force influence line of statically indeterminate structure is the deflected curve of released structure, in which the corresponding constraint is removed and respective unit displacement is imposed. This method and its derivate process for indeterminate structure are completely in agreement with those for determinate structure. Essentially, the direct kinematic method is an energy method. Moreover, a computational approach with a clear concept and unified scheme based on the load-displacement differential relation of beam (i.e., $EIy^{(4)}(x)=q(x)$) is advised to acquire conveniently the closed-form equations of force influence lines of statically indeterminate structures, and exactly draw them. Finally, several examples of force influence lines for indeterminate beams and frame with multiple degrees of indeterminacy demonstrate its general applicability. An additional objective of this work is to help deeply comprehend the qualitative method for constructing force influence line of both statically determinate and indeterminate structure.

2. Formula derivation of force influence line of indeterminate structure

This section firstly proves one lemma on the internal virtual work, and then derives the formulation for constructing force influence lines of statically indeterminate structure by using direct kinematic method based on the principle of virtual displacement. For comparison and completion, the formula derivation of force influence lines of indeterminate structure based on the Betti's theorem of reciprocal works is also introduced.

2.1 One Lemma on the internal virtual work

For a statically indeterminate or determinate structure under arbitrary loads, there is always a released structure by means of removing certain constraint and replacing it with relevant external or internal force, which is equivalent to the original structure.

Lemma 1

If a certain support displacement or generalized displacement occurs at the released structure along the corresponding constraint direction of reaction or internal force, then the internal virtual work (i.e., virtual strain energy) done by the internal forces (M, F_N, F_Q) of original structure undergoing the virtual strains ($\bar{\kappa}$, $\bar{\varepsilon}$, $\bar{\gamma}_0$) of released structure is zero, namely

$$\sum \int (M\bar{\kappa} + F_N\bar{\varepsilon} + F_Q\bar{\gamma}_0)\mathrm{d}x = 0 \tag{1}$$

Proof of Lemma 1

For a statically determinate structure, the released structure by removing some support constraint or internal constraint becomes a mechanism. The imposed support displacement or generalized displacement in the constraint direction of mechanism results in the rigid body displacement without the strain and internal force. Therefore, the total internal virtual work (i.e., virtual strain energy) done by the internal forces of determinate structure undergoing the virtual strains of its respective released structure is zero, i.e., $\sum \int (M\bar{\kappa} + F_N\bar{\varepsilon} + F_Q\bar{\gamma}_0)\mathrm{d}x = 0$.

For a statically indeterminate structure, there is no any support displacement, e.g. as shown in Fig. 1(a). The arbitrary concentrated and distributed loads applying to the original structure produce the moment $M(x)$. Then, a reaction or internal force constraint is removed and the released structure in Fig. 1(b) is obtained. The moment $\bar{M}_1(x)$ of released structure under unit generalized load Z_1=1 in the direction of the removed constraint is exhibited in Fig. 1(d). It is easily known that, there is no relative displacement of original structure corresponding to the unit generalized load Z_1=1 under external loads. Thus, according to the unit-load method the relative displacement Δ_1 with respect to Z_1 is expressed as $\Delta_1 = \sum \int \frac{M\bar{M}_1}{EI}\mathrm{d}x = 0$. It also means that the internal work done by the internal forces of original structure subjected to external loads undergoing the strains of

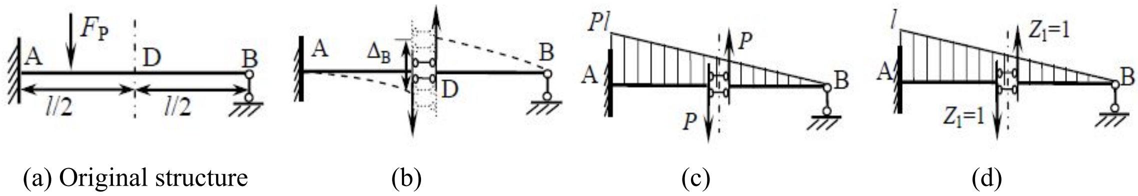


Fig. 1 Statically indeterminate beam (a) Original structure; (b) Released structure with generalized displacement Δ_B; (c) Moment diagram $\bar{M}(x)$ of released structure with generalized displacement Δ_B (equivalent to imposing external load); (d) Moment diagram $\bar{M}_1(x)$ of released structure under unit generalized load

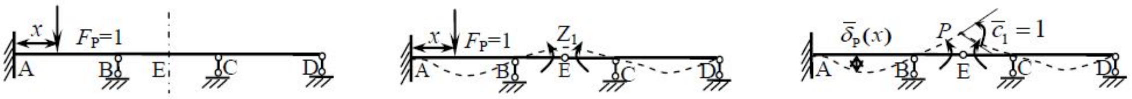
released structure under unit load $Z_1=1$ is zero. In addition, when the generalized virtual displacement Δ_B is imposed at the released structure in the direction of the respective internal constraint, the moment $\bar{M}(x)$ (see Fig. 1(c)) and curvature $\bar{\kappa}(x)$ are generated. For the sake of convenient discussion, only the bending moment and strain of original structure are considered when calculating the virtual work. The internal virtual work done by the internal forces of original structure undergoing the virtual strains of released structure is written as $W_i=\sum\int M\bar{\kappa}\mathrm{d}x=\sum\int\frac{M\bar{M}}{EI}\mathrm{d}x$.

Fig. 1 (c) displays the moment function $\bar{M}(x)$ of released structure with generalized displacement Δ_B, which is equivalent to that under external load P. With the assumption of small deformation and linear elasticity, the moment $\bar{M}(x)$ of released structure with virtual generalized displacement is proportional to the moment $\bar{M}_1(x)$ of released structure under unit generalized force with a constant ratio c, i.e., $\bar{M}(x)=c\bar{M}_1(x)$. Hence, a desired equation of internal virtual work is finally achieved based on the proved expression of relative displacement $\Delta_1=0$, namely, $W_i=\sum\int M\bar{\kappa}\mathrm{d}x=\sum\int\frac{M\bar{M}}{EI}\mathrm{d}x=c\sum\int\frac{M\bar{M}_1}{EI}\mathrm{d}x=0$. The same conclusion on the internal virtual work considering the axial and shear forces and strains can be drawn as well. In summary, the internal virtual work (i.e., virtual strain energy) done by the internal forces of original structure undergoing the virtual strains of released structure is zero, i.e., $\sum\int(M\bar{\kappa}+F_N\bar{\varepsilon}+F_Q\bar{\gamma}_0)\mathrm{d}x=0$. The proof of Lemma 1 is ended.

Lastly, it is particularly pointed out that, the physical meaning of Eq. (1) indicates that the internal virtual work done by the internal forces of determinate or indeterminate structure under external loads undergoing the virtual strains of released structure with a virtual displacement in the direction of the corresponding removed constraint is zero. Essentially, it is because that the displacement of original structure corresponding to the generalized load is zero. Equation (1) is the theoretical premise of kinematic method for constructing influence lines of reaction and internal forces of determinate or indeterminate structure. Further, it is a unified formula applicable to both the statically determinate and indeterminate structures.

2.2 Formula derivation based on the principle of virtual displacement

The influence line of Z_1 standing for the reaction and internal force of statically indeterminate



(a) Original structure

(b) Released structure under moving load and generalized load

(c) Released structure with unit generalized displacement

Fig. 2 Statically indeterminate beam

structure is considered to construct. Herein, Z_1 denotes the moment at E of indeterminate beam. On the one hand, the force state of original structure under vertical unit moving load F_P=1 with the location coordinate x displayed in Fig. 2(a) is equivalent to that of the released structure by relaxing relevant constraint with a pair of external forces Z_1 in Fig. 2(b). Its support reaction force is F_R, and internal forces are $M(x)$, $F_N(x)$, $F_Q(x)$. The external force or generalized load Z_1 of released structure in Fig. 2(b) equals to the corresponding internal force of original structure in Fig. 2(a). Hence, the reactions and internal forces as well as the displacements of released structure in Fig. 2(b) are identical with those of original structure in Fig. 2(a).

On the other hand, the virtual strain state of released structure is considered. Assume that the released structure is exerted by a virtual generalized displacement $\overline{c}_1$=1 relevant to the relaxed constraint, and there are no other support displacements in Fig. 2(c). Then, the released structure yields the virtual displacement $\overline{\delta}_P$ at the point corresponding to the unit moving load and the virtual strains $\overline{\kappa}$, $\overline{\varepsilon}$ and $\overline{\gamma}_0$.

Actually, the principle of virtual displacement is a variant form of the principle of virtual work (Hibbeler 2002, Walls and Elvin 2010, Yang *et al*. 2011, Long and Bao 2012), when the displacement field is kinematically admissible. Using the principle of virtual displacement, the above two states of real forces and virtual displacements are substituted into the virtual displacement equation indicating that the external virtual work equals to the internal virtual work (i.e., virtual strain energy), which is expressed as

$$Z_1 \cdot \overline{c}_1 + F_P \cdot \overline{\delta}_P = \sum \int (M\overline{\kappa} + F_N\overline{\varepsilon} + F_Q\overline{\gamma}_0)\mathrm{d}x \tag{2}$$

Substituting the virtual unit displacement $\overline{c}_1$=1 and real unit moving load F_P=1 into Eq. (2) yields

$$Z_1 \cdot 1 + 1 \cdot \overline{\delta}_P = \sum \int (M\overline{\kappa} + F_N\overline{\varepsilon} + F_Q\overline{\gamma}_0)\mathrm{d}x \tag{3}$$

where

Z_1= external real load acting on the released structure in the direction of $\overline{c}_1$=1

F_P=1= external unit moving load acting on released structure in the direction of $\overline{\delta}_P$

M, F_N, F_Q= internal moment, axial and shear force respectively in released structure caused by the real loads (Z_1 and F_P=1)

$\overline{c}_1$=1= external virtual unit displacement corresponding to the force Z_1 of which the influence line is to be determined

$\overline{\delta}_P$= external virtual displacement in released structure caused by the virtual unit displacement

$\bar{c}_1=1$

$\bar{\kappa}$, $\bar{\varepsilon}$, $\bar{\gamma}_0$= internal virtual bending, axial and shear strain respectively in released structure caused by the virtual unit displacement $\bar{c}_1=1$

In terms of Lemma 1, the right-hand term of Eq. (3) is zero, i.e., the internal virtual work done by the internal forces of statically indeterminate structure under unit moving load and generalized load undergoing the virtual strains of released structure with virtual unit displacement in the direction of respective constraint is zero. Thus, Eq. (3) becomes

$$Z_1(x) = -\bar{\delta}_{\mathrm{P}}(x) \tag{4}$$

Eq. (4) is the formula for constructing influence lines of reaction and internal forces of indeterminate structure by direct kinematic method. It implies that the influence coefficient of original structure just equals to the deflection value of released structure with unit generalized displacement at the acting point of unit moving load, but its sign is opposite. Furthermore, Eq. (4) for indeterminate structure is derived only using the principle of virtual displacement as same as that for determinate structure. Therefore, the direct kinematic method for drawing the force influence lines of statically indeterminate structure derived from the principle of virtual displacement is consistent with that of determinate structure. The direct kinematic method can replace the mixed kinematic method which combines both the force method and the theorem of reciprocal displacements (Kassimali 1999, Hibbeler 2002, Bhavikatti 2005, Li 2010, Long and Bao 2012), to acquire the formula of force influence lines of statically indeterminate structure. Obviously, it is seen that the formula for constructing influence lines of forces of indeterminate structure is a degenerate form of virtual displacement equation, which can be obtained when its term of internal virtual work is zero.

2.3 Formula derivation based on the theorem of reciprocal works

In practice, Eq. (4) for constructing force influence lines of statically indeterminate structure can also be derived from the Betti's theorem of reciprocal works conveniently. Under a unit moving load $F_{\mathrm{P}}=1$, the indeterminate structure generates some deformations and internal forces, for example, as displayed in Fig. 2(a). The state of forces and deformations of original structure equals to that of the released structure by relaxing the relevant constraint and imposing certain external load Z_1 shown in Fig. 2(b), which is defined as state I. Meanwhile, the state of forces and deformations of released structure caused by a generalized displacement in the direction of the relaxed constraint in Fig. 2(c) is called as state II.

The external forces of state I include the unit moving load $F_{\mathrm{P}}=1$ and generalized load Z_1. The displacements of state II include the unit generalized displacement (without support displacement) and the displacement $\bar{\delta}_{\mathrm{P}}$ corresponding to the unit moving load. Hence, the work done by external forces of state I undergoing the displacements of state II is: $W_{12} = 1\cdot\bar{\delta}_{\mathrm{P}} + Z_1\cdot 1$.

For the released structure in Fig. 2(c), the generalized displacement $\bar{c}_1=1$ is produced by the generalized force P. Moreover, the relative displacement of original structure in Fig. 2(a) with respect to the generalized force P is zero, and the relative rotational angle of section E at the equivalent released structure in Fig. 2(b) is also zero. Consequently, based on the fact that the displacement of original structure or the equivalent released structure corresponding to the generalized load Z_1 is zero, the work done by the external forces of state II undergoing the displacements of state I is $W_{21}=P\cdot 0=0$.

With the theorem of reciprocal works $W_{12}=W_{21}$ (Timoshenko and Young 1965, Thompson and Haywood 1986, Leet and Uang 2002, Ghali *et al*. 2003), an equation of reciprocal works is obtained

$$W_{12} = 1 \cdot \overline{\delta}_{\mathrm{P}} + Z_1 \cdot 1 = W_{21} = 0 \tag{5}$$

Apparently, Eq. (4) standing for the force influence lines of indeterminate structure is easily derived from Eq. (5) of reciprocal works, because Eq. (5) can be transformed into Eq. (4) immediately. Note that the theorem of reciprocal works can be similarly applied to derive Eq. (5) and Eq. (4) of force influence line for determinate structure. Finally, it is pointed out that, the formula derivation process of Eq. (4) for the force influence lines of indeterminate and determinate structure using the principle of virtual displacement is more natural and direct than that with the theorem of reciprocal works.

3. Exactly constructing force influence lines of indeterminate structures

Firstly, an exact computational approach based on the load-displacement differential relation of beam (i.e., $EIy^{(4)}(x)=q(x)$) is suggested to construct the force influence lines of statically indeterminate beam and frame structures. Subsequently, three examples for determining the influence lines of reaction and internal force of indeterminate beams and frame are shown.

3.1 Computational approach for force influence lines of indeterminate structures

By virtue of Eq. (4), an exact approach to calculate the force influence lines of indeterminate structure is proposed as follows.

(1) Remove or relax the constraint of original structure corresponding to the reaction or internal force Z of interest for constructing the influence line, and obtain the released structure.

(2) Impose the unit displacement or unit generalized displacement (i.e., relative displacement, and $\delta_1=1$) corresponding to the relaxed constraint of released structure, and take the deflected shape $y(x)$ of released structure as the outline of the force influence line. This is a major advantage of kinematic method for constructing influence lines. One can establish rapidly the general shape of influence lines without the need of careful calculation, so as to further identify the most unfavorable position of moving load.

(3) Utilize the load-displacement differential relation of beam with equal sectional area (where EI denotes flexural stiffness and $q(x)$ is the distributional load, respectively), namely

$$EIy^{(4)}(x)=q(x) \tag{6}$$

Because the distributional load of released structure is zero, namely $q(x)=0$, the analytical expression of deflection curve $y(x)$ can be written as a cubic polynomial

$$y(x)=ax^3+bx^2+cx+d \tag{7}$$

Accordingly, the deflection curve of released structure corresponding to the force influence line is a cubic curve. In terms of the specific case of displacement and force of segments in the beam-type structure, four boundary and continuity conditions for every segment can be obtained.

(4) Substituting the four boundary and continuity conditions for every segment into the deflection curve Eq. (7) and its first, second and third-order differential formulas, the

undetermined coefficients of deflection curves $y(x)$ of all segments can be achieved. Consequently, the closed-form expression of influence line $Z(x)$ of reaction and internal force of indeterminate structure is obtained

$$Z(x)=y(x) \tag{8}$$

The above proposed approach to calculate the influence line expression by direct kinematic method is versatile, exact and convenient for the reaction and internal forces at any section of indeterminate structure. On this basis, one can determine the most unfavorable loadings and the most unfavorable internal forces of indeterminate structure.

Finally, it is pointed out that in the previous works (Kassimali 1999, Hibbeler 2002, Leet and Uang 2002, Ghali *et al*. 2003, Bhavikatti 2005, Long and Bao 2012), the moment-displacement differential relation (i.e., $EI\,y''(x)=M(x)$) is used to solve the force influence line of indeterminate structure. Actually, although one can apply the force method (i.e., flexibility method), the displacement method (i.e., stiffness method), moment distribution method, or especially the closed-form moment distribution method (Dowell 2009, Dowell and Johnson 2011) to address the bending moment of structure with multiple degrees of indeterminacy, these methods are very difficult to acquire an analytical moment function of released structure with removed constraints corresponding to shear and moment. Fortunately, the new approach proposed herein utilizes another differential relation (i.e., $EIy^{(4)}(x)=q(x)=0$) and avoids such a difficulty, so as to achieve the influence line $Z(x)$ of reaction and internal force of indeterminate structure in a general and convenient way.

3.2 Numerical examples

Example 1 Here Examples 7.15 and 7.16 in the reference (Hibbeler 2002) are adopted. Draw influence lines of shear F_{QD} and moment M_D at D for the two-span continuous beam with one degree of indeterminacy and constant cross-section in Fig. 3(a).

(1) Construct the influence line $F_{QD}(x)$

Relax the section D of original structure with a sliding device, and obtain the released system with three segments. Subsequently, impose unit relative displacement at D, and outline the influence line shape of shear F_{QD} in Fig. 3(b).

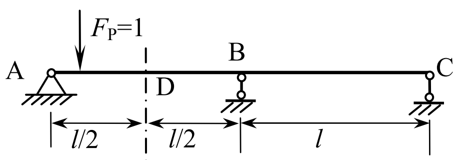


Fig. 3(a) Two-span continuous beam

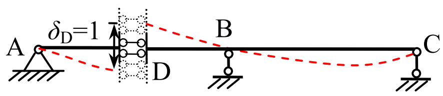


Fig. 3(b) Shape of influence line for F_{QD} of two-span continuous beam

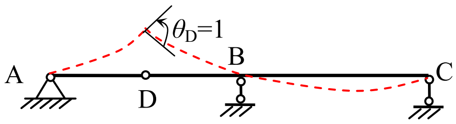


Fig. 3(c) Shape of influence line for moment M_D of two-span continuous beam

Three cubic polynomials $y(x)=ax^3+bx^2+cx+d$ of released structure are written, which contain 12 undetermined coefficients. Twelve boundary and continuity conditions of released structure are listed as:

$$y_1(0)=y_2(l)=y_3(l)=y_3(2l)=0,\ y_2(l/2)-y_1(l/2)=1,\ y_1'(l/2)=y_2'(l/2),$$
$$y_2'(l)=y_3'(l),\ EIy_1''(0)=EIy_3''(2l)=0,\ EIy_1''(l/2)=EIy_2''(l/2),$$
$$EIy_2''(l)=EIy_3''(l),\ EIy_1'''(l/2)=EIy_2'''(l/2)$$

Then, the unknown constants are determined and the equation of influence line $F_{QD}(x)$ is written as follows:

$$y(x)=\begin{cases}\dfrac{1}{4l^3}x^3-\dfrac{5}{4l}x & (0\le x\le l/2)\\ \dfrac{1}{4l^3}x^3-\dfrac{5}{4l}x+1 & (l/2\le x\le l)\\ -\dfrac{1}{4l^3}x^3+\dfrac{3}{2l^2}x^2-\dfrac{11}{4l}x+\dfrac{3}{2} & (l\le x\le 2l)\end{cases}$$

For instance, $y_1(l/2)=-0.5938, y_2(l/2)=0.4062,\ y_3(3l/2)=-0.0938$

(2) Draw the influence line $M_D(x)$

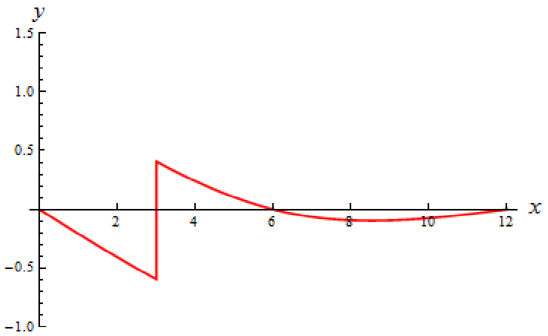
Relax the section D with a pin and get the released system. Meantime, sketch the influence line shape of moment M_D by direct kinematic method, as shown in Fig. 3(c).

Similarly, three cubic polynomials $y(x)=ax^3+bx^2+cx+d$ of released structure with 12 unknown coefficients are listed, and 12 boundary and continuity conditions are written as:

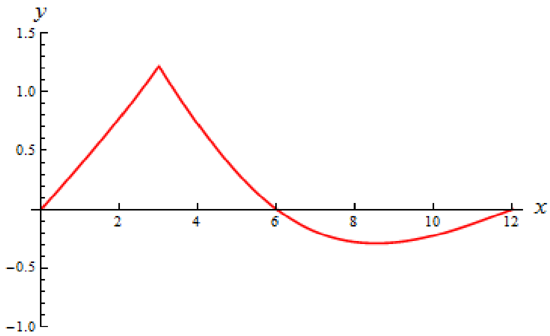
$$y_1(0)=y_2(l)=y_3(l)=y_3(2l)=0,\ y_1(l/2)=y_2(l/2),\ y_1'(l/2)-y_2'(l/2)=1,$$
$$y_2'(l)=y_3'(l),\ EIy_1''(0)=EIy_3''(2l)=0,\ EIy_1''(l/2)=EIy_2''(l/2),$$
$$EIy_2''(l)=EIy_3''(l),\ EIy_1'''(l/2)=EIy_2'''(l/2)$$

Thus, the influence line function $M_D(x)$ is formulated as:

$$y(x)=\begin{cases}\dfrac{1}{8l^2}x^3+\dfrac{3}{8}x & (0\le x\le \dfrac{l}{2})\\ \dfrac{1}{8l^2}x^3-\dfrac{5}{8}x+\dfrac{l}{2} & (\dfrac{l}{2}\le x\le l)\\ -\dfrac{1}{8l^2}x^3+\dfrac{3}{4l}x^2-\dfrac{11}{8}x+\dfrac{3l}{4} & (l\le x\le 2l)\end{cases}$$



(a) Influence line of shear force F_{QD}



(b) Influence line of moment M_D (unit: m)

Fig. 4 Quantitative influence lines of internal forces of two-span continuous beam

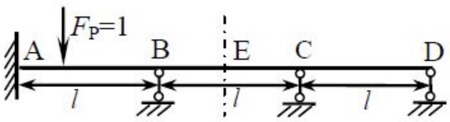


Fig. 5(a) Three-span continuous beam with 3 degrees of indeterminacy

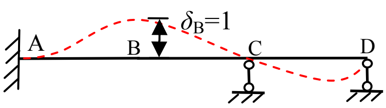


Fig. 5(b) Shape of influence line for F_{RB} of three-span continuous beam

Especially, for l=6 m, $y_1(l/2)$=$y_2(l/2)$=1.2188, $y(3l/2)$=−0.2813

Finally, the plots of influence lines of F_{QD} and M_D are shown in Figs. 4(a) and 4(b) separately. Comparing them with the ones in the reference (Hibbeler 2002), it is evident that the diagram of influence line of F_{QD} (Hibbeler 2002) is not totally correct, because the curvature of segment DB (from x=3 m to x=6 m) in the influence line should be concave upward rather than concave downward.

Example 2 Draw the influence lines of reaction F_{RB} and moment M_B at support B and moment M_E at E for the three-span statically indeterminate beam with 3 degrees of indeterminacy and constant cross-section in Fig. 5(a).

(1) Construct the influence line $F_{RB}(x)$

Remove the support B of original structure, and obtain the released system with three segments. Then, impose unit displacement at B, and outline the influence line shape of reaction F_{RB} in Fig. 5(b).

Twelve boundary and continuity conditions of released structure with 3 segments are listed as:

$$y_1(0)=y_2(2l)=y_3(2l)=y_3(3l)=0,\ \ y_1(l)=y_2(l)=1,\ \ y_1'(0)=0,$$

$$y_1'(l)=y_2'(l)=0,\ \ y_2'(2l)=y_3'(2l),\ \ EIy_2''(2l)=EIy_3''(2l),\ \ EIy_3''(3l)=0$$

As a result, the equation of influence line $F_{RB}(x)$ is written as follows:

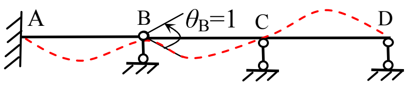


Fig. 5(c) Influence line shape for moment M_B of three-span beam

$$y(x)=\begin{cases} -\dfrac{2}{l^3}x^3+\dfrac{3}{l^2}x^2 & (0\le x\le l) \\ \dfrac{26}{31l^3}x^3-\dfrac{135}{31l^2}x^2+\dfrac{192}{31l}x-\dfrac{52}{31} & (l\le x\le 2l) \\ \dfrac{15}{31l^3}x^3-\dfrac{69}{31l^2}x^2+\dfrac{60}{31l}x+\dfrac{36}{31} & (2l\le x\le 3l) \end{cases}$$

(2) Construct the influence line of moment M_B

Relax the section B with a pin joint and get the released system. Then, outline the shape of influence line of M_B by direct kinematic method shown in Fig. 5(c).

With the 12 boundary and continuity conditions:

$$y_1(0)=y_1(l)=y_2(l)=y_2(2l)=y_3(2l)=y_3(3l)=0,\ \ y_1'(0)=0,\ \ y_1'(l)-y_2'(l)=1,$$

$$y_2'(2l)=y_3'(2l),\ \ EIy_1''(l)=EIy_2''(l),\ \ EIy_2''(2l)=EIy_3''(2l),\ \ EIy_3''(3l)=0$$

the closed-form expression of influence line $M_B(x)$ is obtained:

$$y(x)=\begin{cases} \dfrac{6}{13l^2}x^3-\dfrac{6}{13l}x^2 & (0\le x\le l) \\ -\dfrac{5}{13l^2}x^3+\dfrac{27}{13l}x^2-\dfrac{46}{13}x+\dfrac{24l}{13} & (l\le x\le 2l) \\ \dfrac{1}{13l^2}x^3-\dfrac{9}{13l}x^2+2x-\dfrac{24l}{13} & (2l\le x\le 3l) \end{cases}$$

Assuming l=6 m, the influence line plot of M_B is displayed in Fig. 5(d). In particular, $y(3)$= –0.3462, $y(9)$=–0.5192, and $y(15)$=0.1731. The results are equal to those in the reference (Long and Bao 2012).

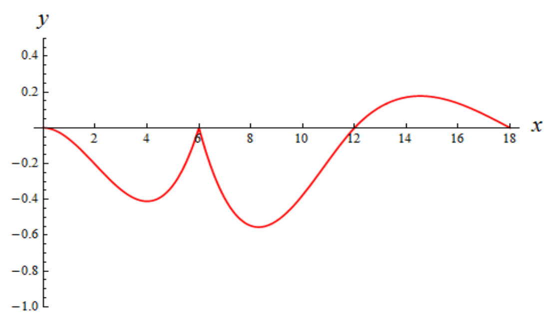


Fig. 5(d) Quantitative influence line for moment M_B of three-span beam (unit: m)

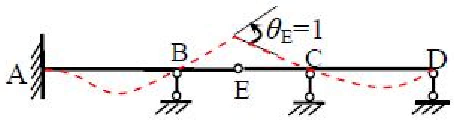


Fig. 5(e) Influence line shape for moment M_E of three-span beam

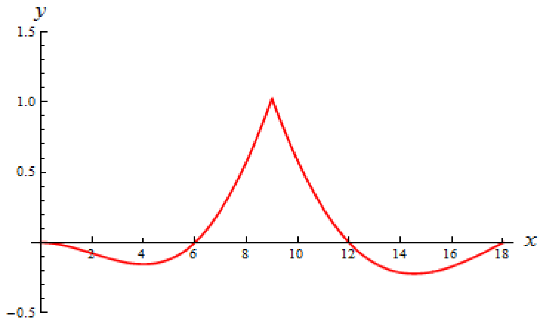


Fig. 5(f) Quantitative influence line for moment M_E of three-span beam (unit: m)

(3) Construct the influence line of moment M_E

Relax the section E at the center of span BC with a hinge and obtain the released system, and then outline the influence line shape of M_E in Fig. 5(e).

Combining with 16 boundary and continuity conditions of four segments:

$$y_1(0)=y_1(l)=y_2(l)=y_3(2l)=y_4(2l)=y_4(3l)=0,\ \ y_2(3l/2)=y_3(3l/2),$$

$$y_1'(0)=0,\ \ y_1'(l)=y_2'(l),\ \ y_2'(3l/2)-y_3'(3l/2)=1,\ \ y_3'(2l)=y_4'(2l),$$

$$EIy_1''(l)=EIy_2''(l),\ \ EIy_2''(3l/2)=EIy_3''(3l/2),\ \ EIy_3''(2l)=EIy_4''(2l),$$

$$EIy_4''(3l)=0,\ \ EIy_2'''(3l/2)=EIy_3'''(3l/2)$$

the influence line formula of $M_E(x)$ is obtained:

$$y(x)=\begin{cases}\dfrac{9}{52l^2}x^3-\dfrac{9}{52l}x^2 & (0\le x\le l)\\[2mm] -\dfrac{1}{52l^2}x^3+\dfrac{21}{52l}x^2-\dfrac{15}{26}x+\dfrac{5l}{26} & (l\le x\le \dfrac{3}{2}l)\\[2mm] -\dfrac{1}{52l^2}x^3+\dfrac{21}{52l}x^2-\dfrac{41}{26}x+\dfrac{22l}{13} & (\dfrac{3}{2}l\le x\le 2l)\\[2mm] -\dfrac{5}{52l^2}x^3+\dfrac{45}{52l}x^2-\dfrac{5}{2}x+\dfrac{30l}{13} & (2l\le x\le 3l)\end{cases}$$

For l=6 m, the influence line plot of M_E is presented in Fig. 5(f), and $y(9)$=1.0240.

Example 3 The four-span continuous frame in Fig. 6(a) is adapted from the reference (Dowell 2009), and draw the influence line of moment M_I at the central section I of span BC.

Firstly, relax the section I of span BC with a hinge and obtain the released system with 8 segments, and then outline the influence line shape of M_I in Fig. 6(b).

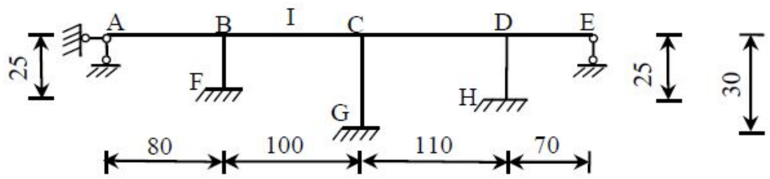


Fig. 6(a) Four-span continuous frame with 9 degrees of indeterminacy (unit: m)

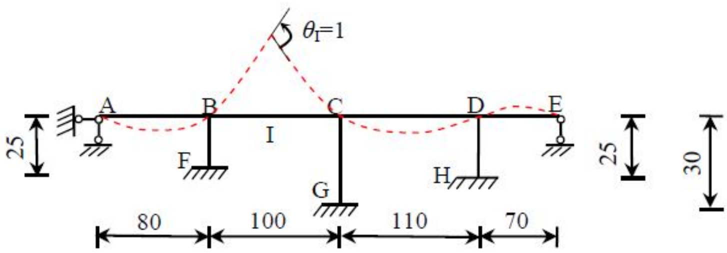


Fig. 6(b) Influence line shape for moment M_I of four-span frame

Thirty-two boundary and continuity conditions of 8 segments (where, member 2, 5 and 7 represents column BF, CG and DH, respectively) are listed as:

$$y_1(0) = y_1(80) = y_3(80) = y_2(25) = y_4(180) = y_6(180) = y_5(30) = y_6(290) = y_7(25)$$

$$= y_8(290) = y_8(360) = 0, \quad y_2(0) = y_5(0) = y_7(0) = 0, \quad y_2'(0) = y_5'(0) = y_7'(0) = 0$$

$$y_1'(80) = y_3'(80) = y_2'(25), y_4'(180) = y_5'(30) = y_6'(180), y_6'(290) = y_7'(25) = y_8'(290),$$

$$EI(y_1''(80) + y_2''(25) - y_3''(80)) = 0, \quad EI(y_4''(180) + y_5''(30) - y_6''(180)) = 0,$$

$$EI(y_6''(290) + y_7''(25) - y_8''(290)) = 0, \quad y_3'(130) - y_4'(130) = 1, \quad y_3(130) = y_4(130),$$

$$EIy_3''(130) = EIy_4''(130), \quad EIy_3'''(130) = EIy_4'''(130), \quad EIy_1''(0) = EIy_8''(360) = 0$$

Finally, the influence line formula of $M_I(x)$ is written as follows and its plot is displayed in Fig. 6(c).

$$y(x) = \begin{cases} \dfrac{13367}{3677441600} x(x-80)(x+80) & (0 \le x \le 80) \\ -\dfrac{1}{45968020000}(x-80)(27457x^2 - 215591720x + 14932892800) & (80 \le x \le 130) \\ -\dfrac{1}{45968020000}(x-180)(27457x^2 - 212846020x + 39835966800) & (130 \le x \le 180) \\ -\dfrac{103}{10112964400}(x-180)(x-290)(391x - 117240) & (180 \le x \le 290) \\ -\dfrac{103}{919360400}(x+260)(x-290)(x-360) & (290 \le x \le 360) \end{cases}$$

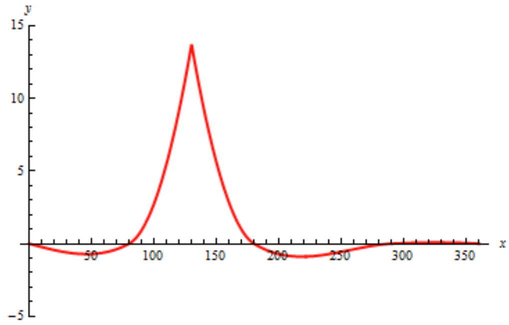


Fig. 6(c) Quantitative influence line for moment M_I of four-span frame (unit: m)

4. Conclusions

The existing kinematic method in some popular references is actually indirect or mixed approach, which combines the force method with the energy method (i.e., theorem of reciprocal displacements) for drawing the shapes of force influence lines of statically indeterminate structure. Moreover, the computational procedure for quantifying the corresponding magnitude of influence line is approximate and cumbersome.

In this paper, the formula of direct kinematic method to construct influence lines of reaction and internal forces of statically indeterminate structure is derived, through applying the principle of virtual displacement. Furthermore, an exact computational approach with a clear logic and unified scheme as well as wide applicability based on the load-displacement differential relation of beam (i.e., $EIy^{(4)}(x)=q(x)$) is advised to determine these influence lines. Finally, three representative examples for calculating the closed-form equations of influence lines of reaction and internal forces of statically indeterminate beams and frame are demonstrated. In a versatile and convenient way, the formulation of force influence line of indeterminate structure is acquired, which is beneficial to the design and state identification of structure and infrastructure engineering.

Acknowledgments

The supports of the National Natural Science Foundation of China (Grant Nos. 51478086 and 11332004), and the Key Laboratory Foundation of Science and Technology Innovation in Shaanxi Province (No. 2013SZS02-K02) are much appreciated.

References

Bhavikatti, S.S. (2005), *Structural Analysis*, 2nd Edition, Vikas Publishing House, New Delhi, India.

Buckley, E. (1997), "Basic influence line equations of continuous beams and rigid frames", *J. Struct. Eng.*, **123**(10), 1416-1420.

Buckley, E. (2003), *The Influence Line Approach to the Analysis of Rigid Frames*, Kluwer Academic Publishers, New York, USA.

Chen, S.F. (2002), "Kinematic method for constructing influence line and theorem of reciprocal reaction-displacement", *Mech. Eng.*, **24**(6), 59-61.

Dowell, R.K. (2009), "Closed-form moment solution for continuous beams and bridge structures", *Eng. Struct.*, **31**, 1880-1887.

Dowell, R.K. and Johnson, T.P. (2011), "Shear and bending flexibility in closed-form moment solutions for continuous beams and bridge structures", *Eng. Struct.*, **33**, 3238-3245.

Ghali, A., Neville, A.M. and Brown, T.G. (2003), *Structural Analysis: a Unified Classical and Matrix Approach*, 5th Edition, Spon Press, New York, USA.

Hibbeler, R.C. (2002), *Structural Analysis*, 5th Edition, Prentice Hall Inc, New Jersey, USA.

Kassimali, A. (1999), *Structural Analysis*, 2nd Edition, Brooks/Cole Publishing Company, Pacific Grove, CA, USA.

Kurrer, K.E. (2008), *The History of the Theory of Structures: from Arch Analysis to Computational Mechanics*, Ernst & Sohn Verlag, Berlin, Germany.

Leet, K. and Uang, C.M. (2002), *Fundamentals of Structural Analysis*, 2nd Edition, McGraw-Hill, New York, USA.

Li, L.K. (2010), *Structural Mechanics*, 5th edition, Higher Education Press, Beijing, China.

Liu, S.M. and Wang, Q.H. (2001), "Problem and suggestion of kinematic method for drawing influence line of internal force of continuous beam", *Mech. Eng.*, **23**(2), 61-63.

Long, Y.Q. and Bao, S.H. (2012), *Structural Mechanics*, 3rd Edition, Higher Education Press, Beijing, China.

Strauss, A., Wendner, R., Bergmeister, K. and Frangopol, D.M. (2011), *Monitoring and influence lines based performance indicators*, Eds. Faber, Köhler and Nishijima, Applications of Statistics and Probability in Civil Engineering, Taylor & Francis Group, London, UK.

Strauss, A., Wendner, R., Frangopol, D.M. and Bergmeister, K. (2012), "Influence line-model correction approach for the assessment of engineering structures using novel monitoring techniques", *Smart Struct. Syst.*, **9**(1), 1-20.

Thompson, F. and Haywood, C.G. (1986), *Structural Analysis Using Virtual Work*, Chapman and Hall Ltd, London, UK.

Timoshenko, S.P. and Young, D.H. (1965), *Theory of Structures*, 2nd Edition, McGraw-Hill, New York, USA.

Walls, R. and Elvin, A. (2010), "Optimizing structures subject to multiple deflection constraints and load cases using the principle of virtual work", *J. Struct. Eng.*, **136**(11), 1444-1452.

Yang, Y.B., Chen, C.T., Lin, T.J. and Hung, C.R. (2011), "Consistent virtual work approach for the nonlinear and postbuckling analysis of steel frames under thermal and mechanical loadings", *Eng. Struct.*, **33**(6), 1870-1882.

Zhao, H., Uddin, N., Shao, X.D., Zhu, P. and Tan, C.J. (2015), "Field-calibrated influence lines for improved axle weight identification with a bridge weigh-in-motion system", *Struct. Infrastruct. Eng.*, **11**(6), 721-743.

Zhu, S.Y., Chen, Z.W., Cai, Q.L., Lei, Y. and Chen, B. (2014), "Locate damage in long-span bridges based on stress influence lines and information fusion technique", *Adv. Struct. Eng.*, **17**(8), 1089-1102.

CC