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Dynamic response of non-uniform Timoshenko beams made of axially FGM subjected to multiple moving point loads

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Abstract. This paper presents a finite element procedure for dynamic analysis of non-uniform Timoshenko beams made of axially Functionally Graded Material (FGM) under multiple moving point loads. The material properties are assumed to vary continuously in the longitudinal direction according to a predefined power law equation. A beam element, taking the effects of shear deformation and cross-sectional variation into account, is formulated by using exact polynomials derived from the governing differential equations of a uniform homogenous Timoshenko beam element. The dynamic responses of the beams are computed by using the implicit Newmark method. The numerical results show that the dynamic characteristics of the beams are greatly influenced by the number of moving point loads. The effects of the distance between the loads, material non-homogeneity, section profiles as well as aspect ratio on the dynamic responses of the beams are also investigated in detail and highlighted.

Keywords: axially FGM; non-uniform beam; FEM; multiple moving point load; dynamic response

1. Introduction

This paper presents a finite element procedure for dynamic analysis of structures subjected to multiple moving point loads which is a great important topic in many engineering fields. The moving load problems have been receiving enormous interests from engineers and researchers for several decades. A wide range of publications on the problems are briefly discussed herein. The early and excellent reference is the book by Fryba (1972), in which a number of closed-form solutions for the moving load problems have been derived by using Fourier and Laplace transforms. The numerical method, especially the finite element method has also widely been employed in analysis of the moving load problems for homogeneous beams (Lin and Trethewey 1990, Thambiranam and Zhuge 1996). Recently, Nguyen and his co-workers derived the finite

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element formulations for studying the dynamic response of homogeneous Bernoulli and Timoshenko beams resting on elastic foundation subjected to a moving harmonic load (Nguyen 2008, Nguyen and Le 2011).

Functionally graded materials (FGMs) are of great importance to many researchers because of its wide range of applications in structural mechanics. FGMs initiated by Japanese scientists in 1984 (Koizumi 1997) can be formed by varying percentage of constituents in any desired direction in order to create new materials of specific physical and mechanical properties. Many investigations on FGM structures subjected to different types of loadings are summarized by Birman and Byrd (2007), only contributions that are most relevant to the present work are discussed below.

Huang and Li (2010) studied the free vibration of non-uniform cross-section beams made of axially FGM. Shahba *et al.* (2011a) employed the exact shape functions from a uniform homogenous Timoshenko beam segment to formulate a finite element formulation for computing natural frequencies and buckling loads of tapered axially FGM Timoshenko beams. Also using the finite element method, Alshorbagy *et al.* (2011), Shahba *et al.* (2011b) studied the free vibration of transversely and axially FGM Bernoulli beams, respectively. Şimşek and Kocatürk (2009), Şimşek (2010), Şimşek *et al.* (2012) studied the dynamic behavior of transversely and axially FGM beams subjected to moving loads by using polynomial series as trial functions for the displacements and rotation in solving the governing equations of motion. Li *et al.* (2013) proposed a method for derivation of exact finite element formulations to model the axially FGM and/or transversally FGM beams with non-uniform Bernoulli-Euler beams. Nguyen *et al*. (2013) studied the vibration of non-uniform transversely FGM beams under a harmonic moving load by using the finite element method. Nguyen (2013), Nguyen and Gan (2014a), Nguyen *et al.* (2014b) derived the finite element formulation for studying the large displacement behavior of FGM beams and frames.

In this paper, the dynamic response of non-uniform FGM Timoshenko beams subjected to multiple moving point loads is investigated by using the finite element method. The beam material properties are assumed to vary in the longitudinal direction by a defined power law equation. Exact polynomials derived from solutions of the governing different equation of a uniform homogenous Timoshenko beam element are employed to interpolate displacements and rotation of the beams. The implicit Newmark method is used to compute the dynamic responses of the beams. The influence of the non-uniform cross sections, the distance between the multiple moving point loads as well as the aspect ratio on the dynamic behavior of the beams is examined and highlighted.

2. Non-uniform FGM beams

A simply supported FGM beam subjected to *nload* point loads with the same amplitude Q_0, moving from left to right with a constant speed, as shown in Fig. 1 is considered. The loads are placed at a regular interval *d*. The beam with total length *L* and a constant cross section height *h*, is assumed to be formed from two different materials. Solid cross section area, $A(x)$, and moment of inertia $I(x)$ are assumed to vary longitudinally in two following manners:

• Type A:

$$A(x) = A_m\left(1-\alpha\left|\frac{x}{L}-\frac{1}{2}\right|\right),\ I(x) = I_m\left(1-\alpha\left|\frac{x}{L}-\frac{1}{2}\right|\right)$$

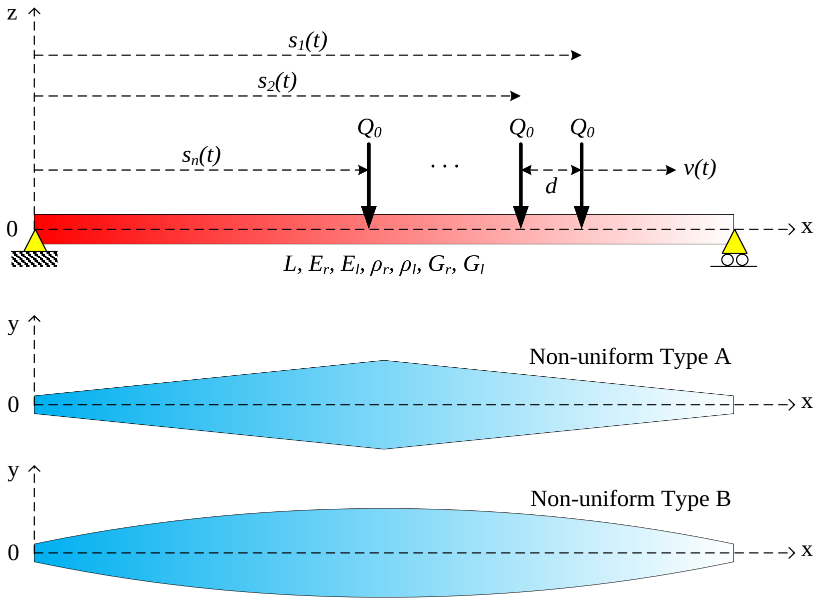


Fig. 1 Non-uniform simply-supported beam made of FGM

• Type B:

$$A(x) = A_m\left[1-\alpha\left(\frac{x}{L}-\frac{1}{2}\right)^2\right],\; I(x) = I_m\left[1-\alpha\left(\frac{x}{L}-\frac{1}{2}\right)^2\right]$$

where A_m, I_m denote the area and moment of inertia of the mid-span section, respectively; $0\leq\alpha<2$ is the non-uniform section parameter, defined how the cross section varies. The effective property P (Young's modulus, shear modulus and mass density) of the beam material is assume to vary in longitudinal direction according to a power law equation as

$$P(x) = (P_l - P_r)\left(1-\frac{x}{L}\right)^n + P_r \tag{1}$$

where n is a non-negative power law index, which defined the distribution of the constituents along the longitudinal direction of the beam. The lower subscripts 'l' and 'r' stand for 'left' and 'right', respectively. As seen from Eq. (1), the left and right end sections of the beam contain pure one material.

3. Finite element formulation

Adopting the first order shear deformation beam theory, the axial and transverse displacements, $u_1(x,z,t)$ and $u_3(x,z,t)$, respectively at any point of the beam are given by

$$\begin{aligned} u_1(x,z,t) &= u(x,t) - z\theta(x,t) \\ u_3(x,z,t) &= w(x,t) \end{aligned} \tag{2}$$

where $u(x,t)$ and $w(x,t)$ are the axial and transverse displacements of the point on the neutral axis x; $\theta(x,t)$ is the rotation of cross section at a point with abscissa x; z is the distance from any arbitrary point to the neutral axis. Based on the assumptions of Hooke's law, the axial strain ε_x, shear strain γ_{xz}, and their corresponding axial and shear stresses, σ_x and τ_{xz}, respectively, resulted from Eq. (2) are as follows

$$\begin{aligned} \varepsilon_x &= u_{,x}(x,t) - z\theta_{,x}(x,t) \;, \quad \gamma_{xz} = w_{,x}(x,t) - \theta(x,t) \\ \sigma_x &= E(x)\varepsilon_x \;, \quad \tau_{xz} = \psi G(x)\gamma_{xz} \end{aligned} \tag{3}$$

where $(...)_{,x}$ denotes the derivative with respect to x, and ψ is the shear correction factor with its value depends on the geometry of the beam cross section.

The differential equations of motion of the beam under the moving loads can be derived by applying Hamilton's principle. By using Eqs. (2)-(3), the strain energy and kinetic energy for the beam can be written in the following forms

$$\begin{aligned} U &= \frac{1}{2}\int_0^l \left[E(x)A(x)\left(u_{,x}\right)^2 + E(x)I(x)\left(\theta_{,x}\right)^2 + \psi G(x)A(x)\left(w_{,x} - \theta\right)^2 \right] dx \\ T &= \frac{1}{2}\int_0^l \left\{ \rho(x)A(x)\left[\left(u_{,t}\right)^2 + \left(w_{,t}\right)^2\right] + \rho(x)I(x)\left(\theta_{,t}\right)^2 \right\} dx \end{aligned} \;, \tag{4}$$

where $E(x)$, $G(x)$, $\rho(x)$ are the Young's modulus, shear modulus and mass density of the material on the section with abscissa x; $(...)_{,t}$ denotes the derivative with respect to t. The potential energy from the moving loads is simply given by

$$V = -\sum_{i-1}^{nload} Q_0 \; w(x,t)\; \delta(x - s_i(t)) \,, \tag{5}$$

where $\delta(.)$ is the Dirac delta function, and $s_i(t)=vt_i$ is the distance from i-th load to the left end of the beam at time t; v is the speed of the moving loads.

Applying Hamilton's principle to Eqs. (4)-(5), one can obtain the following differential equations of motion for the beam in case of ignoring the damping effect in the forms

$$\begin{aligned} &\rho(x)A(x)u_{,tt} - \left[E(x)A(x)u_{,x}\right]_{,x} = 0, \\ &\rho(x)A(x)w_{,tt} - \left[\psi G(x)A(x)\left(w_{,x} - \theta\right)\right]_{,x} = \sum_{i=1}^{nload} Q_i \,, \\ &\rho(x)I(x)\theta_{,tt} - \left[E(x)I(x)\theta_{,x}\right]_{,x} - G(x)A(x)\left(w_{,x} - \theta\right) = 0. \end{aligned} \tag{6}$$

The equations of motion in Eq. (6) with the variable coefficients are hardly solved by analytical methods. Here, the finite element method is used in solving the equations instead of using analytical methods. To this ends, we assume that the beam is being divided into *nel* two-node beam elements with length of l. There are axial and vertical displacements and a rotation at each node, and thus the vector of nodal displacements, **d**, for a generic element contains six components as

$$\mathbf{d} = \left\{ u_i \quad w_i \quad \theta_i \quad u_j \quad w_j \quad \theta_j \right\}^T \,, \tag{7}$$

here and after, a superscript 'T' denotes the transpose of a vector or a matrix. The axial, transverse displacements and cross section rotation inside the element, u, w and θ, respectively are assumed to be interpolated from the nodal displacements according to

$$u = \mathbf{N}_u^T \mathbf{d}, \quad w = \mathbf{N}_w^T \mathbf{d}, \quad \theta = \mathbf{N}_\theta^T \mathbf{d}, \tag{8}$$

where $\mathbf{N}_u=\{N_{u1}\ 0\ 0\ N_{u2}\ 0\ 0\}^T$, $\mathbf{N}_w=\{0\ N_{w1}\ N_{w2}\ 0\ N_{w3}\ N_{w4}\}^T$ and $\mathbf{N}_\theta=\{0\ N_{\theta 1}\ N_{\theta 2}\ 0\ N_{\theta 3}\ N_{\theta 4}\}^T$ are the matrices of interpolating functions for u, w and θ, respectively.

As demonstrated by Nguyen (2013), linear functions and cubic polynomials obtained by solving static governing differential equations of a uniform homogeneous Timoshenko beam element can be employed as interpolation functions for a non-uniform axially FGM beam element. Following this idea, the present work employs the following polynomials as interpolations for the displacement u, w, and the rotation θ

$$N_{u1} = \frac{l-x}{l}, \ N_{u2} = \frac{x}{l} \tag{9}$$

$$\begin{aligned}
N_{w1} &= \frac{1}{1+\lambda}\left[2\left(\frac{x}{l}\right)^3 - 3\left(\frac{x}{l}\right)^2 - \lambda\left(\frac{x}{l}\right) + (1+\lambda)\right] \\
N_{w2} &= \frac{l}{1+\lambda}\left[\left(\frac{x}{l}\right)^3 - \left(2+\frac{\lambda}{2}\right)\left(\frac{x}{l}\right)^2 + \left(1+\frac{\lambda}{2}\right)\left(\frac{x}{l}\right)\right] \\
N_{w3} &= -\frac{1}{1+\lambda}\left[2\left(\frac{x}{l}\right)^3 - 3\left(\frac{x}{l}\right)^2 - \lambda\left(\frac{x}{l}\right)\right] \\
N_{w4} &= \frac{l}{1+\lambda}\left[\left(\frac{x}{l}\right)^3 - \left(1-\frac{\lambda}{2}\right)\left(\frac{x}{l}\right)^2 - \frac{\lambda}{2}\left(\frac{x}{l}\right)\right]
\end{aligned} \tag{10}$$

and

$$\begin{aligned}
N_{\theta 1} &= \frac{6}{(1+\lambda)\, l}\left[\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)\right] \\
N_{\theta 2} &= \frac{1}{1+\lambda}\left[3\left(\frac{x}{l}\right)^2 + (4+\lambda)\left(\frac{x}{l}\right) + (1+\lambda)\right] \\
N_{\theta 3} &= -\frac{6}{(1+\lambda)\, l}\left[\left(\frac{x}{l}\right)^2 - \left(\frac{x}{l}\right)\right] \\
N_{\theta 4} &= \frac{1}{1+\lambda}\left[3\left(\frac{x}{l}\right)^2 - (2-\lambda)\left(\frac{x}{l}\right)\right]
\end{aligned} \tag{11}$$

In Eqs. (10)-(11), λ is the shear deformation parameter, defined as

$$\lambda = \frac{12}{l^2}\frac{E_0\ I_0}{\psi\ G_0\ A_0} \tag{12}$$

with A_0, I_0, E_0, G_0 are the cross section area, moment of inertia, Young's and shear moduli of the uniform homogeneous beam. It should be noted that the interpolation functions stated in Eqs. (10)-(11) were firstly derived by Kosmatka (1995) for a homogeneous Timoshenko beam element. The polynomial functions in Eqs. (10) and (11) have been previously employed by several authors in the derivation of beam finite elements for free and forced vibration analysis of non-uniform axially or transversely FGM beams (Shahba *et al*. 2011a, Nguyen *et al*. 2013). In the computations reported in the next section, the cross section area, moment of inertia of the section at the right node of the element, and Young's and shear moduli of the material at the right end of the beam are designated as A_0, I_0, E_0, G_0, respectively.

Substituting Eqs. (9)-(11) into Eqs. (4)-(5), we can rewrite the strain and kinetic energies for the beam in terms of nodal displacements as

$$U = \frac{1}{2}\sum_{i=1}^{nel} \mathbf{d}^T\mathbf{k}\,\mathbf{d}, \qquad T = \frac{1}{2}\sum_{i=1}^{nel} \dot{\mathbf{d}}^T\mathbf{m}\,\dot{\mathbf{d}}, \tag{13}$$

and the potential of the moving loads has a simple form

$$V = -\sum_{i-1}^{nload} Q_0\,\mathbf{N}_w^T\mathbf{d}\,\delta(x - s_i(t)). \tag{14}$$

In Eq. (13), **k** and **m** denote the stiffness and mass matrices for the element, respectively. The detail expressions for the element stiffness and mass matrices are as follows.

$$\mathbf{k} = (\mathbf{k}_a + \mathbf{k}_b + \mathbf{k}_s), \qquad \mathbf{m} = (\mathbf{m}_u + \mathbf{m}_w + \mathbf{m}_\theta), \tag{15}$$

in which

$$\mathbf{k}_a = \int_0^l \mathbf{N}_{u,x}^T E(x)A(x)\mathbf{N}_{u,x}dx\,,\ \mathbf{k}_b = \int_0^l \mathbf{N}_{\theta,x}^T E(x)I(x)\mathbf{N}_{\theta,x}dx\,, \qquad \mathbf{k}_s = \int_0^l (\mathbf{N}_{w,x}^T - \mathbf{N}_\theta^T)\psi G(x)A(x)(\mathbf{N}_{w,x} - \mathbf{N}_\theta)dx \tag{16}$$

are the stiffness matrices stemming from stretching, bending and shear deformation of the beam element, respectively and

$$\mathbf{m}_u = \int_0^l \mathbf{N}_u^T\rho(x)A(x)\mathbf{N}_u dx\,,\ \mathbf{m}_w = \int_0^l \mathbf{N}_w^T\rho(x)A(x)\mathbf{N}_{w,x}dx\,, \qquad \mathbf{m}_\theta = \int_0^l \mathbf{N}_\theta^T\rho(x)I(x)\mathbf{N}_{\theta,x}dx \tag{17}$$

are the mass matrices corresponding to the axial and transverse translations and cross section rotation, respectively. To improve the accuracy of the finite element formulation, the exact variation of the section profiles is employed in evaluation of integrals in Eqs. (16)-(17).

Using the derived element matrices, the discrete equations of motion for the beam can be written in the form

$$\mathbf{M}\,\ddot{\mathbf{D}} + \mathbf{K}\,\mathbf{D} = \mathbf{F}_{ex} \tag{18}$$

where **M**, **K** are structural mass and stiffness matrices, obtained by assembling the formulated element mass and stiffness matrices in a standard way of the finite element method, respectively; $\mathbf{F}_{ex}$ is the external nodal load vector with the following simply form

$$\mathbf{F}_{ex} = Q_0 \left\{ 0\;0\;0 ... 0\;0 ...\; \mathbf{N}_w\big|_{x_n} ... 0\;0 ...\; \mathbf{N}_w\big|_{x_i} ... 0\;0 ...\; \mathbf{N}_w\big|_{x_1} ...\; 0\;0\;0 \right\}^T \tag{19}$$

The nodal load vector $\mathbf{F}_{ex}$ defined by Eq. (19) contains all zero coefficients except for the elements currently under loading. In addition, the interpolation functions $\mathbf{N}_w|_{xi}$ are evaluated at the current position of the *i*-th load. The system of Eq. (18) can be solved by the direct integration method. The implicit average constant acceleration Newmark method, which ensures the unconditional convergence (Géradin and Rixen, 1997), is adopted in the present work. In the free vibration analysis, the right hand side of Eq. (18) is set to zeros, and a harmonic response, $\mathbf{D} = \bar{\mathbf{D}} \sin \omega t$ is assumed, so that Eq. (18) deduces to an eigenvalue problem as

$$\left(\mathbf{K} - \omega^2\,\mathbf{M}\right)\bar{\mathbf{D}} = \mathbf{0} \tag{20}$$

where ω is the circular frequency, and $\bar{\mathbf{D}}$ is the vibration amplitude. Eq. (20) can be solved by using a standard method for the eigenvalue problem (Géradin and Rixen 1997).

4. Numerical results

A simply-supported beam made of axially FGM subjected to multiple moving loads is investigated. Otherwise stated, the geometric data for the beam area: b_m=0.5 m, h=1.0 m, L=5 m and 20 m, where b_m, h, L are the width of mid-span cross section, height and total length of the beam, respectively. The beam is assumed to be composed of steel and alumina. Young's modulus and mass density of steel respectively are 210 GPa and 7800 kg/m^3, and that of alumina are 390 GPa and 3960 kg/m^3, respectively. The beam material is graded with pure alumina at the left end to pure steel at the right end of the beam. The amplitude of each load is Q_0=100 kN. A Poisson's ratio υ=0.3 and a shear correction factor ψ=5/6 is used in the all the computations in this section. The total time ΔT necessary for a constant moving speed load to across the beam is L/v, where v is the moving speed of the load.

In the computation reported below, a uniform time increment of $dt = \Delta T / 500$ is used for the Newmark method. The following dimensionless parameters representing maximum mid-span deflection and moving load speed are introduced as

$$f_D = \max\left(\frac{w\left(L/2, t\right)}{w_0}\right), \qquad f_v = \frac{\pi\, v}{\omega_1^0}$$

where $w_0 = Q_0 L^3 / 48 E_r I_m$ is the static deflection of a uniform steel beam under a static load Q_0 acting at the mid-span, and $\omega_1^0 = \left(\pi^2 / L^2\right)\sqrt{E_r I_m / \rho_r A_m}$ is the fundamental frequency of the simply supported uniform steel beam. In the below f_D is called the dynamic deflection factor.

4.1 Verification of formulation

In order to verify the accuracy of the derived formulation and the numerical procedure, the fundamental frequency of a uniform FGM beam composed of steel and alumina are computed and compared with the numerical result of Şimşek *et al.* (2012). To this end, for consistence with the work by Şimşek *et al.* (2012), a FGM beam with b=0.4 m, height h=0.9 m and length L=20 m is employed in the computation. Tables 1-2 list values of the first two non-dimensional fundamental frequencies of the beam with various values of the modulus ratio, $E_{ratio}=E_l/E_r$, and the power law index n for an assumed constant value of mass desity, $\rho_{ratio}=\rho_l/\rho_r=1$. Very good agreement between the frequencies obtained in the present work with that of Şimşek *et al.* are given in the Tables. The non-dimensional fundamental frequency in the Tables is defined as,

$$\mu^2 = \frac{\omega^2 \rho_r A_m L^2}{E_r I_m}$$

where ω is the natural frequency of the beam.

To verify the element formulation in further, the maximum dynamic deflection factor and its corresponding speed of an axially FGM beam are evaluated and compared to the published data. The computed result is listed in Table 3 for various values of the index n, where the numerical result from Şimşek *et al.* (2012) is also given. The numerical result shown in the Table was obtained by varying the moving speed with an increment 1m/s. Very good agreements between results are achieved. It should be noted that numerical results listed in Tables 1-3 have been obtained by using ten elements. More than ten elements were employed in the analysis, but no improvement in the numerical results has been observed.

Table 1 First dimensionless frequency parameter of uniform FGM beam

Modulus ratio E_{ratio}	Power law index, n					
	1		2		5	
	Present	Şimşek *et al.* (2012)	Present	Şimşek *et al.* (2012)	Present	Şimşek *et al.* (2012)
0.25	2.7482	2.7532	2.9220	2.9278	3.0772	3.0834
0.5	2.9056	2.9104	3.0069	3.0122	3.0997	3.1052
1.0	3.1350	3.1399	3.1350	3.1399	3.1350	3.1399
2.0	3.4554	3.4611	3.3193	3.3243	3.1877	3.1922
4.0	3.8866	3.8937	3.5737	3.5794	3.2625	3.2667

Table 2 Second dimensionless frequency parameter of uniform FGM beam

Modulus ratio E_{ratio}	Power law index n					
	1		2		5	
	Present	Şimşek *et al.* (2012)	Present	Şimşek *et al.* (2012)	Present	Şimşek *et al.* (2012)
0.25	5.4372	5.4729	5.7283	5.7675	6.0213	6.0639
0.5	5.7685	5.8047	5.9359	5.9739	6.1060	6.1459
1.0	6.2317	6.2703	6.2317	6.2703	6.2317	6.2703
2.0	6.8599	6.9030	6.6372	6.6782	6.4094	6.4482
4.0	7.6893	7.7399	7.1755	7.2208	6.6495	6.6900

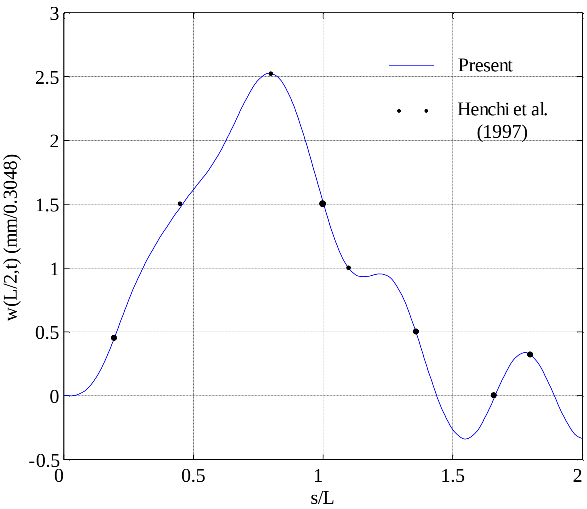


Fig. 2 Time history for mid-span deflection of uniform homogeneous beam under three moving loads (d=L/4)

Table 3 Maximum normalized dynamic deflections factors of the beam and corresponding speeds for α=0, L/h=20

Power law index n	v [m/s]		max(f_D)	
	Present	Şimşek *et al.* (2012)	Present	Şimşek *et al.* (2012)
0.3	219	220	1.0195	1.01947
1	178	179	1.2064	1.20435
3	144	144	1.5146	1.51669
Pure Steel	132	132	1.7386	1.73247

In order to verify the formulation and numerical procedure in modeling multiple moving loads, the time history for mid-span deflection of a uniform homogenous beam subjected to three point loads Q_0=5324.256 N, moving with a constant speed v=22.5 m/s is computed and the result is shown in Fig. 2, where the numerical result obtained by using a dynamic stiffness method of Henchi *et al.* (1997) is depicted by small circles. The beam material properties and geometric data in this computation adopted from the work of Henchi *et al.* (1997) are as follows: L=24.384 m, m=9.576 10^3 kg/m, A=0.576 m, I=2.95 10^{-3} m^4, E=19 10^{11} N/m^2, where L, m, A, I, E are the total length, mass per unit length, section area, moment of inertia and Young's modulus, respectively. A good agreement between the numerical result of the present work with that of Henchi *et al.* (1997) is observed from the figure.

4.2 Fundamental frequency

The effect of the section parameter α on the non-dimensional fundamental frequency μ_1 of the type A FGM beam having aspect ratios, L/h, is shown in Fig. 3 for an index n=3. It can be observed that the fundamental frequency of the beam is considerably affected by the section

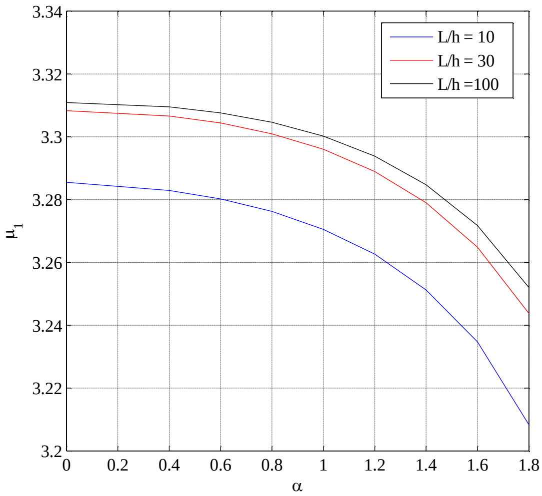
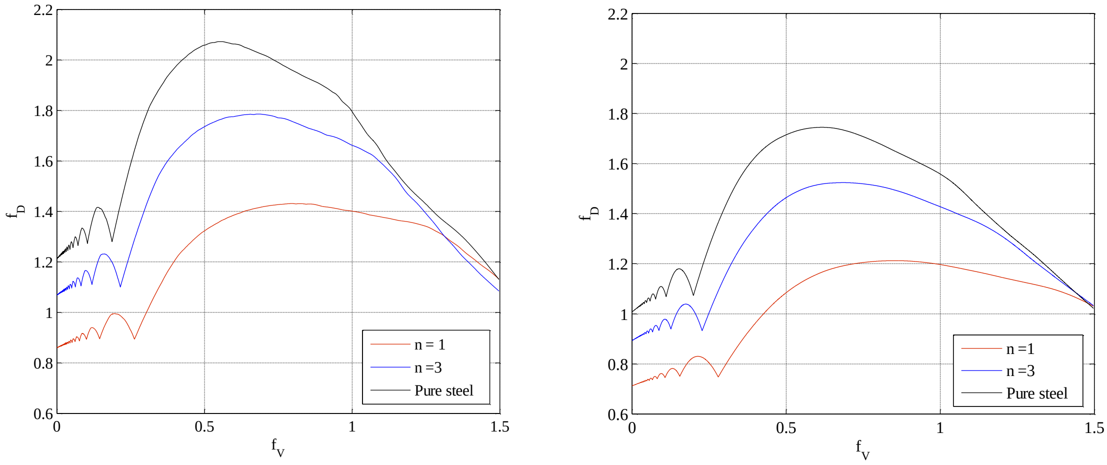


Fig. 3 Effect of section parameter on the dimensionless fundamental frequency of type A FGM beam with different values of the aspect ratio (n=3)

Table 4 Convergence of present element in evaluation fundamental frequency parameter μ_1 of non-uniform FGM beam (n=1, L/h=20)

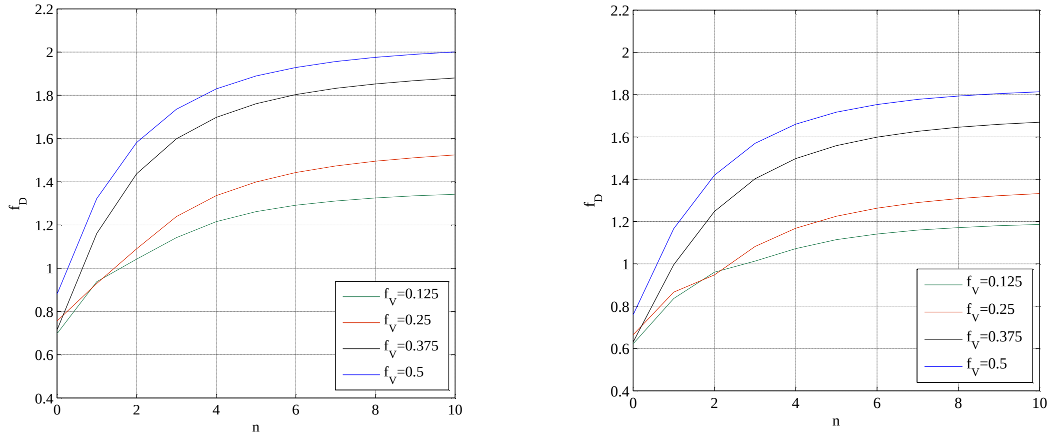
Number of element, *nel*	α 0.5	1.0	1.5	1.8
4	3.5007	3.4923	3.4706	3.4427
8	3.4993	3.4902	3.4668	3.4369
12	3.4991	3.4898	3.4657	3.4346
14	3.4991	3.4897	3.4654	3.4338
16	3.4990	3.4896	3.4652	3.4333
18	3.4990	3.4895	3.4650	3.4328
20	3.4990	3.4894	3.4648	3.4322
22	3.4990	3.4894	3.4647	3.4319
24	3.4990	3.4894	3.4646	3.4315
26	3.4990	3.4894	3.4646	3.4313
28	3.4990	3.4894	3.4646	3.4313

parameter and the aspect ratio. The frequency μ_1 steadily reduces by raising the section parameter α, regardless of the aspect ratio. In addition, the frequency is smaller for a beam associated with a lower aspect ratio. In other words, the shear deformation which has been taken into account in the present work reduces the fundamental frequency of the FGM beam. Thus, the numerical result obtained in this sub-section shows the good ability of the proposed formulation in modeling the shear deformation of the FGM beam. It should be noted that the frequency parameter shown in Fig. 3 has been obtained by using 28 elements, which is much more than the number of elements previously used for the uniform beam. As seen from Table 4, the convergence of the fundamental frequency of the non-uniform FGM beams depends on the section parameter, and the beam with a higher section parameter requires more number of the elements in evaluating the fundamental frequencies.



(a) Type A, one load, *L*/*h*=5 (b) Type A, one load, *L*/*h*=20

Fig. 4 Relation of deflection factor and moving speed parameter of type A beam (α=0.5).

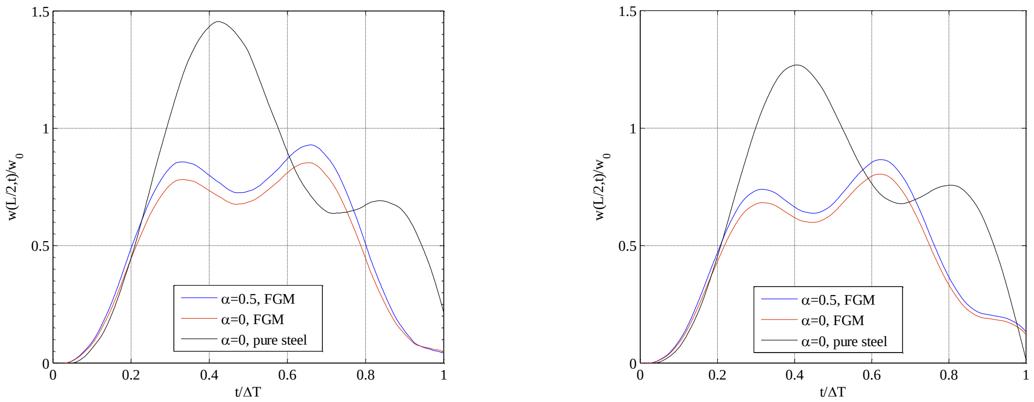


(a) Type A, one load, *L*/*h*=5 (b) Type A, one load, *L*/*h*=20

Fig. 5 Relation of deflection factor and power law index of type A beam (α=0.5)

4.3 Effect of material non-homogeneity

The material non-homogeneity distribution along the longitudinal direction is defined through the power law index n in Eq. (1), and the effect of this index on the dynamic response of the beam is examined in this sub-section. In Figs. 4(a)-(b), the relations between the deflection factor and the speed parameter of Type A beam subjected to a single moving load are depicted for a non-uniform section parameter α=0.5, and for two aspect ratios, *L*/*h*=5 and *L*/*h*=20. As depicted in the figures, the dynamic deflection f_D is higher for a beam associated with a higher index n, regardless of the moving speed and the aspect ratio. This due to the fact that, in refer to the Eq. (1), the beam with a higher index n contains more steel and thus it is softer. The dependency of the deflection factor f_D upon the speed parameter f_v observed from Fig. 4(a)-(b) is similar to that of the homogeneous beams (Olsson 1991), where f_D increases when increasing the speed parameter f_v,



(a) Type A, one load, *L*/*h*=5

(b) Type A, one load, *L*/*h*=20

Fig. 6 Time histories of normalized mid-span deflection of type A beam under a single moving load (n=5, f_v=0.25)

and it then reduces after reaching a peak value, regardless of the index n and the aspect ratio. The effect of the material non-homogeneity and the moving speed on the dynamic deflection factor can also be observed clearly from Figs. 5(a)-(b), where the relation between f_D and the index n of Type A beam are shown for various values of the speed parameter f_v and for a section parameter α=0.5. The effect of the aspect ratio on the dynamic response of the beam can be seen from Figs. 4(a)-(b) and Figs. 5(a)-(b), where f_D is higher for a beam having a smaller aspect ratio. The dynamic deflections in this sub-section (and in the below) have been computed by using 28 elements. More than 28 elements have been used, but no improvement in the results was obtained.

4.4 Effect of section profile

The time histories for mid-span deflection of the Type A beam under a single moving load are shown in Figs. 6(a)-(b) for various values of the section parameter α and for n=5, f_v=0.25. The dynamic response of the beam, as clearly observed from the figures, is greatly affected by the section parameter, where the maximum dynamic deflection is higher for a beam having larger parameter α, regardless of the aspect ratio. Except for the amplitude of the dynamic deflection, the aspect ratio hardly changes the dynamic behavior of the beam.

In Figs. 7(a)-(b), the relation between the maximum deflection parameter, max(f_D), and the section parameter α of the beam having different aspect ratios is depicted for the two types of the section profile and for n=5, f_v =0.25. As shown in both figures, the maximum deflection factor of the beam with Type A section is more sensitive to the non-uniform section parameter α compares to that of the Type B beam, regardless of the number of the moving loads. For a given value of the parameter α and the number of moving loads, the maximum deflection factor of Type A beam is higher than that of Type B beam and the difference becomes larger for a higher value of α. The aspect ratio affects the amplitude of the maximum deflection factor, but it hardly changes the relation between this factor and the non-uniform parameter. The maximum deflection factor of the beam associated with a lower aspect ratio is much more sensitive to the change of the section parameter compares to the beam having a higher aspect ratio.

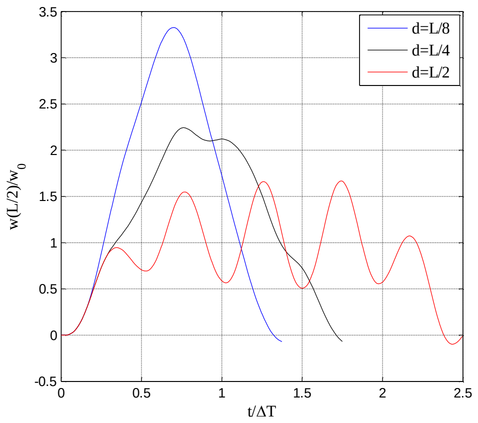


Fig. 8 Time histories for mid-span deflection of type A beam under 4 loads (n=3, f_v=0.25, /h=20, α=0.5)

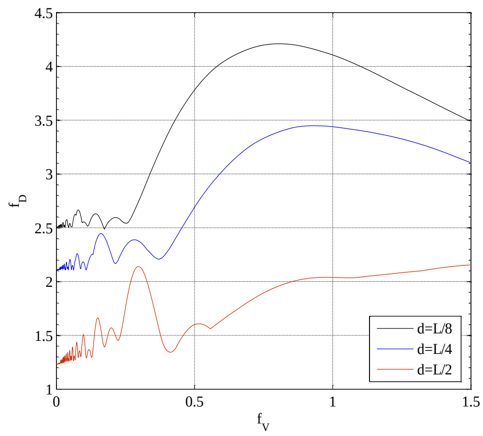
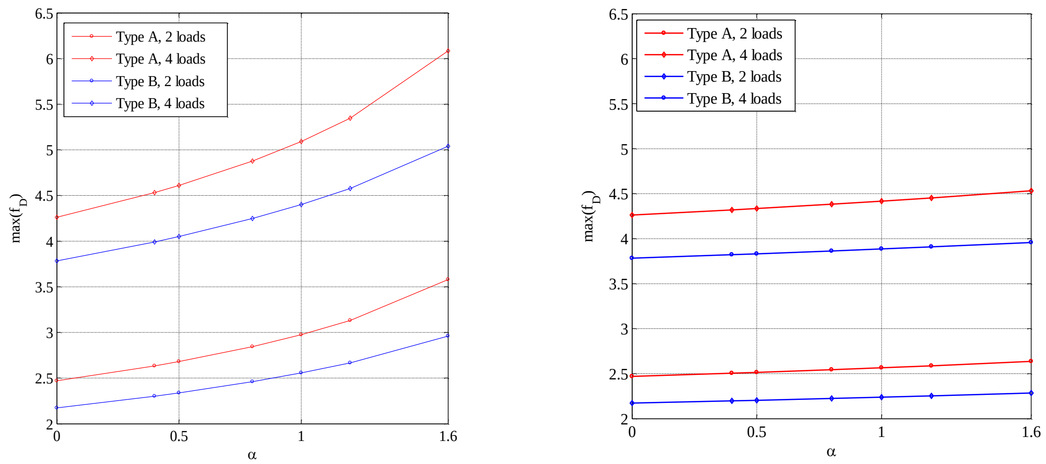


Fig. 9 Deflection factor-speed parameter relation of type A beam under 3 loads, n=3, L/h=20, α=0.5



(a) n=5, f_v=0.25, L/h=5, d=L/10

(b) n=5, f_v=0.25, L/h=20, d=L/10

Fig. 7 Effect of section profile on relation between maximum deflection factor and non-uniform parameter

4.5 Effect of distance between the loads

The time histories for mid-span deflection of the type A beam under four moving loads are depicted in Fig. 8 for various values of the distance between the loads and for n=3, f_v=0.25, L/h=20. In Fig. 9, the relation between the deflection parameter f_D and the moving speed f_v of the type A beam is shown for various values of the distance between the loads, and for n=3, α=0.5, L/h=20. The effect of the distance between the moving loads on the dynamic behavior of the beam is clearly depicted in the figures. The dynamic deflection factor is much larger when the distance between the loads is smaller, regardless of the moving speed.

5. Conclusions

A finite element procedure for analyzing dynamic response of non-uniform axially FGM Timoshenko beams subjected to multiple moving point loads has been presented. A first-order shear deformation beam element taking the effect of the material non-homogeneity, non-uniform cross section, and variation of moving point loads has been formulated by using the shape functions derived from the static governing equations of motion of a uniform homogeneous beam element to interpolate the displacements and rotation degree of freedoms. The exact variations of the section profile were employed in the evaluation of the element stiffness and mass matrices. The dynamic response of the beams has been computed by using the implicit Newmark method. The numerical results have shown that the formulated element is accurate in evaluating the dynamic response of the beams. It has also been demonstrated that the proposed formulated element is capable to model the shear deformation of the axially FGM beams. Parametric studies were carried out to investigate the effect of the material non-homogeneity, loading parameter and the section profile on the dynamic behaviors of the axially FGM beams.

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