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The refined theory of 2D quasicrystal deep beams based on elasticity of quasicrystals

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Abstract. Based on linear elastic theory of quasicrystals, various equations and solutions for quasicrystal beams are deduced systematically and directly from plane problem of two-dimensional quasicrystals. Without employing ad hoc stress or deformation assumptions, the refined theory of beams is explicitly established from the general solution of quasicrystals and the Lur'e symbolic method. In the case of homogeneous boundary conditions, the exact equations and exact solutions for beams are derived, which consist of the fourth-order part and transcendental part. In the case of non-homogeneous boundary conditions, the exact governing differential equations and solutions under normal loadings only and shear loadings only are derived directly from the refined beam theory, respectively. In two illustrative examples of quasicrystal beams, it is shown that the exact or accurate analytical solutions can be obtained in use of the refined theory.

Keywords: deep beams; two-dimensional quasicrystals; the refined theory; general solution

1. Introduction

Quasicrystals (QCs) have become the focus of theoretical and experimental studies in the physics of condensed matter since the first discovery of the icosahedral QC in Al-Mn alloys (Shechtman *et al*. 1984). The discovery of QCs reveals a new symmetry in solids: quasiperiodic symmetry. This changes the traditional concept of classifying solids into two classes: crystals and non-crystals. The electronic structure and the optic, magnetic, thermal and mechanical properties of the material have been extensively investigated in experimental and theoretical analyses (Socolar *et al*. 1986, Ronchetti 1987, Ovidko 1992, Wollgarten *et al*. 1993), which show their complex structure and unusual properties. Elasticity is one of the interesting properties of QCs. Within the framework of the Landau-Lifshitz phenomenological theory, the elastic energies of QCs were formulated (Bak 1985, Levine and Steinhardt 1986). In particular, the field of linear elastic theory of QCs has been formulated for many years (Ding *et al*. 1993, Hu *et al*. 1996). It provides us with a fundamental theory based on the notion of a continuum model to describe the elastic behavior of QCs. Great progress has been made in the fields of the mechanic involving the elasticity and defects, see review article for detail (Hu *et al*. 2000, Fan and Mai 2004).

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Among approximately 200 individual QCs observed to date, two-dimensional (2D) QCs with fine thermal stability play an important role in this kind of matter (Fan 2011). A 2D QC refers not to a real plane but to a three-dimensional (3D) solid with 2D quasiperiodic and one-dimensional periodic structure (Stadnik 1999). There are ten systems, i.e., triclinic, monoclinic, orthorhombic, tetragonal, trigonal, hexagonal, pentagonal, decagonal, octagonal systems and 57 point groups (Hu *et al*. 1996). Among them, six systems and 31 point groups are of crystal rotational symmetry, four systems and 26 point groups are of non-crystal rotational symmetry. According to Janssen's treatment (Janssen1992), Hu *et al.* (1996) have discussed the point groups of all 2D QCs in detail.

Slender and thin bodies are one of the most well known structures of vital significance in the structural design and therefore received extensive study from scientific workers. Various theories are proposed by many authors with the help of some ad hoc assumptions. Additionally some observations are made on the formulation of the theories of beams due to Timoshenko (1921), Levinson (1981), Wang *et al*. (2000) and the original work of Bernoulli and Euler. Efforts have been made to the exact solutions of beams without ad hoc assumption. Cheng (1979) presented a method for the solution of 3D elasticity equations. With this method refined theories of several plates were explicitly established (Barrett and Ellis 1988, Wang 1990, Gao and Ricoeur 2011, Zhao *et al*. 2013). Moreover, Gao and coauthors indicated that applications of Cheng's method are quite successful in various beams (Gao and Wang 2004, 2005, Gao *et al*. 2007).

In order to study the deformation and mechanical/physical behavior of the new solid phase, the general framework of QC beam theories should be set up explicitly. In the present paper, from linear elastic theory of 2D QCs, the refined theory of deep beams is derived by using the general solution of 2D hexagonal QCs (Gao and Zhao 2009) and the Lur'e symbolic method (Lur'e 1964) without ad hoc assumptions. The exact governing equations for the beams without transverse surface loadings and under transverse loadings are derived directly from the beam theory. Finally, two examples are examined to illustrate the application of the refined theory of QC beams.

2. Basic equations and the general solution

2D QCs can be described as a 3D cut of a five-dimensional (5D) hypercubic crystal (Stadnik, 1999). The 5D hyperspace may be divided into two subspaces, the parallel or physical space $E^{\parallel}$ and the perpendicular or mathematical space $E^{\perp}$. In a fixed rectangular coordinate system (x_1, x_2, x_3), let 2D QCs be quasi-periodic in x_1–x_2 plane and periodic in x_3-direction. We consider a straight QC beam of narrow rectangular cross-section as a plane stress problem and assume that the beam length in x_1-direction is denoted by l, the beam width in x_2-direction is set to be unit, and the beam height in x_3-direction is h.

For the plane stress problem, the displacement field in 2D QCs consists of two components, u_α (α=1, 3) and w_1. The phonon-type displacement field $u_\alpha \in E^{\parallel}$ is analogous to the field in conventional crystals. The elementary excitation associated with the phonon mode is propagating. The introduction of the phason gives a macro-description of the quasiperiodicity of the new solid phase. The phason-type displacement field $w_1 \in E^{\perp}$ is peculiar to QCs. The phason phase characterizes the degree of freedom associated with relative translation of incommensurate sub-lattices, or equivalently, with special atomic rearrangements. Within the framework of the elastic theory of QCs (Lubensky *et al*. 1985), both phonon and phason fields are considered as continuous averaged field variables. Corresponding to the phonon and phason parameters, there are two stress fields $\sigma_{\alpha\beta}$ (β=1, 3) and $H_{1\beta}$ associated with two strain fields $\varepsilon_{\alpha\beta}$ and $w_{1\beta}$, respectively, the latter being a new parameter in QC elasticity which is also asymmetric for most classes of QCs except the cubic QCs.

According to 2D QCs elastic theory (Hu *et al*. 1996), the strain-displacement relations for the plane stress state of 2D QCs are given by

$$\varepsilon_{\alpha\beta} = \frac{1}{2}(\partial_\beta u_\alpha + \partial_\alpha u_\beta), \quad w_{1\beta} = \partial_\beta w_1. \tag{1}$$

In the absence of body forces, the static equilibrium equations are

$$\partial_\beta \sigma_{\alpha\beta} = 0, \quad \partial_\beta H_{1\beta} = 0. \tag{2}$$

For 2D hexagonal QCs, the point groups 6*mm*, 622, $\bar{6}$ *m*2 and 6/*mmm* belong to Laue class 10, whose constitutive relations read

$$\begin{aligned}
&\sigma_{11} = C_{11}\varepsilon_{11} + C_{13}\varepsilon_{33} + R_1 w_{11}, \quad \sigma_{33} = C_{13}\varepsilon_{11} + C_{33}\varepsilon_{33} + R_3 w_{11}, \\
&\sigma_{13} = \sigma_{31} = 2C_{44}\varepsilon_{13} + R_4 w_{13}, \\
&H_{11} = R_1\varepsilon_{11} + R_3\varepsilon_{33} + K_1 w_{11}, \quad H_{13} = 2R_4\varepsilon_{13} + K_4 w_{13}.
\end{aligned} \tag{3}$$

There are 9 independent constants C_{mn}, K_m and R_m in Eq. (3)

$$\begin{aligned}
&C_{11} = \bar{C}_{11} + \bar{C}_{12}\frac{\bar{R}_1\bar{R}_2 - \bar{C}_{12}\bar{K}_1}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2} + \bar{R}_2\frac{\bar{C}_{12}\bar{R}_1 - \bar{C}_{11}\bar{R}_2}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2}, \\
&C_{33} = \bar{C}_{33} + \bar{C}_{13}\frac{\bar{R}_1\bar{R}_3 - \bar{C}_{13}\bar{K}_1}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2} + \bar{R}_3\frac{\bar{C}_{13}\bar{R}_1 - \bar{C}_{11}\bar{R}_3}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2}, \\
&C_{13} = \bar{C}_{13} + \bar{C}_{12}\frac{\bar{R}_1\bar{R}_3 - \bar{C}_{13}\bar{K}_1}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2} + \bar{R}_2\frac{\bar{C}_{13}\bar{R}_1 - \bar{C}_{11}\bar{R}_3}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2}, \quad C_{44} = \bar{C}_{44}, \\
&R_1 = \bar{R}_1 + \bar{C}_{12}\frac{\bar{R}_1\bar{K}_2 - \bar{R}_2\bar{K}_1}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2} + \bar{R}_2\frac{\bar{R}_1\bar{R}_2 - \bar{C}_{11}\bar{K}_2}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2}, \\
&R_3 = \bar{R}_3 + \bar{C}_{13}\frac{\bar{R}_1\bar{K}_2 - \bar{R}_2\bar{K}_1}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2} + \bar{R}_3\frac{\bar{R}_1\bar{R}_2 - \bar{C}_{11}\bar{K}_2}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2}, \quad R_4 = \bar{R}_4, \\
&K_1 = \bar{K}_1 + \bar{R}_2\frac{\bar{R}_1\bar{K}_2 - \bar{R}_2\bar{K}_1}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2} + \bar{K}_2\frac{\bar{R}_1\bar{R}_2 - \bar{C}_{11}\bar{K}_2}{\bar{C}_{11}\bar{K}_1 - \bar{R}_1^2}, \quad K_4 = \bar{K}_4,
\end{aligned}$$

which are expressed by 5 elastic constants $\bar{C}_{mn}$ of phonon fields, 3 constants $\bar{K}_m$ of phason fields and 4 constants $\bar{R}_m$ of phonon-phason coupling fields. In the absence of body forces, elimination of the stresses in terms of the displacements from the equilibrium equations (3) yields the simultaneous equations

$$\begin{aligned}
&(C_{11}\partial_1^2 + C_{44}\partial_3^2)u_1 + (C_{13} + C_{44})\partial_1\partial_3 u_3 + (R_1\partial_1^2 + R_4\partial_3^2)w_1 = 0, \\
&(C_{13} + C_{44})\partial_1\partial_3 u_1 + (C_{44}\partial_1^2 + C_{33}\partial_3^2)u_3 + (R_3 + R_4)\partial_1\partial_3 w_1 = 0, \\
&(R_1\partial_1^2 + R_4\partial_3^2)u_1 + (R_3 + R_4)\partial_1\partial_3 u_3 + (K_1\partial_1^2 + K_4\partial_3^2)w_1 = 0.
\end{aligned} \tag{4}$$

According to the general solution of plane elasticity of 2D hexagonal QCs with distinct eigenvalues (Gao and Zhao 2009), the components of displacements are expressed in terms of three potential functions ψ_i

$$u_1 = \delta_{Ii}\partial_1\psi_i,\quad u_3 = k_{1i}\partial_3\psi_i,\quad w_1 = k_{2i}\partial_1\psi_i, \tag{5}$$

where i=1, 2, 3. δ_{ij} is the Kronecker delta symbol, and the following summation convention has been used throughout this paper: the Einstein summation over repeated lower case indices from 1 to 3 is applied, while upper case indices take on the same numbers as the corresponding lower case ones but are not summed. The constants k_{1i} and k_{2i} are

$$k_{1i} = \frac{\left[R_4(C_{13}+C_{44}) - C_{44}(R_3+R_4)\right]s_i^2 + \left[C_{11}(R_3+R_4) - R_1(C_{13}+C_{44})\right]\delta_{Ii}}{C_{33}R_4 s_i^4 + (C_{13}R_3 + C_{13}R_4 + C_{44}R_3 - C_{33}R_1)s_i^2 + C_{44}R_1\delta_{Ii}},$$

$$k_{2i} = \frac{-C_{33}C_{44}s_i^4 + (C_{11}C_{33} - 2C_{13}C_{44} - C_{13}^2)s_i^2 - C_{11}C_{44}\delta_{Ii}}{C_{33}R_4 s_i^4 + (C_{13}R_3 + C_{13}R_4 + C_{44}R_3 - C_{33}R_1)s_i^2 + C_{44}R_1\delta_{Ii}}.$$

Besides, the potential functions ψ_i satisfy the equations

$$\nabla_I^2\psi_i = \partial_1^2\psi_i + \frac{1}{s_I^2}\partial_3^2\psi_i = 0, \tag{6}$$

where s_i^2 are three eigenvalues of the cubic algebra equation of s^2. The values of k_{1i}, k_{2i} and s_i^2 are related by the following expressions

$$\begin{aligned}\frac{C_{44}\delta_{Ii} + (C_{13}+C_{44})k_{1i} + R_4 k_{2i}}{C_{11}\delta_{Ii} + R_1 k_{2i}} &= \frac{C_{33}k_{1i}}{(C_{13}+C_{44})\delta_{Ii} + C_{44}k_{1i} + (R_3+R_4)k_{2i}} \\ = \frac{R_4\delta_{Ii} + (R_3+R_4)k_{1i} + K_4 k_{2i}}{R_1\delta_{Ii} + K_1 k_{2i}} &= \frac{1}{s_i^2}.\end{aligned} \tag{7}$$

For the sake of conciseness, the refined theory of 2D QC beams will be given only to the case of distinct eigenvalues s_i^2 in the following context. For the cases of equal eigenvalues, the corresponding derivations can be obtained in use of a similar analysis technique, although the general solution will take a more complicated form for these cases (Gao and Zhao 2009).

3. The refined theory

The problem of QC beams may be decomposed into two fundamental problems: the extension of a beam and the bending of a beam. In the case of bending of a beam, the beam is subjected only to a set of anti-symmetrical loadings and edge conditions, thus only odd functions of x_3 are required for u_1 and w_1, and even functions of x_3 for u_3. For the Lur'e symbolic method (Lur'e 1964), treating Eq. (6) as an ordinary differential equation in x_3 with constant coefficients, one obtains the following symbolic solution of Eq. (6)

$$\psi_i(x_1, x_3) = \frac{\sin(s_i x_3 \partial_1)}{s_I\partial_1} g_i(x_1), \tag{8}$$

where g_i are unknown functions of x_1 which can be determined from the remaining boundary conditions; and the trigonometric differential operators sin $(s_i x_3 \partial_1)/(s_i \partial_1)$ and cos $(s_i x_3 \partial_1)$ must be interpreted as representing series in powers of $(s_i x_3 \partial_1)^2$, i.e.

$$\begin{aligned} \frac{\sin(s_i x_3 \partial_1)}{s_i \partial_1} &= x_3 \left(\delta_{Ii} - \frac{1}{3!} s_i^2 x_3^2 \partial_1^2 + \frac{1}{5!} s_i^4 x_3^4 \partial_1^4 - \cdots \right), \\ \cos(s_i x_3 \partial_1) &= \left(\delta_{Ii} - \frac{1}{2!} s_i^2 x_3^2 \partial_1^2 + \frac{1}{4!} s_i^4 x_3^4 \partial_1^4 - \cdots \right). \end{aligned} \tag{9}$$

Substituting Eq. (8) into Eq. (5), one obtains

$$u_1 = \frac{\sin(s_i x_3 \partial_1)}{s_I} g_i, \quad u_3 = k_{1i} \cos(s_i x_3 \partial_1) g_i, \quad w_1 = k_{2i} \frac{\sin(s_i x_3 \partial_1)}{s_I} g_i. \tag{10}$$

Since the functions g_i are not defined, all results expressed by them have no explicit physical sense. Now we introduce the angles of rotation ψ, φ and the deflection w of the mid-plane, which are widely used in other beam theories

$$\psi = -\partial_3 u_1 \Big|_{x_3=0} = -\delta_{Ii} \partial_1 g_i, \quad w = u_3 \Big|_{x_3=0} = k_{1i} g_i, \quad \varphi = -\partial_3 w_1 \Big|_{x_3=0} = -k_{2i} \partial_1 g_i. \tag{11}$$

From Eq. (11), one obtains

$$g_i = \frac{1}{A} \left(l_{i1} \frac{\psi}{\partial_1} + l_{i2} w + l_{i3} \frac{\varphi}{\partial_1} \right), \tag{12}$$

where the parameters l_{ij} and A are defined by

$$l_{i1} = \varepsilon_{ijk} k_{2j} k_{1k}, \quad l_{i2} = \varepsilon_{ijk} \delta_{Kk} k_{2j}, \quad l_{i3} = \varepsilon_{ijk} \delta_{Jj} k_{1k}, \quad A = \varepsilon_{ijk} \delta_{Ii} k_{1j} k_{2k}.$$

From Eqs. (10) and (12), the final expressions for the displacements are

$$\begin{aligned} &Au_1 = \frac{\sin(s_i x_3 \partial_1)}{s_I \partial_1} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi), \quad Au_3 = \frac{\cos(s_i x_3 \partial_1)}{\partial_1} k_{1i} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi), \\ &Aw_1 = \frac{\sin(s_i x_3 \partial_1)}{s_I \partial_1} k_{2i} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi). \end{aligned} \tag{13}$$

By using the generalized Hooke's law in Eq. (3), expressions (13) can be used to determine the stress components as

$$\begin{aligned} A\sigma_{13} &= \cos(s_i x_3 \partial_1) m_{1i} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi), \\ A\sigma_{11} &= \frac{\sin(s_i x_3 \partial_1)}{s_I} s_I^2 m_{1i} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi), \\ A\sigma_{33} &= \frac{\sin(s_i x_3 \partial_1)}{s_I} m_{2i} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi), \\ AH_{13} &= \cos(s_i x_3 \partial_1) m_{3i} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi), \\ AH_{11} &= \frac{\sin(s_i x_3 \partial_1)}{s_I} s_I^2 m_{3i} (l_{i1} \psi + l_{i2} \partial_1 w + l_{i3} \varphi), \end{aligned} \tag{14}$$

where the parameters m_{ij} are available

$$m_{1i} = -m_{2i} = C_{44}(\delta_{Ii} + k_{1i}) + R_4 k_{2i}, \quad m_{3i} = R_4(\delta_{Ii} + k_{1i}) + K_4 k_{2i}.$$

In the following two sections, discussion will be given to the refined theory in the cases of homogeneous boundary conditions and non-homogeneous boundary conditions, respectively.

4. Homogeneous boundary conditions

The classical bending problems consider only homogeneous boundary conditions, there are

$$\sigma_{13} = 0, \quad \sigma_{33} = 0, \quad H_{13} = 0 \quad (x_3 = \pm h/2). \tag{15}$$

which lead to the following linear differential matrix equation for the angles of rotation and the deflection of the mid-plane

$$\begin{bmatrix} m_{1i} l_{i1} CS_I & m_{1i} l_{i2} CS_I \partial_1 & m_{1i} l_{i3} CS_I \\ m_{2i} l_{i1} \dfrac{SN_I}{s_I} & m_{2i} l_{i2} \dfrac{SN_I}{s_I} \partial_1 & m_{2i} l_{i3} \dfrac{SN_I}{s_I} \\ m_{3i} l_{i1} CS_I & m_{3i} l_{i2} CS_I \partial_1 & m_{3i} l_{i3} CS_I \end{bmatrix} \begin{bmatrix} \psi \\ w \\ \varphi \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}, \tag{16}$$

where the differential operators SN_i and CS_i are defined by

$$SN_i = \sin\left(\frac{s_i h \partial_1}{2}\right), \quad CS_i = \cos\left(\frac{s_i h \partial_1}{2}\right).$$

Let L_{ij} and L_0 be the elements and the determinant of the 3×3 matrix of the Eq. (16), respectively, there is

$$L_0 = A^2 \varepsilon_{ijk} m_{2i} m_{3j} m_{1k} \frac{SN_I CS_J CS_K}{s_I \partial_1^3} \partial_1^4, \tag{17}$$

where ε_{ijk} is the Levi-Civita permutation symbol, and the relationship $\varepsilon_{ijk} m_{2i} m_{3j} m_{1k}=0$ comes into existence. In virtue of the Lur'e symbolic method (Lur'e 1964), the solution of Eq. (16) is

$$\begin{bmatrix} \psi \\ w \\ \varphi \end{bmatrix} = \begin{bmatrix} L_{22}L_{33} - L_{23}L_{32} & L_{13}L_{32} - L_{12}L_{33} & L_{12}L_{23} - L_{13}L_{22} \\ L_{23}L_{31} - L_{21}L_{33} & L_{11}L_{33} - L_{13}L_{31} & L_{13}L_{21} - L_{11}L_{23} \\ L_{21}L_{32} - L_{22}L_{31} & L_{12}L_{31} - L_{11}L_{32} & L_{11}L_{22} - L_{12}L_{21} \end{bmatrix} \begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{bmatrix}, \tag{18}$$

and ξ_p (p=1, 2, 3) satisfy

$$L_0 \xi_p = 0. \tag{19}$$

According to Appendix B of Gao and Wang (2004), it is proved that the solutions of Eq. (19) can be decomposed into two parts which are governed by a fourth-order equation and a transcendental equation, respectively. That is, ξ_p can be rewritten as

$$\xi_p = \xi_p^{(1)} + \xi_p^{(2)}, \tag{20}$$

where the superscripts "(1)" and "(2)" indicate the fourth-order part and the transcendental part, respectively, and $\xi_p^{(1)}$ and $\xi_p^{(2)}$ have to satisfy the following governing differential equations of the beam problem, respectively

$$\partial_1^4 \xi_p^{(1)} = 0, \quad T_0 \xi_p^{(2)} = \varepsilon_{ijk} m_{2i} m_{3j} m_{1k} \frac{SN_I CS_J CS_K}{s_I \partial_1^3} \xi_p^{(2)} = 0, \tag{21}$$

where T_0 is the transcendental differential operator. The angles of rotation and the deflection of the mid-plane of beams can be also decomposed into two parts, respectively

$$\psi = \psi^{(1)} + \psi^{(2)}, \quad w = w^{(1)} + w^{(2)}, \quad \varphi = \varphi^{(1)} + \varphi^{(2)}. \tag{22}$$

In the following two subsections, the solutions of Eq. (22) will be investigated in detail to show how the refined theory could be established based on them.

4.1 The fourth-order equation and fourth-order solution

$\xi_p^{(1)}$ satisfy the following fourth-order equation

$$\partial_1^4 \xi_p^{(1)} = 0, \tag{23}$$

and the solutions of $\psi^{(1)}$, $w^{(1)}$ and $\varphi^{(1)}$ become

$$\begin{bmatrix} \psi^{(1)} \\ w^{(1)} \\ \varphi^{(1)} \end{bmatrix} = \begin{bmatrix} L_{22}L_{33} - L_{23}L_{32} & L_{13}L_{32} - L_{12}L_{33} & L_{12}L_{23} - L_{13}L_{22} \\ L_{23}L_{31} - L_{21}L_{33} & L_{11}L_{33} - L_{13}L_{31} & L_{13}L_{21} - L_{11}L_{23} \\ L_{21}L_{32} - L_{22}L_{31} & L_{12}L_{31} - L_{11}L_{32} & L_{11}L_{22} - L_{12}L_{21} \end{bmatrix} \begin{bmatrix} \xi_1^{(1)} \\ \xi_2^{(1)} \\ \xi_3^{(1)} \end{bmatrix}. \tag{24}$$

By using Eqs. (23) and (24), the result turns out to be

$$\partial_1^4 w^{(1)} = 0. \tag{25}$$

After tedious manipulation from Eq. (16), one obtains

$$\begin{aligned} \varepsilon_{ijk} k_{1i} m_{1j} m_{2k} \frac{SN_J CS_K}{s_J} \psi^{(1)} &= -\varepsilon_{ijk} \delta_{1i} m_{1j} m_{2k} \frac{SN_J CS_K}{s_J} \partial_1 w^{(1)}, \\ \varepsilon_{ijk} k_{1i} m_{1j} m_{2k} \frac{SN_J CS_K}{s_J} \varphi^{(1)} &= \varepsilon_{ijk} k_{2i} m_{1j} m_{2k} \frac{SN_J CS_K}{s_J} \partial_1 w^{(1)}. \end{aligned} \tag{26}$$

Taking account of Taylor series of the trigonometric differential operators (9) and with the help of Eq. (25), Eq. (26) can be simplified as

$$\begin{aligned} \left(1 - \frac{h^2 \partial_1^2}{40} \frac{\bar{B}}{B}\right) \psi^{(1)} &= \frac{C}{B} \left(1 - \frac{h^2 \partial_1^2}{40} \frac{\bar{C}}{C}\right) \partial_1 w^{(1)}, \\ \left(1 - \frac{h^2 \partial_1^2}{40} \frac{\bar{B}}{B}\right) \varphi^{(1)} &= \frac{E}{B} \left(1 - \frac{h^2 \partial_1^2}{40} \frac{\bar{E}}{E}\right) \partial_1 w^{(1)}, \end{aligned} \tag{27}$$

where the parameters $B, \bar{B}, C, \bar{C}, E, \bar{E}$ in Eq. (27) read

$$
\begin{aligned}
&B=\varepsilon_{ijk}k_{1i}m_{1j}m_{2k}s_J^2,\ \ \bar{B}=\varepsilon_{ijk}k_{1i}m_{1j}m_{2k}s_J^4,\\
&C=-\varepsilon_{ijk}\delta_{Ii}m_{1j}m_{2k}s_J^2,\ \ \bar{C}=-\varepsilon_{ijk}\delta_{Ii}m_{1j}m_{2k}s_J^4,\\
&E=\varepsilon_{ijk}k_{2i}m_{1j}m_{2k}s_J^2,\ \ \bar{E}=\varepsilon_{ijk}k_{2i}m_{1j}m_{2k}s_J^4.
\end{aligned}
$$

Taking the operator $1+h^2\partial_1^2\bar{B}/(40B)$ on both sides of Eq. (27), there is

$$
\psi^{(1)}=\frac{C}{B}\left[1-\frac{h^2\partial_1^2}{40}\left(\frac{\bar{C}}{C}-\frac{\bar{B}}{B}\right)\right]\partial_1 w\ ,\quad \varphi^{(1)}=\frac{E}{B}\left[1-\frac{h^2\partial_1^2}{40}\left(\frac{\bar{E}}{E}-\frac{\bar{B}}{B}\right)\right]\partial_1 w\ , \tag{28}
$$

and from Eq. (13), the total displacements can be found to be

$$
\begin{aligned}
u_1&=\frac{x_3}{A}\left\{\left(1-\frac{s_I^2x_3^2\partial_1^2}{6}\right)\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)\right.\\
&\left.\quad-\frac{h^2\partial_1^2}{40}\left[l_{i1}\frac{C}{B}\left(\frac{\bar{C}}{C}-\frac{\bar{B}}{B}\right)+l_{i3}\frac{E}{B}\left(\frac{\bar{E}}{E}-\frac{\bar{B}}{B}\right)\right]\right\}\delta_{Ii}\partial_1 w^{(1)},\\
u_3&=\frac{1}{A}\left\{\left(1-\frac{s_I^2x_3^2\partial_1^2}{2}\right)\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)\right.\\
&\left.\quad-\frac{h^2\partial_1^2}{40}\left[l_{i1}\frac{C}{B}\left(\frac{\bar{C}}{C}-\frac{\bar{B}}{B}\right)+l_{i3}\frac{E}{B}\left(\frac{\bar{E}}{E}-\frac{\bar{B}}{B}\right)\right]\right\}k_{1i}w^{(1)},\\
w_1&=\frac{x_3}{A}\left\{\left(1-\frac{s_I^2x_3^2\partial_1^2}{6}\right)\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)\right.\\
&\left.\quad-\frac{h^2\partial_1^2}{40}\left[l_{i1}\frac{C}{B}\left(\frac{\bar{C}}{C}-\frac{\bar{B}}{B}\right)+l_{i3}\frac{E}{B}\left(\frac{\bar{E}}{E}-\frac{\bar{B}}{B}\right)\right]\right\}k_{2i}\partial_1 w^{(1)}.
\end{aligned} \tag{29}
$$

The stress components can be found to be

$$
\begin{aligned}
\sigma_{13}&=\frac{1}{A}\left\{\left(1-\frac{1}{2}s_I^2x_3^2\partial_1^2\right)\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)\right.\\
&\left.\quad-\frac{h^2\partial_1^2}{40}\left[l_{i1}\frac{C}{B}\left(\frac{\bar{C}}{C}-\frac{\bar{B}}{B}\right)+l_{i3}\frac{E}{B}\left(\frac{\bar{E}}{E}-\frac{\bar{B}}{B}\right)\right]\right\}m_{1i}\partial_1 w^{(1)},\\
\sigma_{11}&=\frac{x_3}{A}\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)s_I^2m_{1i}\partial_1^2w^{(1)},\ \sigma_{33}=\frac{x_3}{A}\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)m_{2i}\partial_1^2w^{(1)},\\
H_{13}&=\frac{1}{A}\left\{\left(1-\frac{1}{2}s_I^2x_3^2\partial_1^2\right)\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)\right.\\
&\left.\quad-\frac{h^2\partial_1^2}{40}\left[l_{i1}\frac{C}{B}\left(\frac{\bar{C}}{C}-\frac{\bar{B}}{B}\right)+l_{i3}\frac{E}{B}\left(\frac{\bar{E}}{E}-\frac{\bar{B}}{B}\right)\right]\right\}m_{3i}\partial_1 w^{(1)},\\
H_{11}&=\frac{x_3}{A}\left(l_{i1}\frac{C}{B}+l_{i2}+l_{i3}\frac{E}{B}\right)s_I^2m_{3i}\partial_1^2w^{(1)}.
\end{aligned} \tag{30}
$$

Eqs. (29) and (30) constitute the first-order theory of 2D QC beams with the differential governing Eq. (25), which can satisfy two edge conditions along the boundary of beams and coincide with the corresponding expressions of QC.

4.2 The transcendental equation and transcendental solution

$\xi_p^{(2)}$ satisfy the following transcendental equation

$$T_0 \xi_p^{(2)} = 0, \tag{31}$$

and the solutions of $\psi^{(2)}$, $w^{(2)}$ and $\varphi^{(2)}$ become

$$\begin{bmatrix} \psi^{(2)} \\ w^{(2)} \\ \varphi^{(2)} \end{bmatrix} = \begin{bmatrix} L_{22}L_{33} - L_{23}L_{32} & L_{13}L_{32} - L_{12}L_{33} & L_{12}L_{23} - L_{13}L_{22} \\ L_{23}L_{31} - L_{21}L_{33} & L_{11}L_{33} - L_{13}L_{31} & L_{13}L_{21} - L_{11}L_{23} \\ L_{21}L_{32} - L_{22}L_{31} & L_{12}L_{31} - L_{11}L_{32} & L_{11}L_{22} - L_{12}L_{21} \end{bmatrix} \begin{bmatrix} \xi_1^{(2)} \\ \xi_2^{(2)} \\ \xi_3^{(2)} \end{bmatrix}. \tag{32}$$

By using Eqs. (31) and (32), the result turns out to be

$$T_0 w^{(2)} = 0. \tag{33}$$

To reduce Eq. (33) to applicable differential equation, the transcendental differential operator T_0 in Eq. (33) must be replaced by an infinite number of simply algebraic operators associated with the eigenvalues of T_0. The eigenvalues of T_0 can be found by solving the equation

$$T_0(\lambda) = 0, \tag{34}$$

which is yielded by substituting $\partial_1{}^2$ by λ^2 in T_0. The differential operator corresponding to an individual eigenvalue λ then becomes $\partial_1{}^2 - \lambda^2$. The differential equations associated with the eigenvalues λ_n are

$$\partial_1^2 \psi_n = \lambda_N^2 \psi_n, \;\; \partial_1^2 w_n = \lambda_N^2 w_n, \;\; \partial_1^2 \varphi_n = \lambda_N^2 \varphi_n. \tag{35}$$

From Eq. (26), the corresponding angles of rotation of the mid-plane of beams with respect to w_n are expressed as

$$\begin{aligned} \psi_n &= -\frac{\varepsilon_{ijk}\delta_{1i} m_{1j} m_{2k} \sin\dfrac{s_J h\lambda_n}{2}\cos\dfrac{s_K h\lambda_N}{2} \Big/ s_J}{\varepsilon_{ijk} k_{1i} m_{1j} m_{2k} \sin\dfrac{s_J h\lambda_n}{2}\cos\dfrac{s_K h\lambda_N}{2} \Big/ s_J} \partial_1 w_n, \\ \varphi_n &= \frac{\varepsilon_{ijk} k_{2i} m_{1j} m_{2k} \sin\dfrac{s_J h\lambda_n}{2}\cos\dfrac{s_K h\lambda_N}{2} \Big/ s_J}{\varepsilon_{ijk} k_{1i} m_{1j} m_{2k} \sin\dfrac{s_J h\lambda_n}{2}\cos\dfrac{s_K h\lambda_N}{2} \Big/ s_J} \partial_1 w_n. \end{aligned} \tag{36}$$

Therefore, all the expressions of displacements and stresses for QC beams can be acquired in terms of the deflection of the mid-plane. By combining this transcendental solution with the fourth-order solution, a refined theory for 2D QC beams with free faces can be established, as shown in the two differential governing Eqs. (25) and (33).

5. Non-homogeneous boundary conditions

For non-homogeneous boundary conditions, we discuss two special cases of boundary conditions, that is, normal surface loadings only and shear surface loadings only.

5.1 Two surfaces subjected to normal loadings only

Now let us consider the case where QC beams are subjected only to normal loadings, i.e.

$$\sigma_{13}=0,\quad \sigma_{33}=\pm q_2(x_1),\ H_{13}=0\ \ (x_3=\pm h/2). \tag{37}$$

Substituting the stress expressions in Eq. (14) into the boundary conditions (37) of beams, we get the following matrix equation

$$\begin{bmatrix} m_{1i}l_{i1}CS_I & m_{1i}l_{i2}CS_I\partial_1 & m_{1i}l_{i3}CS_I \\ m_{2i}l_{i1}\dfrac{SN_I}{s_I} & m_{2i}l_{i2}\dfrac{SN_I}{s_I}\partial_1 & m_{2i}l_{i3}\dfrac{SN_I}{s_I} \\ m_{3i}l_{i1}CS_I & m_{3i}l_{i2}CS_I\partial_1 & m_{3i}l_{i3}CS_I \end{bmatrix}\begin{bmatrix}\psi \\ w \\ \varphi\end{bmatrix}=\begin{bmatrix}0 \\ Aq \\ 0\end{bmatrix}. \tag{38}$$

In virtue of the Lur'e symbolic method (Lur'e 1964), the solution of Eq. (38) is

$$\begin{bmatrix}L_0\psi \\ L_0 w \\ L_0\varphi\end{bmatrix}=\begin{bmatrix} L_{22}L_{33}-L_{23}L_{32} & L_{13}L_{32}-L_{12}L_{33} & L_{12}L_{23}-L_{13}L_{22} \\ L_{23}L_{31}-L_{21}L_{33} & L_{11}L_{33}-L_{13}L_{31} & L_{13}L_{21}-L_{11}L_{23} \\ L_{21}L_{32}-L_{22}L_{31} & L_{12}L_{31}-L_{11}L_{32} & L_{11}L_{22}-L_{12}L_{21}\end{bmatrix}\begin{bmatrix}0 \\ Aq \\ 0\end{bmatrix}. \tag{39}$$

After tedious manipulation from Eq. (39), one obtains

$$\frac{L_0}{A^2}\begin{pmatrix}\psi \\ w \\ \varphi\end{pmatrix}=\varepsilon_{ijk}\begin{pmatrix}\delta_{Ii}\partial_1 \\ -k_{1i} \\ -k_{2i}\partial_1\end{pmatrix}m_{3j}m_{1k}CS_J CS_K q. \tag{40}$$

Eq. (40) is the exact governing equation for the angles of rotation ψ, φ and the deflection w of the mid-plane of beams subjected to the normal loadings.

5.2 Two surfaces subjected to shear loadings only

The other case is that QC beams are subjected only to shear surface loadings, i.e.

$$\sigma_{13}=\tau_1(x_1),\quad \sigma_{33}=0,\ H_{13}=\tau_2(x_1)\ \ (x_3=\pm h/2). \tag{41}$$

Substitution of Eq. (14) into the boundary conditions (41) leads to the following matrix equation

$$\begin{bmatrix} m_{1i}l_{i1}CS_I & m_{1i}l_{i2}CS_I\partial_1 & m_{1i}l_{i3}CS_I \\ m_{2i}l_{i1}\dfrac{SN_I}{s_I} & m_{2i}l_{i2}\dfrac{SN_I}{s_I}\partial_1 & m_{2i}l_{i3}\dfrac{SN_I}{s_I} \\ m_{3i}l_{i1}CS_I & m_{3i}l_{i2}CS_I\partial_1 & m_{3i}l_{i3}CS_I \end{bmatrix}\begin{bmatrix}\psi \\ w \\ \varphi\end{bmatrix}=\begin{bmatrix}A\tau_1 \\ 0 \\ A\tau_2\end{bmatrix}. \tag{42}$$

After the same manipulation as in the preceding case, the exact governing equation for ψ, φ and w subjected to the shear loadings can be reached as follows

$$\frac{L_0}{A^2}\begin{pmatrix}\psi\\ w\\ \varphi\end{pmatrix}=\varepsilon_{ijk}\begin{pmatrix}\delta_{Ii}\partial_1\\ -k_{1i}\\ -k_{2i}\partial_1\end{pmatrix}\frac{SN_J}{s_J}CS_K\,(m_{2j}m_{3k}\tau_1-m_{1j}m_{2k}\tau_2). \tag{43}$$

Combining Eqs. (40) and (43), we obtain the exact governing equations for the angles of rotation ψ, φ and the deflection w of the mid-plane of beams subjected to the most general loadings at the top and bottom surfaces of QC beams. In the similar way, all the expressions of displacements and stresses for QC beams can be obtained in terms of the mid-plane displacement functions.

6. Examples

To illustrate the applications of the refined theory developed in the previous sections, we present the following two examples: simply supported beams with a sinusoidal distributed load and a constant distributed load, respectively. It should be noted that the same examples for transversely isotropic elastic beams (Gao and Zhao 2007) and elastic beams (Timoshenko and Goodier 1970, Gao and Wang 2005) have been discussed.

6.1 The simply supported beam with a sinusoidal distributed load

Considering a QC beam of uniform cross-section, which is simply supported at x_1=0 and x_1=l, and is subjected to a sinusoidal distributed load along the length of beams, namely, q=$q_0\sin\rho x_1$, where ρ=$n\pi/l$, n is an integer and q_0 is a constant. For isotropic elastic beams, the same example was considered by utilizing the Airy stress function method (Timoshenko and Goodier 1970).

Since the load q is sinusoidal distributed along the length of beams, from the exact governing differential Eq. (40) and the Taylor series of the trigonometric differential operators in Eq. (9), ψ, φ and w have the form as

$$\begin{aligned}
\psi&=\frac{A^2}{L_0'}\varepsilon_{ijk}\delta_{Ii}m_{3j}m_{1k}CH_JCH_Kq_0\cos\rho x_1-\frac{\varepsilon_{ijk}\delta_{Ii}m_{3j}m_{1k}}{\varepsilon_{ijk}k_{1i}m_{3j}m_{1k}}(3A_1x^2+2A_2x+A_3+\bar{A}_1),\\
w&=-\frac{A^2}{\rho L_0'}\varepsilon_{ijk}k_{1i}m_{3j}m_{1k}CH_JCH_Kq_0\sin\rho x_1+A_1x^3+A_2x^2+A_3x+A_4,\\
\varphi&=-\frac{A^2}{L_0'}\varepsilon_{ijk}k_{2i}m_{3j}m_{1k}CH_JCH_Kq_0\cos\rho x_1+\frac{\varepsilon_{ijk}k_{2i}m_{3j}m_{1k}}{\varepsilon_{ijk}k_{1i}m_{3j}m_{1k}}(3A_1x^2+2A_2x+A_3+\tilde{A}_1),
\end{aligned} \tag{44}$$

where A_1, A_2, A_3 and A_4 are unknown constants, and $\bar{A}$ and $\tilde{A}$ associate with A_1

$$\bar{A}=\frac{3}{4}\rho^2h^2\left[\frac{\varepsilon_{ijk}k_{1i}m_{3j}m_{1k}(s_J^2+s_K^2)}{\varepsilon_{ijk}k_{1i}m_{3j}m_{1k}}-\frac{\varepsilon_{ijk}\delta_{Ii}m_{3j}m_{1k}(s_J^2+s_K^2)}{\varepsilon_{ijk}\delta_{Ii}m_{3j}m_{1k}}\right]A_1,$$

$$\tilde{A}=\frac{3}{4}\rho^2h^2\left[\frac{\varepsilon_{ijk}k_{1i}m_{3j}m_{1k}(s_J^2+s_K^2)}{\varepsilon_{ijk}k_{1i}m_{3j}m_{1k}}-\frac{\varepsilon_{ijk}k_{2i}m_{3j}m_{1k}(s_J^2+s_K^2)}{\varepsilon_{ijk}k_{2i}m_{3j}m_{1k}}\right]A_1,$$

$$L_0'=-A^2\varepsilon_{ijk}m_{2i}m_{3j}m_{1k}\frac{SH_I CH_J CH_K}{s_I},\quad SH_i=\sinh\left(\frac{s_ih\rho}{2}\right),\quad CH_i=\cosh\left(\frac{s_ih\rho}{2}\right).$$

Employing the Betti-Rayleigh reciprocal theorem, a set of edge conditions of 2D QC strip bodies was established by adopting the decay analysis technique (Gao 2009). Together with the boundary conditions of the simply supported beams on two ends

$$\int_{-h/2}^{h/2}x_3\sigma_{11}\mathrm{d}x_3=0,\quad w=0\quad (x_1=0,\ l),\tag{45}$$

these unknown constants can be determined as

$$A_1=A_2=A_3=A_4=0.\tag{46}$$

Substituting these expressions into Eqs. (13) and (14) leads to all the expressions of displacements and stresses for QC beams

$$\begin{aligned}&u_1=G_i\frac{\sinh(s_ix_3\rho)}{s_I\rho}q_0\cos\rho x_1,\quad u_3=G_i\frac{\cosh(s_Ix_3\rho)}{\rho}k_{1i}q_0\sin\rho x_1,\\&w_1=G_i\frac{\sinh(s_Ix_3\rho)}{s_I\rho}k_{2i}q_0\cos\rho x_1,\end{aligned}\tag{47}$$

$$\begin{aligned}&\sigma_{13}=G_i\cosh(s_Ix_3\rho)m_{1i}q_0\cos\rho x_1,\quad \sigma_{11}=-G_i\frac{\sinh(s_Ix_3\rho)}{s_I}s_I^2m_{1i}q_0\sin\rho x_1,\\&\sigma_{33}=-G_i\frac{\sinh(s_Ix_3\rho)}{s_I}m_{2i}q_0\sin\rho x_1,\\&H_{13}=G_i\cosh(s_Ix_3\rho)m_{3i}q_0\cos\rho x_1,\quad H_{11}=-G_i\frac{\sinh(s_Ix_3\rho)}{s_I}s_I^2m_{3i}q_0\sin\rho x_1,\end{aligned}\tag{48}$$

where

$$G_i=\frac{A}{L_0'}\varepsilon_{ijk}\left(l_{I1}-l_{I2}k_{1I}-l_{I3}k_{2I}\right)m_{3j}m_{1k}CH_JCH_K.$$

6.2 The simply supported beam with a constant distributed load

The other example is a bending beam of uniform cross-section which is simply supported at $x_1=\pm l$ and which carries a uniformly distributed load of intensity $q=q_0$. Substituting the Taylor series of the trigonometric differential operators in Eq. (9) into Eq. (40), by dropping all the terms associated with h^2 and the higher orders, the result turns out to be

$$D\partial_1^4 w = q_0,\quad \psi = \left(1 - \frac{D_1}{8} h^2 \partial_1^2\right)\partial_1 w,\quad \varphi = -\frac{D_2}{8} h^2 \partial_1^3 w, \tag{49}$$

where D is the flexural rigidity of the QC beam,

$$D = \frac{h^3 \varepsilon_{ijk} m_{2i} m_{3j} m_{1k}\left(s_I^2 + 3s_J^2 + 3s_K^2\right)}{48\varepsilon_{ijk} k_{1i} m_{3j} m_{1k}},$$

$$D_1 = \frac{\varepsilon_{ijk}\delta_{Ii} m_{3j} m_{1k}\left(s_J^2 + s_K^2\right)}{\varepsilon_{ijk}\delta_{Ii} m_{3j} m_{1k}} + \frac{\varepsilon_{ijk} k_{1i} m_{3j} m_{1k}\left(s_J^2 + s_K^2\right)}{\varepsilon_{ijk} k_{1i} m_{3j} m_{1k}},$$

$$D_2 = \frac{\varepsilon_{ijk} k_{2i} m_{3j} m_{1k}\left(s_J^2 + s_K^2\right)}{\varepsilon_{ijk} k_{1i} m_{3j} m_{1k}}.$$

From Eqs. (14) and (49) the bending moment $M(x_1)$ of the beam can be found to be

$$M = \int_{-h/2}^{h/2} x_3 \cdot \sigma_{11} \mathrm{d}x_3 = \alpha\left(1 + \frac{\beta}{\alpha} h^2 \partial_1^2\right)\partial_1^2 w, \tag{50}$$

where

$$\alpha = \frac{h^3}{24} m_{1i} s_I^2\left(l_{i1} + l_{i2}\right),\quad \beta = -\frac{h^3}{192} m_{1i} s_I^2\left(D_1 l_{i1} + D_2 l_{i2}\right) - \frac{h^3}{480} m_{1i} s_I^4\left(l_{i1} + l_{i3}\right).$$

The boundary conditions for the beam are

$$M(\pm l) = 0,\quad w(\pm l) = \tag{51}$$

From Eqs. (49)-(51), the solution for the mid-plane deflection is

$$w = \frac{q_0 l^4}{24D}\left(\frac{x_1^4}{l^4} - 6\frac{x_1^2}{l^2} + 5\right) + \frac{q_0 l^2 h^2 \beta}{2D\alpha}\left(1 - \frac{x_1^2}{l^2}\right). \tag{52}$$

Up to here, these examples show that the exact or accurate solutions may be obtained by applying the refined theory deduced herein.

6.3 The degenerated form of QC beams

Determination of the independent elastic constants C_{ij}, K_i and R_i for different kinds of QCs depends on their symmetries with the group representation theory (Ding *et al*. 1993, Hu *et al*. 1996, Hu *et al*. 2000). It is noted that, although C_{ij} in QCs can be measured by some experimental methods, K_i are difficult to measure (Tanaka *et al*. 1996). Significant progress in this area has been made by Jeong and Steinhardt (Jeong and Steinhardt 1993), who evaluated K_i of decagonal QCs by Monte Carlo simulation. The values of K_i are of the same order of magnitude as C_{ij} obtained by resonant ultrasound spectroscopy (Chernikov *et al*. 1998). There are no data available for R_i which, based on the estimation of some experts (Edagawa 2007, Takeuchi and Edagawa 2007) working in the field of QCs, hold lower values than K_i.

Unfortunately, material constants for 2D hexagonal QCs are not available presently, so numerical examples cannot be given here. Alternatively, we will discuss a degenerated form of QC beam to investigate its validity, i.e., a 2D QC beam reduces to a transversely isotropic elastic beam. In this case, no phonon-phason field coupling effect is taken into account, i.e., $R_1=R_3=R_4=0$. Hence the governing equations (1)-(3) reduce to two groups of equations for uncoupled phonon and phason field problems, respectively. s_1^2 and s_2^2 relate only to elastic constants in the phonon field, while s_3^2 associates only with elastic constants in the phason field. The constants k_{1i} and k_{2i} degenerated from expressions (7) reduce to

$$k_{1n} = \frac{C_{11} - C_{44} s_n^2}{\left(C_{13} + C_{44}\right) s_n^2} = \frac{C_{13} + C_{44}}{C_{33} s_n^2 - C_{44}}, \quad k_{2n} = 0,$$

where n=1, 2. On the other hand, $k_{13}=0$ and $k_{23}\neq 0$, which associates with s_3^2. Since the analysis in the following calculation does not involve k_{23} except the requirement $k_{23}\neq 0$, it suffices to discuss only k_{1n} and s_n^2. For the transversely isotropic elastic beams, the flexural rigidity D and some parameters have the forms

$$D = \frac{C_{44} h^3 \left(1 + k_{11}\right)\left(1 + k_{12}\right)\left(s_2^2 - s_1^2\right)}{k_{11} - k_{12}}, \quad \frac{\beta}{\alpha} = \frac{\left(4k_{11} + k_{12} + 5\right) s_2^2 - \left(k_{11} + 4k_{12} + 5\right) s_1^2}{40\left(k_{11} + k_{12}\right)}. \tag{53}$$

Noticeably, the solution of the mid-plane deflection in Eq. (52) described by Eq. (53) coincides with the corresponding one given by Gao *et al*. (2007). Therefore, the exact theory of 2D QC beams can be degenerated into that of transversely isotropic elastic beams by omitting the phonon-phason field coupling effect. For further simplification, the parameters of isotropic elastic beams have the forms

$$D = \frac{Eh^3}{12}, \quad \frac{\beta}{\alpha} = \frac{8 + 5\nu}{40}. \tag{54}$$

We recover the analytical solutions for isotropic elastic beams (Timoshenko and Goodier 1970, Gao and Wang 2005).

In comparison with the theories of elastic beams (Timoshenko and Goodier 1970, Gao and Wang 2005, Gao *et al*. 2007), the existence of phason field influences strongly the deformation and mechanical behavior of QC materials. Owing to the introduction of the phason field, a theoretical description of the deformed state of QC beams requires a combined consideration of interrelated phonon and phason fields, so the beam theory of QCs is more complex than that of the conventional crystals. The refined theory provides important information for studying the mechanical behaviours of the new solid phase and understanding clearly the interplay of the interaction between the phonon and phason activity.

Of course, the theoretical prediction needs to be verified by experimental observation. Experiments are very important to measure elastic constants, deformation, mechanical/physical behavior, etc. of the materials. In principle, experimental methods can also be used to determine the displacement and stress fields of the materials, but there are few such results to date. So there are difficulties in comparing theoretical solutions with test data apart from those obtained in the present paper. Therefore, the refined theory derived here by an analytical approach provides theoretical models to gain insight into the physical nature of this class of materials and to study possible applications of QCs.

7. Conclusions

Without any ad hoc assumptions, a refined theory of 2D QC deep beams has been deduced systematically and directly from linear elastic theory of QC by using the general solution and the Lur'e symbolic method. For the homogeneous beams, the refined theory is exact in the sense that a solution of the theory satisfies all the equations in elastic theory and consists of two parts: the fourth-order part and the transcendental part. For the non-homogeneous beams, two special cases of boundary conditions are considered, that is, normal surface loadings only and shear surface loadings only, and the exact governing differential equations and solutions are derived directly from the beam theory. Meanwhile, as two illustrative examples, explicit expressions of analytical solutions are obtained for QC beams subjected to a sinusoidal distributed vertical load and a constant distributed load, respectively.

By the refined theory, calculation of stresses or deformations can be carried out by two independent parts. Therefore, it can increase the possibility of solving complicated beam problems, simplify the calculation and improve the efficiency of the computation. In addition, because the refined theory is derived directly from linear elastic theory of QC without requirement of any ad hoc assumptions concerning the deformation or the stress state, results based on it are of high accuracy, appeal to application and help to describe problems in an incisive way.

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