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A method for predicting approximate lateral deflections in thin glass plates

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Abstract. In the present paper a three-dimensional non-linear truss element and a short computer program for the modeling and predicting approximate lateral deflections in thin glass plates by the method of incremental loading are proposed. Due to the out-of-plane large deflections of thin glass plates compared to the plate thickness within each loading increment, the equilibrium and stiffness conditions are written with respect to the deformed structure. An application is presented on a thin fully tempered monolithic rectangular glass plate, laterally supported around its perimeter subjected to uniform wind pressure. The results of the analysis are compared with published experimental results and found to have satisfactory approximation. It is also observed that the large deflections of a glass plate lead to a part substitution of the bending plate behavior by a tensioned membrane behavior which is favorable.

Keywords: truss model; glass plate; tempered glass; geometric nonlinearity; wind pressure; computer program

1. Introduction

Glass is an important material which has been used in buildings for centuries. But the recent use of larger areas of this exciting material in civil and architectural engineering structures such as glass facades, shells, pyramids and shelters of glass have placed greater demands on the glass as a structural material (Charles 1958, Hooper 1973, Behr *et al.* 1993, Norville *et al.* 1998, Norville and Minor 2000, Foraboschi 2007).

The ability to produce glass of improved mechanical strength – as is the heat strengthened and the fully tempered glass which are two or four times stronger than annealed glass respectively– allows in modern construction to provide improved aesthetics and designs even under adverse conditions (heavy loads due to wind pressure or snow, temperature changes, effects of blast, strong earthquakes etc).

Several past studies of theoretical and experimental investigations and failure tests from many researchers, provide sufficient data and information to define the behavior and strength of

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monolithic glass lites, laminated glass panes and insulated glass units (Behr *et al.* 1985, Vallabhan and Chou 1986, Norville *et al.* 1993, Kotalakidis *et al.* 1999, Minor and Norville 2006, Pankhardt 2008, Iizumi and Kopp 2009, Pankhardt and Balazs 2010, Gavanski and Kopp 2011, Hooper *et al.* 2012). Since thin glass plates are usually used in buildings subjected to lateral wind pressure loads, the most important property for load resistance is bending characteristics. So the response of these glass plates is such that the out-of-plane deflections are large compared to the plate thickness resulting in geometric nonlinearities (von Karman plates) (Vallabhan 1983, Vallabhan *et al.* 1985, Vallabhan *et al.* 1987). On the contrary, there is no inherent nonlinearity of stress-strain law for the glass which for usual loads, remains practically within the linear elastic region until the first cracking due to tension occurs.

For the non-linear analysis and predicting approximate lateral deflections in thin glass plates, first a general finite-element computer program can be used and then apply the method of incremental loading in which within each loading increment, even a small one, the equilibrium and stiffness conditions are written with respect to the deformed structure. The usual finite elements for the spatial discretization of a structure have complicated stiffness matrices and pose difficulties, particularly in handing non-linear problems (Argyris 1978, Felippa 2009, Taylor 2011).

To avoid the complexities of expensive analyses with 2D and 3D finite-element computations, a three-dimensional non-linear truss element for the modeling of thin monolithic glass plates is presented in this study. The truss models have been proved reliable by comparison of their results with other published data, experimental or numerical (Absi 1978, Vecchio and Collins 1993, ASCE-ACI 1998, Papadopoulos *et al.* 2009, Xenidis *et al.* 2013). Thanks to its very simple geometry, a truss model can describe in a simple way the geometric nonlinearities through easily writing the equilibrium equations with respect to the deformed structure and by updating the global stiffness matrix within each step of an incremental loading procedure. For this reason a very short computer program in a FORTRAN 2000 code has been developed. This short, fully documented computer program compared to the often used very large general purpose computer programs, exhibits the advantages of more simplicity, clarity and transparency of assumptions.

To demonstrate the ability of the method to predict approximate lateral deflections in non-linear glass structures, the proposed truss model and the computer program are applied on a thin fully tempered monolithic rectangular glass plate, laterally supported around its perimeter subjected to uniform wind pressure perpendicular to its plane. The results obtained of the analysis compare very well with published experimental results (Norville *et al.* 1993).

2. Modeling procedure

2.1 The proposed truss model

We consider a thin monolithic rectangular glass plate as shown in Fig. 1(a), with size ℓ_y by ℓ_z and thickness d laterally supported around its perimeter (Norville *et al.* 1993 in Fig. 3, Himansu Sekhar Pal 1986 in Figs. 2.1, 2.2, 2.3 and 2.4) subjected to uniform wind pressure w, perpendicular to its plane. The magnitude of the uniform wind pressure (i.e., *maxw*) is assumed as the critical loading (i.e., failure pressure), based on which the deflections and the tensions of bars of the glass plate model due to bending will be measured at the failure moment.

Because of the double symmetry between structure and loading we can study only one quarter

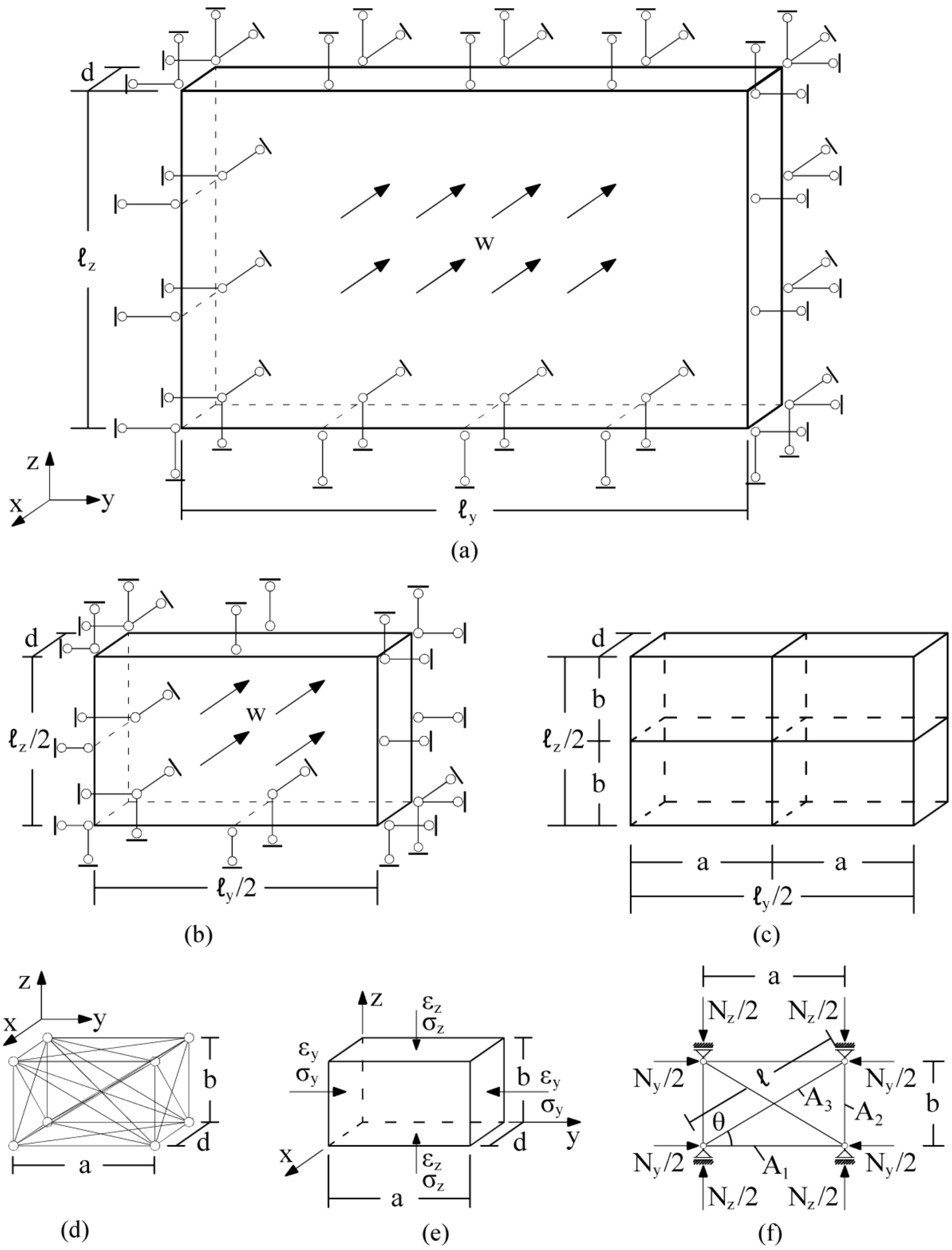

Fig. 1 (a) Rectangular glass plate laterally supported around its perimeter subjected to uniform wind pressure, (b) The quarter of the glass plate under study, (c) Its discretization into elementary rectangular solids, (d) The 3D rectangular truss element, (e) The continuum glass plate element, (f) The corresponding 2D truss element

of the glass plate (Fig. 1(b)) using the appropriate supports on the axes of symmetry. In the case of a square glass plate there is also symmetry as to the diagonals. The quarter of the studied glass plate can be subdivided into elementary rectangular solids of parallel sides (i.e., parallelepiped) with dimensions *a*, *b* and d (Fig. 1(c)). Each one of these solids is modeled as a three-dimensional truss (Fig. 1(d)) where all its sides and its diagonals are bars.

2.2 Cross-sectional areas of bars

The bars of the 3D elementary truss model of the glass plate (Fig. 1(d)) can be divided into 7 groups: 4 bars to the sides of each one of the three axes *x*, *y*, *z* of the solid, 4 bars to the diagonals of the sides parallel to each one of the three major planes of the solid *xy*, *xz*, *yz* and finally 4 bars to the internal diagonals of the solid (in total 7•4=28 bars). The respective cross-sections of the above mentioned 7 groups of bars will be denominated as A_x, A_y, A_z, A_{xy}, A_{xz}, A_{yz}, A_{xyz}. Due to the small thickness of the glass plate, relatively to its other two sides, it is considered here that in the stress-strain behavior on the *x* axis (which is the one of the thickness *d*) only the bars on the *x* direction contribute. So result

$$A_x = (a/2)\cdot(b/2) = (a\cdot b)/4 \qquad (1)$$

The cross-sections of remaining bars can be determined by the biaxial stress-strain behavior, inside the plane of the glass plate. Thanks again to the small thickness of the glass plate, it is reasonable to group in couples the cross-sections of the six remaining bars of the plate as A_y=A_{xy}, A_z=A_{xz} and A_{yz}=A_{xyz}. So, the cross-sectional areas of bars are derived from the combination of the relations that express the linear elastic isotropic biaxial behavior of the stress-strain law of modeled solid and of the force-displacement relations on the nodes of the truss model, assuming a value of *v*=0.22 (Norville *et al.* 1993) for the Poisson ratio of glass. In order to determine the cross-sectional areas A_1=A_y=A_{xy}, A_2=A_z=A_{xz} and A_3=A_{yz}=A_{xyz} of bars of the 2D truss element as shown in Fig. 1(f), we have to compare it to the corresponding continuum glass plate element of Fig. 1(e), as regards two representative stress-strain states in the linear elastic region. The stress-strain relations of the continuum glass plate element in the initial linear elastic isotropic state are

$$\begin{bmatrix}\sigma_y\\ \sigma_z\end{bmatrix} = \left(\frac{E}{1-\nu^2}\right)\cdot\begin{bmatrix}1 & \nu\\ \nu & 1\end{bmatrix}\cdot\begin{bmatrix}\varepsilon_y\\ \varepsilon_z\end{bmatrix} \qquad (2)$$

where *E* the elasticity modulus and *v* the Poisson ratio of glass.

$$\text{For } \nu = 0.22 \rightarrow 1-\nu^2 = 1-0.22^2 = 0.95 \approx 1.00$$

By considering the longitudinal deformation ($\varepsilon_z = 0$, $\varepsilon_y = \Delta a/a$) arise $\sigma_y = E\cdot\varepsilon_y$ and $\sigma_z = \nu\cdot E\cdot\varepsilon_y = 0.22\cdot E\cdot\varepsilon_y$

For the continuum glass plate element of Fig. 1(e), is valid

$$N_y = \sigma_y\cdot b\cdot d \;\rightarrow\; \frac{N_y}{b\cdot d} = E\cdot\varepsilon_y = E\cdot\frac{\Delta a}{a} \;\rightarrow\; N_y = \left(\frac{E\cdot b\cdot d}{a}\right)\cdot\Delta a \qquad (3)$$

and

$$N_z = \sigma_z\cdot a\cdot d \;\rightarrow\; \frac{N_z}{a\cdot d} = \nu\cdot E\cdot\varepsilon_y = \nu\cdot E\cdot\frac{\Delta a}{a} \;\rightarrow\; N_z = \left(\frac{0.22\cdot E\cdot a\cdot d}{a}\right)\cdot\Delta a \qquad (4)$$

Whereas, for the 2D truss element of Fig. 1(f) in respect to the *z* axis, we have

$$N_z = \left(\frac{8 \cdot E \cdot A_3}{\ell}\right) \cdot \Delta a \cdot \cos\theta \cdot \sin\theta = \frac{8 \cdot E \cdot A_3}{a} \cdot \Delta a \cdot \cos^2\theta \cdot \sin\theta \tag{5}$$

where $\ell = a/\cos\theta$

By combining the above Eqs. (4) and (5), we obtain

$$A_3 = \frac{a \cdot d}{36.36 \cdot \cos^2\theta \cdot \sin\theta} = \frac{\ell \cdot d}{18.18 \cdot 2 \cdot \cos\theta \cdot \sin\theta} \Rightarrow A_3 = 0.055 \cdot \left(\frac{\ell \cdot d}{\sin 2\theta}\right) \tag{6}$$

For the same truss element in respect to the y axis, we have

$$N_y = \left(\frac{8 \cdot E \cdot A_1}{a}\right) \cdot \Delta a + \left(\frac{8 \cdot E \cdot A_3}{\ell}\right) \cdot \Delta a \cdot \cos\theta \cdot \cos\theta \Rightarrow N_y = \left(\frac{8 \cdot E \cdot A_1}{a}\right) \cdot \Delta a + \left(\frac{8 \cdot E \cdot A_3}{a/\cos\theta}\right) \cdot \Delta a \cdot \cos^2\theta$$
$$\Rightarrow N_y = \frac{8 \cdot E}{a} \cdot \Delta a \cdot \left(A_1 + A_3 \cdot \cos^3\theta\right) \tag{7}$$

By combining the above Eqs. (6) and (7), we obtain

$$N_y = \frac{8 \cdot E}{a} \cdot \Delta a \cdot \left(A_1 + \frac{1}{36.36} \cdot \frac{a \cdot d}{tg\theta}\right) \tag{8}$$

and finally, by combining the equations (3) and (8) we conclude

$$8 \cdot \left(A_1 + \frac{1}{36.36} \cdot \frac{a \cdot d}{tg\theta}\right) = b \cdot d \Rightarrow 8 \cdot A_1 + \frac{1}{4.545} \cdot \left(\frac{a \cdot d}{tg\theta}\right) = b \cdot d \Rightarrow 8 \cdot A_1 = b \cdot d - \frac{1}{4.545} \cdot \left(\frac{a \cdot d}{tg\theta}\right) \Rightarrow$$
$$\Rightarrow A_1 = \frac{1}{8} \cdot \left(b \cdot d - 0.22 \cdot \frac{a \cdot d}{tg\theta}\right) \tag{9}$$

From similar considerations for the transverse deformation (ε_y=0, ε_z=$\Delta b/b$) result

$$A_2 = \frac{1}{8} \cdot \left(a \cdot d - 0.22 \cdot \frac{b \cdot d}{ctg\theta}\right) \tag{10}$$

2.3 Stress-strain behavior of bars

The bars of the proposed truss model may follow the non-linear uniaxial stress-strain behavior of glass, which include cracking at tension, plastic yield, softening and fracture under compression. Because the material of the plate is assumed to be elastic with small strains, the stresses of glass plate are within the linear elastic region as long as there hasn’t been a crack due to tension. For this reason, during the non-linear analysis we can use the linear elastic stress-strain law for the bars of truss model until the point of cracking due to tension occurs, when the incremental loading of the glass plate is interrupted.

2.4 Algorithm for the non-linear analysis

In order to take into account geometric nonlinearities, the equilibrium equations are written and the global stiffness matrix is updated, with respect to the deformed truss within each step of incremental loading. The local stiffness matrix **k** of a bar in 2D, with respect to reference axes *yz*, is written as

$$\mathbf{k} = \mathbf{k}_e + \mathbf{k}_g = \frac{EA}{a_0} \cdot \begin{bmatrix} c_y^2 & c_y c_z \\ c_y c_z & c_z^2 \end{bmatrix} + \frac{N}{a} \cdot \begin{bmatrix} c_z^2 & -c_y c_z \\ -c_y c_z & c_y^2 \end{bmatrix} \tag{11}$$

where $\mathbf{k_e}$ the elastic stiffness, $\mathbf{k_g}$ the geometric stiffness, *E* the elasticity modulus, *A* the section area, a_o the undeformed length, a the present length, *N* the axial force and c_y, c_z the direction cosines of the bar.

Whereas, the global stiffness matrix of a truss is written as

$$\mathbf{K}_G = \mathbf{B} \cdot \mathrm{diag}(k_i) \cdot \mathbf{B}^T \quad i = 1,....,n_b \tag{12}$$

where **B** is the Boolean linkage matrix and n_b the number of bars of the truss.

Based on the proposed algorithm, a very short computer program in a FORTRAN 2000 code has been developed and is used to solve the three-dimensional truss element with geometric non-linear behavior by the method of incremental loading.

2.5 The short computer program

In the proposed fully documented computer program, due to small thickness of the glass plate when compared with the other two dimensions, the bars in the thickness direction of the plate have very small length and very large cross-sections. This fact implies a very high axial stiffness from the other bars and mainly when compared to the stiffness with respect to the deflections of the glass plate, perpendicular to its plane.

This problem, which create errors in rounding small differences of large numbers, is countered with two techniques: on the one hand the variables are denoted with double accuracy and on the other the stiffness of the bars in the thickness direction of the glass plate is artificially diminished by e.g., 1000 times. In Appendix A is presented the listing of the program.

3. Numerical example

The three-dimensional non-linear truss model and the computer program for the incremental loading, are applied on a specific thin fully tempered monolithic rectangular glass plate with nominal dimensions ℓ_y=1930 mm by ℓ_z=965 mm and thickness *d*=6 mm (Norville *et al.* 1993, in Table 1), laterally supported around its perimeter that is subjected to uniform wind pressure w perpendicular to its plane (Fig. 2(a)). The Young modulus of glass is assumed *E*=71700 MPa and the tensile strength due to bending is taken σ_{tu}=120 MPa (Pankhardt 2010, in Table 3.1). The cross-sectional areas of bars of the elementary rectangular modeled solid of glass plate according the equations (1), (6), (9) and (10) of section 2.2 are A_x=14550.39 mm^2, A_1=141.13 mm^2, A_2=141.13 mm^2 and A_3=112.59 mm^2. The linear elastic stress-strain law for glass is applicable, until the first tension induced cracking occurs and the loading is implemented in increments of 0.5 kPa (Fig. 2(b)).

Required are the deflections and the tensions of bars of the glass plate model due to bending for

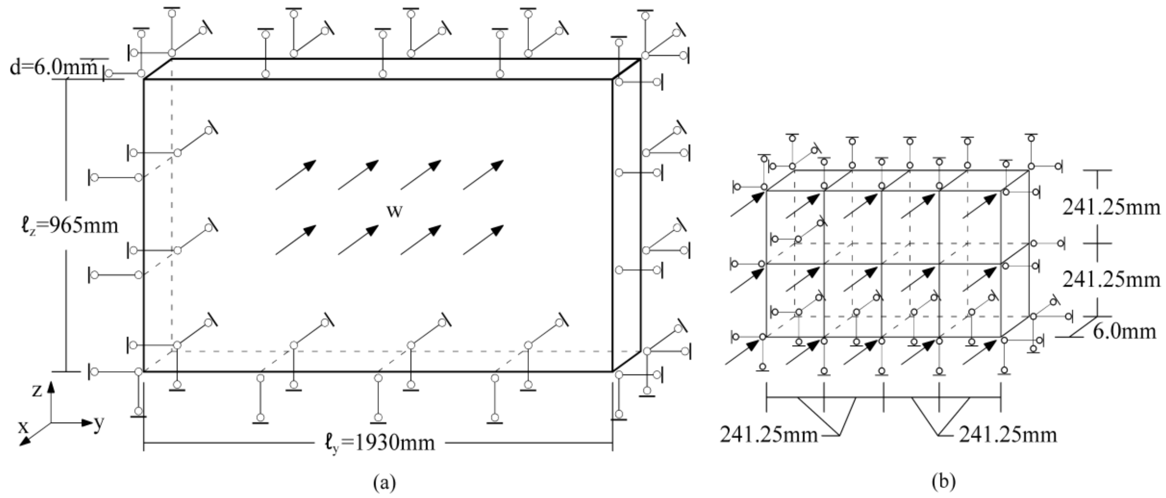

Fig. 2 Given data of the numerical example (a) Geometry of the glass plate, (b) The quarter of glass plate under study and its discretization into elementary rectangular solids with the loading to implemented in increments of 0.5 kPa

Table 1 Comparisons of test results by Norville *et al.* (1993) for three specimens of glass plates with corresponding results of the proposed truss model

	Norville *et al*. (1993) Specimens			Proposed truss model
	1	2	3	
Failure wind pressure in kPa	20.2	22.1	23.2	21.4
Central deflection at the moment of failure in mm	43.7	46.0	47.5	48.8

every step of loading as well as the magnitude of the uniform wind pressure (i.e., maxw) which is the critical loading (i.e., failure pressure), where the first tension induced cracking of the glass plate will occur and the maximum deflection (i.e., maxυ) at the center of the glass plate at the failure moment.

We observe that for uniform wind pressure max$w$=21.4 kPa the first tension induced cracking of a bar of the glass plate model appears (Fig. 3(b)), while the maximum deflection at the central area at the moment of failure is maxυ=48.8 mm (Fig. 3(a)). These results from the analysis are compared to published experimental results (Norville *et al.* 1993, in Table 3), where for three specimens of monolithic glass plates with the same geometrical, mechanical and loading characteristics as in the present application, there were measured at the failure moment, values of uniform wind pressure max$w$=20.2, 22.1 and 23.2 kPa and respective maximum deflections at the central area of the glass plates with values of maxυ=43.7, 46 and 47.5 mm. The experimental test results and the results obtained of the analysis are summarized in the above Table 1.

Therefore it is observed a satisfactory correlation between the results of the analysis of the proposed truss model to the experimental test results. Based on the results of the application, the diagram of the deformed average thickness for the examined glass plate at the moment of failure has been drawn (Fig. 3(a)), whereas in Fig. 3(b) the tensions of bars of truss model are noticed at the same moment, with value greater than 60 MPa.

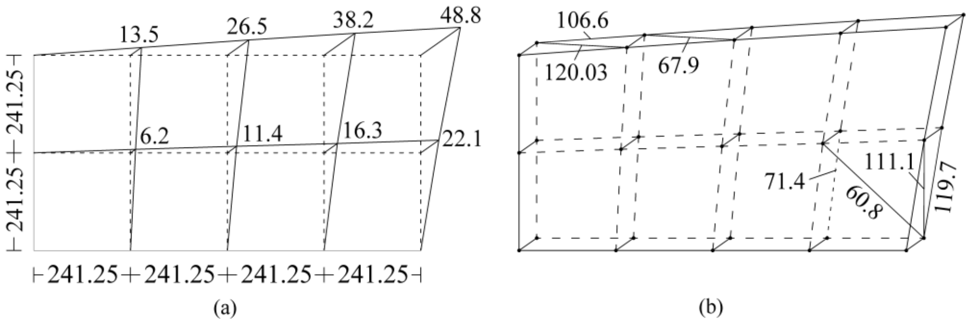

Fig. 3 State of the studied quarter of glass plate at the moment of failure (a) Deformed average thickness of glass plate (deflections in mm), (b) Stresses of segments (in MPa) with value greater than 60 MPa

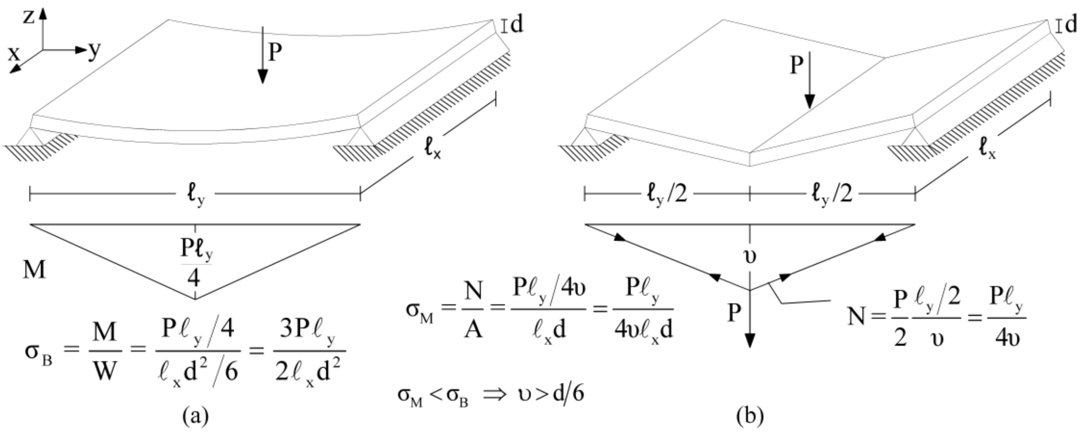

Fig. 4 (a) Bending plate behavior of a glass plate, (b) Tensioned membrane behavior of a glass plate

4. Notable remark

When the loading of a glass plate is applied as a whole from the beginning and not in increments, then is used the initial stiffness matrix that is applicable to the undeformed structure and is not taken into account the geometric nonlinearity due to large displacements.

However, the incremental loading and the application of stiffness and equilibrium conditions with respect to the deformed structure are not only more realistic but also result to smaller stresses, strains and deflections. This is due to the fact that as the deflections of the glass plate grow bigger, the bigger is the substitution of the bending plate behavior by a tensioned membrane behavior (Vallabhan 1983) which is favorable, causing lesser stresses, strains and deflections. This notable remark is explained in Fig. 4, where it is seen that to result the membrane stress σ_M smaller than the higher bending stress σ_B of the plate , is enough the higher deflection υ in the middle of the plate to be bigger of the plate thickness, with value $\upsilon > d/6$.

5. Conclusions

Due to the small thickness of monolithic glass plates, their large dimensions and the heavy wind pressure loads, the out-of-plane deflections are large compared to the plate thickness (von Karman plates) and for this reason a geometric non-linear behavior of the bended glass plate appears. On the contrary, usually there is no inherent nonlinearity of stress-strain law for the glass, which remains practically within the linear elastic region until the first cracking occurs. For the above reasons the approximate lateral deflections in thin glass plates are achieved using the method of incremental loading and the equilibrium and stiffness conditions are written with respect to the deformed structure for every small rise in loading. A three-dimensional non-linear truss model and a short computer program are applied on a fully tempered rectangular glass plate, laterally supported around its perimeter subjected to uniform wind pressure perpendicular to its plane. The results of the analysis are compared and found to have a satisfactory approximation with published experimental results. So the proposed truss model seems to prove useful for predicting approximate lateral deflections in thin monolithic glass plates while the method is simple and the computation time is very small for a good satisfactory solution. Finally it is observed, that the larger the deflections, the greater the substitution of the bending plate behavior by a tensioned membrane behavior. This is favorable because leads to smaller stresses, strains and deflections as it was explained by a simple example.

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Appendix A: Listing of the computer program.

Table A.1 Main program

```
Program GEOMETRIC_NONLINEAR_TRUSS_3D
   Use DATA_MODULE
   Implicit none
   Integer(4) K,KB,L,R,ISMAX,ILOAD,I
   Real(8) LX,LY,LZ,STRESS,MAX_STRESS,MAX_DISPL
   OPEN(100,FILE="C:\............txt") ! Open input data file
   OPEN(200,FILE="C:\............txt") ! Open output data file
   !!! READING INPUT: START !!!
   READ(100, '(1x,I2,1x,I3,1x,F4.1,1x,F4.0,1x,F6.1,1x,F4.1,1x,F6.1,1x,F9.7,1x,F9.7,1x,F9.7)')
   & NN,NB,SMAX,NSTEPS,ELASTO,NI,MAXLOAD,A_L,B_L,D_L
   Allocate (IX(NN)) ; Allocate (IY(NN)) ; Allocate (IZ(NN)) ; Allocate (IXYZ(3*NN))  ; Allocate (N(NB))
   Allocate (U(3*NN)) ; Allocate (X(NN))  ; Allocate (Y(NN))  ; Allocate (Z(NN)) ;
   Allocate (STRESS_MAT(NB))
   Allocate (K_MAT(3*NN,3*NN)) ;  Allocate (FX(NN)) ; Allocate (FY(NN)) ; Allocate (FZ(NN)) ;
   Allocate (PX(NN))
   Allocate (PY(NN)) ; Allocate (PZ(NN)) ; Allocate (P(3*NN))  ; Allocate (CX(NB)) ; Allocate (CY(NB))
   Allocate (CZ(NB)) ; Allocate (KL(NB)) ; Allocate (KR(NB))  ; Allocate (A_TYPE(NB)) ;
   Allocate (A(NB))
   Allocate (L0(NB)) ; Allocate (LE(NB)) ; Allocate (UX(NN)) ; Allocate (UY(NN)) ; Allocate (UZ(NN))
   Allocate (DL(NB)) ; Allocate (E(NB)) ; Allocate (SUX(NN)) ; Allocate (SUY(NN)) ;
   Allocate (SUZ(NN))
   Do K=1,NN ; READ(100, '(1x,I1,1x,I1,1x,I1,1x,F7.5,1x,F7.5,1x,F7.5,1x,F9.5,1x,F9.5,1x,F9.5)' ) &
       IX(K), IY(K), IZ(K), X(K), Y(K), Z(K), PX(K), PY(K), PZ(K) ; End Do
   Do K=1,NB ; READ(100, '(1x,I2,1x,I2,1x,I3)' ) KL(K), KR(K), A_TYPE(K) ; End Do
   !!! READING INPUT: END !!!
   IXYZ=0 ; IXYZ(1)=IX(1) ; IXYZ(2)=IY(1) ; IXYZ(3)=IZ(1) ; K=1
   Do I=3,3*NN-3,3 ; IXYZ(I+1)=IX(I-K)  ; IXYZ(I+2)=IY(I-K) ; IXYZ(I+3)=IZ(I-K) ; K=K+2 ; End Do
   CALL AREA_CALCULATION  ! Calculation of Area of the cross-sections of bars
   Do KB=1,NB ; L=KL(KB) ; R=KR(KB) ; LX=X(R)-X(L) ; LY=Y(R)-Y(L) ; LZ=Z(R)-Z(L) ;
       L0(KB) = Sqrt(LX*LX+LY*LY+LZ*LZ) ; End Do
   Do K=1,NN ; FX(K)=0. ; FY(K)=0.  ; FZ(K)=0. ; SUX(K)=0. ; SUY(K)=0. ; SUZ(K)=0. ; End Do
   Do K=1,(3*NN) ; P(K)=0. ; End Do ; ISMAX=0 ; UX_r=0 ;  UY_r=0 ; UZ_r=0 ; Tot_r_n=0 ; DORSM=0
   ! Calculation of total number of the restrained degrees of freedom
   Do I=1,NN ; UX_r=UX_r + IX(I) ; UY_r=UY_r+IY(I) ; UZ_r=UZ_r+IZ(I) ; End Do ;
   Tot_r_n=UX_r+UY_r+UZ_r
   DORSM = 3*NN - Tot_r_n ! Dimension of the Restrained Stiffness Matrix (and of the restrained
   load matrix)
   Allocate (K_MAT_R(DORSM,DORSM)) ; Allocate (P_R(DORSM)) ;
   Allocate (K_Aux_MAT(DORSM,DORSM)) ;
   Allocate (P_Aux(DORSM)) ; Allocate (U_R(DORSM)) ; Allocate (U_Aux(DORSM))
   !!! BEGINNING OF THE ALGORITHM !!!
   LOAD = 1.0/NSTEPS ; ILOAD=1
   MAIN_LOOP: Do WHILE (ISMAX==0)
      CALL LOAD_MAT  ! FORMING THE LOAD MATRIX
      CALL STIFF_MAT  ! FORMING THE STIFFNESS MATRIX
      CALL GAUSS          ! SOLUTION OF THE EQUILIBRIUM EQUATIONS
```

Table A.1 Continued

```
Do K=1,NN            ! CALCULATION OF DISPLACEMENTS AND NEW COORDINATES
OF NODES
      UX(K)=U(3*K-2) ; UY(K)=U(3*K-1) ; UZ(K)=U(3*K) ; U(3*K-2)=0. ; U(3*K-1)=0. ; U(3*K)=0.
      X(K)=X(K)+UX(K) ; Y(K)=Y(K)+UY(K) ; Z(K)=Z(K)+UZ(K) ; End Do
    Do I=1,NB ! CALCULATION OF ELONGATIONS AND FORCES OF BARS
      L=KL(I) ; R=KR(I) ; LX=X(R)-X(L) ; LY=Y(R)-Y(L) ; LZ=Z(R)-Z(L)
      LE(I)=sqrt(LX*LX+LY*LY+LZ*LZ) ; DL(I)=LE(I)-L0(I) ; E(I)=DL(I)/L0(I)
      STRESS=ELASTO*E(I) ; If (STRESS.gt.SMAX) Then ; ISMAX=1 ; End If
      STRESS_MAT(I)=STRESS ; N(I)=STRESS*A(I) ; CX(I)=LX/LE(I) ; CY(I)=LY/LE(I) ;
      CZ(I)=LZ/LE(I)
      FX(L)=FX(L)+N(I)*CX(I) ; FY(L)=FY(L)+N(I)*CY(I) ; FZ(L)=FZ(L)+N(I)*CZ(I)
      FX(R)=FX(R)-N(I)*CX(I) ; FY(R)=FY(R)-N(I)*CY(I) ; FZ(R)=FZ(R)-N(I)*CZ(I) ; End Do
    MAX_STRESS=STRESS_MAT(1) ; Do I=1,NB ; If (STRESS_MAT(I).gt.MAX_STRESS) Then ;
    MAX_STRESS=STRESS_MAT(I) ; End If ; End Do
    ! CALCULATION OF TOTAL DISPLACEMENTS OF NODES
    Do K=1,NN ; SUX(K)=SUX(K)+UX(K) ; SUY(K)=SUY(K)+UY(K) ; SUZ(K)=SUZ(K)+UZ(K) ;
    End Do
    MAX_DISPL=abs(SUX(1)) ; Do I=1,NN ; If (abs(SUX(I)).gt.MAX_DISPL) Then ;
    MAX_DISPL=abs(SUX(I)) ;
   End If ; End Do
    ILOAD=ILOAD+1 ; LOAD=LOAD+(1.0/NSTEPS)
  End Do MAIN_LOOP
! OUTPUT PRINTING
  WRITE (200, '(1x,A,1x,F6.2)') "ULTIMATE WIND PRESSURE w (kPa):", (LOAD*MAXLOAD)
  WRITE (200, '(1x,A,1x,F6.2)') "MAXIMUM DISPLACEMENT (mm):", (MAX_DISPL*1000.)
  WRITE (200, '(1x,A)') "DISPLACEMENTS OF NODES AT THE MOMENT OF FAILURE (mm):"
  Do K=1,NN ; WRITE (200, '(1x,A,1x,I2,1x,A,1x,F6.1)') "NODE", K, ":", (abs(SUX(K))*1000.) ; End Do
  WRITE (200, '(1x,A)') "STATE OF BARS AT THE MOMENT OF FAILURE"
  WRITE (200, '(11x,A,5x,A,5x,A)') "LENGTH(m)","STRESS(MPa)", "AXIAL FORCE(kN)"
  Do I=1,NB ; if (STRESS_MAT(I).gt.(0.5*SMAX)) then ; WRITE (200,
'(1x,A,1x,I3,1x,A,1x,F8.3,8x,F6.1,11x,F6.2)') &
              "BAR",I,":",LE(I),(STRESS_MAT(I)*10.),N(I); end if ; End Do
End Program
```

Table A.2 Module DATA_MODULE

```
Module DATA_MODULE
  Implicit none
  INTEGER(4) NN,NB,DORSM,UX_r,UY_r,UZ_r,Tot_r_n
  REAL(8) SMAX,ELASTO,NI,NSTEPS,MAXLOAD,A_L,B_L,D_L,LOAD
  Real(8), Allocatable :: X(:)    ; Real(8), Allocatable :: Y(:)     ; Real(8), Allocatable :: Z(:)
  Integer(4), Allocatable :: IX(:) ; Integer(4), Allocatable :: IY(:)  ; Integer(4), Allocatable :: IZ(:)
  Real(8), Allocatable :: PX(:)   ; Real(8), Allocatable :: PY(:)    ; Real(8), Allocatable :: PZ(:)
  Real(8), Allocatable :: FX(:)   ; Real(8), Allocatable :: FY(:)    ; Real(8), Allocatable :: FZ(:)
  Real(8), Allocatable :: CX(:)   ; Real(8), Allocatable :: CY(:)    ; Real(8), Allocatable :: CZ(:)
  Real(8), Allocatable :: P(:)    ; Real(8), Allocatable :: P_Aux(:)  ; Integer(4), Allocatable :: A_TYPE(:)
  Integer(4), Allocatable :: KL(:) ; Integer(4), Allocatable :: KR(:)  ; Real(8), Allocatable :: L0(:)
  Real(8), Allocatable :: LE(:)   ; Real(8), Allocatable :: K_MAT(:,:) ; Real(8), Allocatable ::
  K_Aux_MAT(:,:)
```

Table A.2 Continued

```
Real(8), Allocatable :: U_R(:)   ; Real(8), Allocatable :: U(:)      ; Real(8), Allocatable :: U_Aux(:)
   Real(8), Allocatable :: UY(:)    ; Real(8), Allocatable :: UZ(:)     ; Real(8), Allocatable :: DL(:)
   Real(8), Allocatable :: E(:)     ; Real(8), Allocatable :: N(:)      ; Real(8), Allocatable :: SUX(:)
   Real(8), Allocatable :: SUY(:)   ; Real(8), Allocatable :: SUZ(:)    ; Real(8), Allocatable ::
   STRESS_MAT(:)
   Real(8), Allocatable :: A(:)     ; Real(8), Allocatable :: K_MAT_R(:,:) ; Real(8), Allocatable :: P_R(:)
   Integer(4), Allocatable :: IXYZ(:) ; Real(8), Allocatable :: UX(:)
End module
```

Table A.3 Subroutine GAUSS

```
Subroutine GAUSS
   Use DATA_MODULE
   Implicit none
   Integer(4) NM1,I,J,I1,K,IL
   Real(8) COEFF
   Do I=1,DORSM ; Do J=1,DORSM ; K_Aux_MAT(I,J)=K_MAT_R(I,J) ; End Do ; End Do
   Do I=1,DORSM ; P_Aux(I)=P_R(I) ; End Do
   NM1=DORSM-1
   Do I=1,NM1 ; I1=I+1 ; Do J=I1,DORSM
         COEFF=-(K_Aux_MAT(J,I))/(K_Aux_MAT(I,I)) ; P_Aux(J)=P_Aux(J)+P_Aux(I)*COEFF
         Do K=1,DORSM ; K_Aux_MAT(J,K)=K_Aux_MAT(J,K)+K_Aux_MAT(I,K)*COEFF ; End Do ;
         End Do ; End Do
   U_Aux(DORSM)=P_Aux(DORSM)/K_Aux_MAT(DORSM,DORSM)
   Do I=1,NM1 ; IL=DORSM-I ; I1=IL+1 ; U_Aux(IL)=P_Aux(IL)
      Do J=I1,DORSM ; U_Aux(IL)=U_Aux(IL)-K_Aux_MAT(IL,J)*U_Aux(J) ; End Do
      U_Aux(IL)=U_Aux(IL)/K_Aux_MAT(IL,IL) ; End Do
   Do I=1,DORSM ; U_R(I)=U_Aux(I) ; End Do
   U=0.0 ; K=1 ; Do I=1,(3*NN) ; IF(IXYZ(I)==1) CYCLE ; U(I)=U_R(K) ; K=K+1 ; End Do
End Subroutine
```

Table A.4 Subroutine LOAD_MAT

```
Subroutine LOAD_MAT
   Use DATA_MODULE
   Implicit none
   Integer(4) I,K
   Do K=1,NN ; P(3*K-2)=(PX(K)/NSTEPS)+FX(K)*(1-IX(K)) ; P(3*K-1)=(PY(K)/NSTEPS)+FY(K)*
   (1-IY(K))
      P(3*K)=(PZ(K)/NSTEPS)+FZ(K)*(1-IZ(K)) ; FX(K)=PX(K)*LOAD ; FY(K)=PY(K)*LOAD ;
      FZ(K)=PZ(K)*LOAD
   End Do
   ! Consideration of the constrained degrees of freedom
   K=1 ; Do I=1,3*NN ; IF(IXYZ(I)==1) CYCLE ; P_R(K)=P(I) ; K=K+1 ; END Do
End Subroutine
```

Table A.5 Subroutine STIFF_MAT

```
Subroutine STIFF_MAT
Use DATA_MODULE
Implicit none
Integer(4) I,J,L,K,R
Real(8) LX,LY,LZ,STIF0,STIFX,STIFY,STIFZ,STIFXY,STIFXZ,STIFYZ
! Initialization
K_MAT=0. ; K_MAT_R = 0. ; STIF0=0. ; STIFX=0. ; STIFY=0. ; STIFZ=0. ; STIFXY=0. ; STIFXZ=0. ;
STIFYZ=0.
Do I=1,NB
L=KL(I) ; R=KR(I) ; LX=X(R)-X(L) ; LY=Y(R)-Y(L) ; LZ=Z(R)-Z(L)
LE(I)=Sqrt(LX*LX+LY*LY+LZ*LZ) ; CX(I)=LX/LE(I) ; CY(I)=LY/LE(I) ; CZ(I)=LZ/LE(I)
STIF0=ELASTO*(A(I)/L0(I)) ; STIFX =STIF0*(CX(I)**2.) ; STIFY=STIF0*(CY(I)**2.) ;
STIFZ=STIF0*(CZ(I)**2.)
STIFXY = STIF0*(CX(I)*CY(I)) ; STIFXZ = STIF0*(CX(I)*CZ(I)) ; STIFYZ = STIF0*(CY(I)*CZ(I))
K_MAT((3*L-2),(3*L-2))=K_MAT((3*L-2),(3*L-2))+STIFX  ; K_MAT((3*L-2),(3*L-1))
=K_MAT((3*L-2),(3*L-1))+STIFXY
K_MAT((3*L-2),(3*L  ))=K_MAT((3*L-2),(3*L  ))+STIFXZ ; K_MAT((3*L-1),(3*L-2))
=K_MAT((3*L-1),(3*L-2))+STIFXY
K_MAT((3*L-1),(3*L-1))=K_MAT((3*L-1),(3*L-1))+STIFY  ; K_MAT((3*L-1),(3*L  ))
=K_MAT((3*L-1),(3*L  ))+STIFYZ
K_MAT((3*L  ),(3*L-2))=K_MAT((3*L  ),(3*L-2))+STIFXZ ; K_MAT((3*L  ),(3*L-1))
=K_MAT((3*L  ),(3*L-1))+STIFYZ
K_MAT((3*L  ),(3*L  ))=K_MAT((3*L  ),(3*L  ))+STIFZ  ; K_MAT((3*L-2),(3*R-2))
=K_MAT((3*L-2),(3*R-2))-STIFX
K_MAT((3*L-2),(3*R-1))=K_MAT((3*L-2),(3*R-1))-STIFXY ; K_MAT((3*L-2),(3*R  ))
=K_MAT((3*L-2),(3*R  ))-STIFXZ
K_MAT((3*L-1),(3*R-2))=K_MAT((3*L-1),(3*R-2))-STIFXY ;
K_MAT((3*L-1),(3*R-1))=K_MAT((3*L-1),(3*R-1))-STIFY
K_MAT((3*L-1),(3*R  ))=K_MAT((3*L-1),(3*R  ))-STIFYZ ; K_MAT((3*L  ),(3*R-2))
=K_MAT((3*L  ),(3*R-2))-STIFXZ
K_MAT((3*L  ),(3*R-1))=K_MAT((3*L  ),(3*R-1))-STIFYZ ; K_MAT((3*L  ),(3*R  ))
=K_MAT((3*L  ),(3*R  ))-STIFZ
K_MAT((3*R-2),(3*L-2))=K_MAT((3*R-2),(3*L-2))-STIFX  ; K_MAT((3*R-2),(3*L-1))
=K_MAT((3*R-2),(3*L-1))-STIFXY
K_MAT((3*R-2),(3*L  ))=K_MAT((3*R-2),(3*L  ))-STIFXZ ; K_MAT((3*R-1),(3*L-2))
=K_MAT((3*R-1),(3*L-2))-STIFXY
K_MAT((3*R-1),(3*L-1))=K_MAT((3*R-1),(3*L-1))-STIFY  ; K_MAT((3*R-1),(3*L  ))
=K_MAT((3*R-1),(3*L  ))-STIFYZ
K_MAT((3*R  ),(3*L-2))=K_MAT((3*R  ),(3*L-2))-STIFXZ ; K_MAT((3*R  ),(3*L-1))
=K_MAT((3*R  ),(3*L-1))-STIFYZ
K_MAT((3*R  ),(3*L  ))=K_MAT((3*R  ),(3*L  ))-STIFZ  ; K_MAT((3*R-2),(3*R-2))
=K_MAT((3*R-2),(3*R-2))+STIFX
K_MAT((3*R-2),(3*R-1))=K_MAT((3*R-2),(3*R-1))+STIFXY ;
K_MAT((3*R-2),(3*R  ))=K_MAT((3*R-2),(3*R  ))+STIFXZ
K_MAT((3*R-1),(3*R-2))=K_MAT((3*R-1),(3*R-2))+STIFXY ;
K_MAT((3*R-1),(3*R-1))=K_MAT((3*R-1),(3*R-1))+STIFY
K_MAT((3*R-1),(3*R  ))=K_MAT((3*R-1),(3*R  ))+STIFYZ ; K_MAT((3*R  ),(3*R-2))
=K_MAT((3*R  ),(3*R-2))+STIFXZ
```

Table A.5 Continued

```
K_MAT((3*R  ),(3*R-1))=K_MAT((3*R  ),(3*R-1))+STIFYZ ; K_MAT((3*R  ),(3*R  ))
=K_MAT((3*R  ),(3*R  ))+STIFZ
End Do
 ! Consideration of the constrained degrees of freedom
   K=1
   Do I=1,3*NN ; L=1 ; IF(IXYZ(I)==1) CYCLE
        Do J=1,3*NN ; IF(IXYZ(J)==1) CYCLE ; K_MAT_R(K,L)=K_MAT(I,J) ; L=L+1 ;
        End Do ; K=K+1
   End Do
End Subroutine
```

Table A.6 Subroutine AREA_CALCULATION

```
Subroutine AREA_CALCULATION
   Use DATA_MODULE
   Implicit none
   Real(8) AX, A1, A2, A3, tg_theta, ctg_theta, cos_theta, sin_theta, sin_2theta, EL
   Integer(4) KB,Area_type
   ! Initialization
   AX=0. ; A1=0. ; A2=0. ; A3=0. ; tg_theta=0. ; ctg_theta=0. ; cos_theta=0. ; sin_theta=0. ;
   sin_2theta=0. ; EL=0. ; A=0.
   ! Auxial Calculations
   cos_theta=A_L/(sqrt(A_L*A_L + B_L*B_L)) ; sin_theta=B_L/(sqrt(A_L*A_L + B_L*B_L))
   sin_2theta=2.*cos_theta*sin_theta ; tg_theta=B_L/A_L ; ctg_theta=1./tg_theta ; EL=A_L/cos_theta
   ! Calculation of area parameters
   AX= ((0.5*A_L)*(0.5*B_L))*10000. ; A1= ((1./8.)*((B_L*D_L)-NI*((A_L*D_L)/(tg_theta))))*10000.
   A2= ((1./8.)*((A_L*D_L)-NI*((B_L*D_L)/(ctg_theta))))*10000. ;
   A3= ((NI/4.)*((EL*D_L)/(sin_2theta)))*10000.
   ! Calculation of area of the cross-section for all bars
   Do KB=1,NB ; Area_type = A_TYPE(KB)
      Select case(Area_type)
      case(101) ; A(KB)=AX    ; case(201) ; A(KB)=2.*AX ; case(401) ; A(KB)=4.*AX ;
      case(102) ; A(KB)=A1
      case(202) ; A(KB)=2.*A1 ; case(103) ; A(KB)=A2    ; case(203) ; A(KB)=2.*A2 ;
      case(123) ; A(KB)=A3
      case(112) ; A(KB)=A1    ; case(212) ; A(KB)=2.*A1 ; case(113) ; A(KB)=A2    ;
      case(213) ; A(KB)=2.*A2
      End Select
   End Do
End Subroutine
```

Description of the basic parameters

Input parameters

NN = The number of nodes of the truss.
NB = The number of bar elements of the truss.

ELAST0 = Modulus of Elasticity (kN/cm^2).
NSTEPS = The number of steps of the incremental procedure.
SMAX = The strength of the glass in tension due to bending (kN/cm^2).
MAXLOAD= The final value of the wind pressure (kN/m^2).
A(NB)= Cross-section area of each bar (cm^2).
[X(NN)], [Y(NN)], [Z(NN)]= Initial coordinates of the nodes (m).
[IX(NN)], [IY(NN)], [IZ(NN)]= Vectors with elements which indicate the type of restraint of nodes along X, Y and Z axes: IX (or IY, or IZ)=0 for unrestrained nodes, IX (or IY, or IZ)=1 for restrained nodes.
[KL(NB)] = The index number of the node of the left edge of each bar.
[KR(NB)] = The index number of the node of the right edge of each bar.
[PX(NN)], [PY(NN)], [PZ(NN)] = External forces of nodes (kN).

Output parameters

MAX_DISPL= The maximum displacement at the moment of failure in axis parallel to wind pressure (mm).
STRESS_MAT(NB)= Matrix with stresses of bars at the moment of failure (MPa).
N(I)= The axial forces of bars at the moment of failure (kN).
SUX(K), SUY(K), SUZ(K)= Displacements of nodes at the moment of failure (mm).
LOAD=(1/NSTEPS)*MAXLOAD