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Construction of the shape functions of beam vibrations for analysis of the rectangular plates by Kantorovich-Vlasov's method

Emmanuel E.T. Olodo* and Gerard Degan[a]

Laboratory of Applied Mechanics and Energetic (LEMA), University of Abomey-Calavi, Benin

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Abstract. For analysis of the plates and membranes by numerical or analytical methods, the question of choice of the system of functions satisfying the different boundary conditions remains a major challenge to address. It is to this issue that is dedicated this work based on an approach of choice of combinations of trigonometric functions, which are shape functions of a bended beam with the boundary conditions corresponding to the plate support mode. To do this, the shape functions of beam vibrations for strength analysis of the rectangular plates by Kantorovich-Vlasov's method is considered.Using the properties of quasi-orthogonality of those functions allowed assessing to differential equation for every member of the series. Therefore it's proposed some new forms of integration of the beam functions, in order to simplify the problem.

Keywords: shape functions ; beam vibrations ; beam functions ; Kantorovich-Vlasov's method

1. Introduction

Rectangular plates are one on of the most important structural elements that are of interest in the field of mechanics, civil engineering and structural engineering. The study of their free vibration behaviour is very important in the structural designers. Many researches for dynamic analyses of thin rectangular plates are available in the literature. Misra (2012) studied the free vibration analysis of isotropic plate using multiquadric radial basis function.The symplectic duel method which gives exact solutions for the free vibration analysis of rectangular thin plates was used by Yufeng and Liu (2009) to study SSSS plate problems. Ezeh *et al*. (2013) have presented a variational solution for free vibration analysis of thin rectangular flat isotropic SSSS plate based on classical plate theory, using Taylor series. Since the origin of plate, Fourier and trigonometric series has been used to treat plate problems as shape function. There are many employed scholar approaches in solving the problem of plates like energy method, numerical method and equilibrium method. All these approaches give approximate results in most difficult situations.

There are several variational approaches for the analysis of the plates of which the best known are those for Ritz, Galerkin, Lagrange, etc. Besides these approaches, there are a number of

*Corresponding author, Assistant Professor, E-mail: olodoe@live.fr
[a]Professor, E-mail : ger_degan@yahoo.fr

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approximate ways of solving variational problem of plates related to boundary value problems. In this work, we are interested in the Kantorovich-Vlasov's method in order to propose a method that can contribute to accuracy improvement of solution procedure that falls between the exact solution of Sophie Germain plate differential equation and variational approaches. The Kantorovich-Vlasov variational method, which is a modification of the Ritz method, allows reducing the initial three-dimensional elasticity problem or bidimensional problem of the theory of thick plates, to one-dimensional problem involving a system of ordinary differential equations. In Kantorovich-Vlasov's method, the rectangular plate is modeled by two beams. In the direction *Ox* beam allows to choose the functions $X_i(x)$ and in the direction *Oy* it allows to find functions $w_i(y)$ through solution of Cauchy problem for beams bending.

Different approaches are used to analyze forced beam vibration at high amplitudes (Caddemi *et al*. 2013, Ning *et al*. 2013, Bors *et al*. 2013). Analytical models based on use of harmonic balance method and Galerkin method (with a single mode) (Azzar *et al*. 2002, Daya *et al*. 2004) have been developed. Although, for the study of the elasto-dynamic behaviour of continuous systems, the discrete form based on Galerkin method is an approach at first sight, more powerful than formulation based on the discrete write of the differential equations of movement, he quickly suggests its limitations in presence of complex geometries, its main disadvantage resides in fact that shape functions that characterize the real and virtual displacement approached fields, are defined globally on the whole studied area. Kazakov (2012) proposed study related to elastodynamic infinite elements, based on modified Bessel shape functions, appropriate for Soil-Srtucture Interaction. In this work it is demonstrated that, the application of the elastodynamical infinite elements is the easier and appropriate way to achieve an adequate simulation including basic aspects of Soil-Structure Interaction. Other interesting work, related to the vibration behaviour of the beams, are presented in Curnier (2005), Ekwaro-Osire *et al*. (2001), Han *et al*. (1999), Rakotomanana (2004). Many authors have used the Timoshenko beam (Jelenic *et al*. 2011, Triantafyllou *et al*. 2011) to describe vibration phenomena of the 1D structures. Interesting results were obtained also for the study of Rayleigh cantilever bridges Xi (2013). But the difficulty of the Rayleigh method is to choose the vector or the shape function to test. It is actually forced to seek a solution in a space of solutions to one dimension: vectors proportional to the chosen vector, shape functions proportional to the chosen shape function.

The purpose of this work is to propose a method of construction of shape functions of beam vibrations for strength analysis of a rectangular plate by Kantorovich-Vlasov's approach.

2. Materials and method

We will use the system of dynamic beams functions which are the shape functions of the beam vibrations. Indeed, these functions are based on the system of Krylov's functions (Timoshenko 1985).

Krylov's functions:

Consider the homogeneous differential equation

$$B\frac{d^4w_0}{dx^4} + kw_0 = 0 \tag{1}$$

The analytical solution of Eq. (1) will have the following form (Krylov 1931, Iakimov 1971)

$$w_0(x) = C_1F_1(\beta x) + C_2F_2(\beta x) + C_3F_3(\beta x) + C_4F_4(\beta x) \tag{2}$$

Here, $\beta = \sqrt[4]{\frac{k}{4B}}$; C_1, C_2, C_3, C_4 – undetermined coefficients,
F_1, F_2, F_3 et F_4 are the Krylov's functions that can be expressed as follows

$$F_1(\beta x) = ch\beta x \cdot cos\beta x = \frac{e^{\beta x} + e^{-\beta x}}{2} cos\beta x$$

$$F_2(\beta x) = \frac{1}{2\beta}(ch\beta x \cdot sin\beta x + sh\beta x \cdot cos\beta x = \frac{1}{2\beta}\left(\frac{e^{\beta x} + e^{-\beta x}}{2} sin\beta x + \frac{e^{\beta x} - e^{-\beta x}}{2} cos\beta x\right)$$

$$F_3(\beta x) = \frac{1}{2\beta^2} sh\beta x \cdot sin\beta x = \frac{1}{2\beta^2}\frac{e^{\beta x} - e^{-\beta x}}{2} sin\beta x$$

$$F_4(\beta x) = \frac{1}{4\beta^4}(ch\beta x \cdot sin\beta x - sh\beta x \cdot cos\beta x) = \frac{1}{4\beta^4}\left(\frac{e^{\beta x}+e^{-\beta x}}{2} sin\beta x - \frac{e^{\beta x}-e^{-\beta x}}{2} cos\beta x\right) \quad (3)$$

One can simplify writing these functions by expressing variables without unit. For that, let's writing:
$\xi = \frac{x}{a}$; $\eta = \frac{y}{b}$; a and b are the dimensions of the rectangular plate. In unitless coordinates, the Krylov's functions are then wrote as follows

$$F_{1m}(\xi) = \frac{1}{2}(ch\gamma_m\xi + cos\gamma_m\xi);$$
$$F_{2m}(\xi) = \frac{1}{2}(sh\gamma_m\xi + sin\gamma_m\xi);$$
$$F_{3m}(\xi) = \frac{1}{2}(ch\gamma_m\xi - cos\gamma_m\xi);$$
$$F_{4m}(\xi) = \frac{1}{2}(sh\gamma_m\xi - sin\gamma_m\xi); \quad (4)$$

m- the number of terms of the series,
γ_m- are the eigenvalues of the differential equation of the beam vibrations i.e., coefficients depending on the beam boundary conditions and the number m of the terms of the series.

Dynamic beams functions are determined by the following formula (5)

$$Y_m(\eta) = F_{im}(\eta) - C_m F_{jm}(\eta), \quad (5)$$

i, *j* are selected so that they satisfy the beam boundary conditions at its left end (η = 0), C_m is chosen so that it satisfies one of the beam boundary conditions at its right end (η = 1). This last boundary condition satisfaction leads to a transcendent functional equation from which one can determine the coefficients γ_m.

The orthogonality and quasi-orthogonality properties of shape functions for beam vibrations are known (Timoshenko 1985).

$$\int_0^1 Y_m(\eta)Y_n(\eta)d\eta = \frac{1}{\gamma_m^4}\int_0^1 Y_m^{IV}(\eta)Y_n(\eta)d\eta = \frac{1}{\gamma_m^4}\int_0^1 Y_m''(\eta)Y_n''(\eta)d\eta = d_m^2 \cdot \delta_{mn}$$

$$\delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases} \qquad \frac{1}{\gamma_m^2}\int_0^1 Y_m''(\eta)Y_n(\eta)d\eta = -d_m''^2 \quad (6)$$

In case of beams having supports at the ends, the quasi-orthogonality property allows approximating integral (6) in form of : $\left|\int_0^1 Y_m''(\eta)Y_n(\eta)d\eta\right| << \gamma_m^2 d_m''^2$.

For the calculation of the plates by variational methods, because of quasi-orthogonality of the dynamic beams functions, the non-diagonal terms of the system become negligible and thus, yield a system of algebraic equations or system of differential equations. In this case, several opportunities for using this property become possible for analysis of plates by Kantorovich-Vlasov's method.

Indeed, consider the following differential equation for bending of plates

$$\nabla^4 w - \frac{q}{D} = 0 \tag{7}$$

where $q=q(x,y)$ represents the distributed load and D the flexural stiffness.

One can find the solution by single trigonometric series (Levy's Method). The solution of plate problems by a single trigonometric series may be considered as a specific application of the rigorous solutions. To obtain a particular solution by Levy's method, it is required that two opposite edges of the plate be simply supported and it is assume that the plate is infinitely long in the other direction.We assume that the edges at x=0 and $x=a$ are simply supported and that the origin of the coordinate system is moved to x=0 and $y = \frac{b}{2}$.

If we introduce coordinates without units by writing:

$\xi = \frac{x}{a}, \eta = \frac{y}{b}$ and $\lambda = \frac{a}{b}$ (a and b are the dimensions of the rectangular plate), the Eq. (7) in no dimensional coordinates (without units) will have the following form

$$\tilde{\nabla}^4 w - \frac{qa^4}{D} = 0 \tag{8}$$

where, $\tilde{\nabla}^4 \ldots = \frac{\partial^4 \ldots}{\partial \xi^4} + 2\lambda^2 \frac{\partial^4 \ldots}{\partial \xi^2 \partial \eta^2} + \lambda^4 \frac{\partial^4 \ldots}{\partial \eta^4}$.

In this case, the Levy's method allows us to find the solution of eqn (8) in the following form, (the boundary conditions being satisfied at longitudinal edges b of the plate

$$w(\xi,\eta) = \frac{q_0 a^4}{D} \sum_{m=1}^{\infty} X_m(\xi) Y_m(\eta)$$

$$(y = 0, \eta = 0 \; and \; y = b, \eta = 1) \tag{9}$$

$\xi = \frac{x}{a}$, $\eta=y/b$, in no dimensional coordinates $(0 \le \eta \le 1)$; $q_0 -$ a value corresponding to any load ; $Y_m(\eta)$ the beams functions satisfying the conditions to the longitudinal plate boundaries; $X_m(\xi)$ the function from the resolution of the differential equation based on Kantorovich-Vlasov's method taking into account the quasi orthogonality properties of beam functions (5)

$$X_m^{IV}(\xi) - 2\lambda^2 \gamma_m^2 b_m^2 \cdot X_m''(\xi) + \lambda^4 \gamma_m^4 \cdot X_m(\xi) = C_m(\xi) \tag{10}$$

where, $\lambda = \frac{a}{b}$; $C_m = \frac{1}{d_m^2} \int_0^1 \frac{q(\xi,\eta)}{q_0} Y_m(\eta) d\eta$, $b_m^2 = \frac{d_m''^2}{d_m^2}$.

3. Results and discussion

From differential Eq. (10) derive characteristic roots

$$\Gamma_{1-4} = \pm(\alpha_m \pm i\beta_m) \tag{11}$$

where, $\propto_m = \lambda\gamma_m\sqrt{\frac{1+b_m^2}{2}}$, $\quad \beta_m = \lambda\gamma_m\sqrt{\frac{1-b_m^2}{2}}$.

Except in the cases where supports are rolled to the longitudinal boundaries of the plate, the roots (11) are complex ($b_m^2 < 1$)when the Kantorovich-Vlasov's method solution coincides with that of Levy's method.

The main difficulty is related to calculation of integrals of quadratic beams functions and their products by the derivatives of these functions.

When *m≠n*, the resolution passes by integration by parts, and in particular one can show the orthogonality of the dynamic beams functions.

It is shown in Timoshenko (1985), since a limit transition yields the integral of quadratic beams functions in the following form

$$d_m^2 = \int_0^1 Y_m^2(\xi)d\xi = \frac{1}{4}\left[Y_m^2(1) + \frac{1}{\gamma_m^4}Y_m^{''2}(1) - 2Y_m'(1)Y_m'''(1)\right] \tag{12}$$

so the quadratic form of the beams functions is expressed using the values of these functions and their derivatives at the left end of the beam. By this same limit transition, we can calculate the integral of the product of the beam function by its second derivative.

In this way, it can be calculate quadratic dynamic beams functions. In all cases, this presents no difficulty for calculations on computer. However, the use of analytical methods for the study of these functions is not always convenient because they represent rather complicated combinations of Krylov's functions.

It therefore arises the question of why the integrals are expressed by the values of the functions on the right end of the beam and not on the left end. The answer is simple. In particular, for the beams to the symmetric boundary conditions, from symmetry conditions (or reverse symmetry), combinations of the values of the dynamic beam functions to left (*ξ*=0) and right (*ξ*=0) boundaries of the beam, in the formula (12) are equal.

We can consider that this condition is justified for all conditions for which there exist a beam support. We have

$$d_m^2 = \int_0^1 Y_m^2(\xi)d\xi = \frac{1}{4}\left[Y_m^2(0) + \frac{1}{\gamma_m^4}Y_m^{''2}(0) - 2Y_m'(0)Y_0'''(0)\right] \tag{13}$$

This last formula allows, thinking to the simple Krylov's functions values, when $\xi = 0$, to get quickly with some reliability, expressions of their quadratures.

For example, in considering a beam embedded in the left end with support rolled at the right end, we have the following boundary conditions:

$-Y_m(0) = Y_m''(0) = 0, Y_m(1) = Y''(1) = 0$.

The shape function of beam vibrations are determined by the following formula:

$Y_m(\eta) = F_{3m}(\eta) - C_m F_4(\eta), C_m = \frac{F_{3m}(1)}{F_{4m}(1)}$.

$Y_m'(\eta) = \gamma_m(F_{2m}(\eta) - C_m F_3(\eta))$, $Y_m''(\eta) = \gamma_m^2(F_{1m}(\eta) - C_m F_2(\eta))$,

The γ_m coefficient satisfies the functional equation:

$F_{1m}(1)F_{4m}(1) - F_{2m}(1)F_{4m}(1) = 0$, whose roots have the following values:

$$\gamma_1 = 3{,}9266;\ \gamma_2 = 7{,}0685;\ \gamma_3 = 10{,}2108;\ \gamma_4 = 13{,}3518;$$

$\gamma_m = (m + \frac{1}{4})\pi$ if $m > 4$.

We get the quadrature value by formula (13):

$d_m^2 = \frac{1}{4\gamma_m^4} Y_m^{''2}(0) = \frac{1}{4}$.

The use of formula (12) would lead to much more tedious calculations. The integral of the product of the dynamic beam function by its second derivative may be calculated by the following formula

$$d_m^{''2} = \frac{1}{\gamma_m^2} \int_0^1 Y_m(\eta) Y_m^{''}(\eta) d\eta = -\frac{1}{4} \left[\frac{1}{\gamma_m} \left(Y_m(\eta) Y_m^{'}(\eta) + \frac{1}{\gamma_m^4} Y_m^{''}(\eta) Y_m^{'''}(\eta) \right) {}_0^1 - \left(Y_m^{'2}(0) + \frac{1}{\gamma_m^4} Y_m^{'''2}(0) \right) \right] \tag{14}$$

Even if formula (14) seems more complicated than the formula (13) and still has values of the function at the right end of the beam, it can be further simplified. For symmetrical conditions of support, it can be written in the following form

$$d_m^{''2} = \frac{1}{2\gamma_m} \left(Y_m(0) Y_m^{'}(0) + \frac{1}{\gamma_m^4} Y_m^{''}(0) Y_m^{'''}(0) \right) - \frac{1}{4} \left(Y_m^{'2}(0) + \frac{1}{\gamma_m^4} Y_m^{'''2}(0) \right) \tag{15}$$

Using formulas (14) and (15), one can get formulas for different types of supports, expressed from the Krylov's functions.

For *m*>3, they can be replaced by simplified asymptotic functions. Thus, for the studied beam, taking into account the boundary conditions we have

$$d_m^{''2} = -\frac{1}{4\gamma_m^5} \left(\frac{1}{\gamma_m} Y_m^{''}(0) Y_m^{'''}(0) + Y_m^{'''2}(0) \right) = \frac{1}{4} C_m \left(C_m - \frac{1}{\alpha_m} \right) \tag{16}$$

$d_1^{''2} = 0{,}18667$; $d_2^{''2} = 0{,}21463$; $d_3^{''2} = 0{,}22551$; $d_4^{''2} = 0{,}23128$: $d_5^{''2} = 0{,}23484$.

For *m*> 2, we can use the following asymptotic formula:

$d_m^{''2} = \frac{1}{4}\left(1 - \frac{1}{\gamma_m}\right)$, $d_3^{''2} = \frac{1}{4}\left(1 - \frac{1}{10{,}2108}\right) = 0{,}225516$.

4. Conclusions

From this work, we can retain the following:

1. It was suggested a method of construction of shape functions of beam vibrations for strength analysis of thin rectangular plates using Kantorovich-Vlasov's approach.
2. It has shown that to build the shape functions of beam vibrations for strength analysis of thin plates, one can use the system of the Krylov's functions (1).
3. The use of the properties of quasi orthogonality of the shape functions of beam vibrations allows to neglect the non-diagonal terms of the system and to establish a system of independent algebraic or differential equations.
4. It is proposed new forms of integration of the dynamic beam functions.
5. The proposed method is a contribution to the development of Fourier transform method of separation of variables, it allows obtaining an analytical approached solution to problems at boundaries of differential equations in partial derivatives.
6. It is finally shown that, one can significantly simplify the method of applying a number of terms of the series, under various boundary conditions, using simple asymptotic formulas.

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