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Moment Lyapunov exponents of the Parametrical Hill's equation under the excitation of two correlated wideband noises

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Abstract. The Lyapunov exponent and moment Lyapunov exponents of Hill's equation with frequency and damping coefficient fluctuated by correlated wideband random processes are studied in this paper. The method of stochastic averaging, both the first-order and the second-order, is applied. The averaged Itô differential equation governing the *pth* norm is established and the *pth* moment Lyapunov exponents and Lyapunov exponent are then obtained. This method is applied to the study of the almost-sure and the moment stability of the stationary solution of the thin simply supported beam subjected to time-varying axial compressions and damping which are small intensity correlated stochastic excitations. The validity of the approximate results is checked by the numerical Monte Carlo simulation method for this stochastic system.

Keywords: elastic beam; eigenvalues; stochastic averaging; stochastic stability; Monte Carlo method

1. Introduction

The dynamic stability of elastic systems under two non-correlated random excitations has been investigated by many authors. There are numerous engineering structures that are subjected to the action of such loadings. The dynamic stability of these engineering structures is governed in general by the stability of the trivial solution of the stochastic differential equation of the form

$$\ddot{q}(t) + 2[\zeta + g(t)]\dot{q}(t) + [1 + f(t)]q(t) = 0 \qquad (1)$$

where $g(t)$ and $f(t)$ are the stochastic processes and ζ is the damping coefficient.

Kapitaniak (1986) studied the non–Markovian process defined by Hill's Eq. (1) with frequency and damping coefficient fluctuated by a non-white noise stochastic process. The stability of the first and second-order moments of the solution process was given by the well-known condition of stability of the differential equation with constant coefficients. Kozin and Wu (1973) obtained numerically sufficient almost-sure asymptotic stability boundaries when only one of stochastic processes $g(t)$ and $f(t)$ was present. Ariaratnam and Ly (1989) obtained optimal results when

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both $g(t)$ and $f(t)$ were present, by solving the envelope of the boundaries. The regions of the almost-sure asymptotic stability were obtained for arbitrary ergodic processes as well as for ergodic Gaussian processes. The moment Lyapunov exponents, which are defined by

$$\Lambda_q(p) = \lim_{t\to\infty} \frac{1}{t} \log E\left[\left\|q(t; q_{0,}\dot{q}_0)\right\|^p\right] \tag{2}$$

characterize the moment stability of a stochastic dynamical system with state vector $\boldsymbol{q}(t; q_{0,}\dot{q}_0) = \{q(t; q_{0,}\dot{q}_0), \dot{q}(t; q_{0,}\dot{q}_0)\}^T$, where $E[\cdot]$ denotes the expected value and $\|\cdot\|$ denotes a suitable vector norm. The *pth* moment of the response of the system is asymptotically stable if $\Lambda_q(p) < 0$. Moreover, $\Lambda_q(p)$ is a convex function of p and $\Lambda'_q(0)$ is equal to the largest Lyapunov exponent λ_q, which is defined by

$$\lambda_q = \lim_{t\to\infty} \frac{1}{t} \log\left\|q(t; q_{0,}\dot{q}_0)\right\| \tag{3}$$

and describes the almost-sure or simple stability of the system. Generally speaking, the Lyapunov exponent is easier to obtain. However, in general, the almost-sure stability cannot assure the moment stability. Therefore, it is important to obtain the moment Lyapunov exponents of stochastic systems so that the complete properties of dynamic stability can be described. The method of averaging for stochastic dynamic systems was proposed by Stratonovich (1963) and developed by Khasminskii (1966). The purpose is to approximate the solution of a stochastic dynamic system by a Markov diffusion process which satisfies the Itô stochastic differential equation when the excitation is a wideband process. After this approximation, it may be easier to obtain the solution or its dynamic properties of the approximated or averaged system. Ariaratnam and Tam (1979) applied a method of stochastic averaging to obtain the steady-state values of the response amplitude considering a linear second-order oscillator subjected to both parametric and forced excitations of small intensity. The moments of the solutions of the averaged equations were evaluated and conditions for stability in the moments were obtained. It was found that only those values of the spectral densities of the parametric excitations which correspond to zero frequency and are twice the system natural frequency influence the stability. Ling *et al*. (2011) investigated the response and stability of a single degree-of-freedom viscoelastic system with strongly nonlinear stiffness under excitations of wide band noise. Firstly, terms associated with the viscoelasticity were approximately equivalent to damping and stiffness forces; the viscoelastic system was approximately transformed to a single degree-of-freedom system without viscoelasticity. Then, with the application of the method of stochastic averaging of the first order, the averaged Itô differential equation was obtained. The stationary response and the largest Lyapunov exponent can be analytically expressed. Bai and Zhang (2012) applied the stochastic averaging method of the first order for quasi-integrable-Hamiltonian systems to obtain the averaged equations and formulated the expression for the largest Lyapunov exponent. The necessary and sufficient conditions for the almost sure asymptotic stability of the rotor system were presented approximately. The largest Lyapunov exponent was evaluated and employed to determine the region of almost sure asymptotic stability of rotor systems with random axial loads. In the paper Liu *et al.* (2013), the asymptotic Lyapunov stability with probability one of n-degree-of-freedom quasi non-integrable Hamiltonian systems subjected to weakly parametric excitations of combined Gaussian and Poisson white noises was studied by using the largest Lyapunov exponent. Hijawi *et al.* (1997a, b) developed a unified second-order stochastic averaging approach to treat dynamic systems with weak stiffness and inertia nonlinearities. The mathematical modeling of the governing equation and the first and second-order averaging methods were used to determine the response statistics and stochastic stability. The results were compared with those obtained by Gaussian and non-Gaussian closures and by Monte Carlo simulation. Rozycki and Zembaty (2011) presented an analysis of a stochastic eigenvalue problem of plane bar structures. Particular attention was paid to the effect of spatial variations of the flexural properties of the structure on the first four eigenvalues. The stochastic eigenvalue problem was solved independently by the stochastic finite element method and Monte Carlo techniques. In the paper Huang and Xie (2008), the method of averaging, both the first order and second order, was used to obtain the differential equation governing the *p*th moment. The moment stability of a viscoelastic system can be determined by solving the averaged equation. Kozić *et al*. (2008) obtained explicit expressions for an asymptotic expansion of the moment and almost-sure stability boundaries of the simply supported beam which was subjected to the axial compressions and varying damping which were two uncorrelated random processes. Papers Li *et al*. (2013), Xu *et al.* (2011, 2012, 2013) investigate the stochastic stability for nonlinear system with Lévy process and correlated Gaussian colored noises based on Lyapunov exponents. A method of equivalent linearization is proposed to reduce and simplify the original systems. The mean square responses are carried out to verify the effectiveness of the proposed approach, then the Lyapunov exponents will be defined and derived to explore the stochastic stability. A novel structural damage detection method with a new damage index was recently proposed by authors Zhang *et al.* (2011) based on the statistical moments of dynamic responses of shear building structures subject to white noise ground motion. The statistical moment based damage detection method was theoretically extended with general application. This method is more versatile and can identify damage locations and damage severities of many types of building structures under various external excitations. Liu *et al*. (2012) investigated dynamic responses of axially moving viscoelastic beam subjected to a randomly disordered periodic excitation. Based on the largest Lyapunov exponent, the almost sure stability of the trivial steady-state solution was examined. The authors then obtained the first-order and the second-order steady-state moments for the non-trivial steady-state solutions. In the paper Ku *et al.* (2013) provides a three-stage identification procedure as a solution to the problem of harmonic and white noise excitations in the acceleration responses of a linear dynamic system. This procedure combines the uses of the mode indicator function, the complex mode indication function, the enhanced frequency response function, an iterative rational fraction polynomial method and mode shape inspection for the correlation-related functions of the force-embedded acceleration responses. The procedure is verified via numerical simulation.

In this paper, the method of stochastic averaging, both the first and the second order, will be used to determine the response statistics. The averaged Itô differential equation governing the *p*th norm is determined and the *p*th moment Lyapunov exponents are then obtained for an elastic structure in the first mode subjected to two correlated wideband random processes. The variations of the moment Lyapunov exponents with the change of different parameters of the system are discussed and compared with the results from the paper Kozić *et al*. (2008). Furthermore, these results are compared with those obtained by the Monte Carlo simulation.

2. Discretization of the equation of the motion

We will now present an example which gives the best illustration of the theoretical results. In this sense, consider the elastic beam subjected to stochastically fluctuating axial compressions and damping force. It is assumed that the boundaries are simply supported. The motion of the beam governed by the partial differential equation, considered by Pavlović *et al*. (2005), by introducing a small parameter ε, is given by

$$L(w)=\frac{\partial^2 w}{\partial t^2}+2\left[\varepsilon\alpha+\sqrt{\varepsilon}g(t)\right]\frac{\partial w}{\partial t}+\frac{\partial^4 w}{\partial z^4}+\left[f_0+\sqrt{\varepsilon}f(t)\right]\frac{\partial^2 w}{\partial z^2}=0 \tag{4}$$

with the following homogeneous boundary conditions

$$\left.\begin{matrix} z=0 \\ z=1 \end{matrix}\right\}, \qquad w(z,t)=0, \qquad \frac{\partial^2 w(z,t)}{\partial z^2}=0. \tag{5}$$

The quantities α and f_0 in Eq. (4) are the positive constants, $g(t)$ and $f(t)$ are the wideband stationary correlated noises with zero mean. In order to further simplify Eq. (4), a mode of the Galerkin method will be used for reducing Eq. (4) to a corresponding ordinary differential equation representing only the time varying part of the solution. Consider the shape function $\sin\pi z$, which satisfies the boundary conditions (5), the first mode of the transverse motion of the beam can be described by

$$w(z,t)=q(t)\sin\pi z. \tag{6}$$

Furthermore, Galerkin's method requires that

$$\int_0^1 L(w)\delta w dz=0. \tag{7}$$

By substituting (4) and (6) into (7) and evaluating the integral as indicated, it follows that the given shape function will satisfy the following ordinary differential equation

$$\ddot{q}(t)+2\varepsilon\alpha\dot{q}(t)+\omega_0^2\left[1-\sqrt{\varepsilon}\xi_1(t)\right]q(t)+\sqrt{\varepsilon}\omega_0\xi_2(t)\dot{q}(t)=0, \tag{8}$$

where $\omega_0^2=\pi^4-\pi^2 f_0$ and

$$\xi_1(t)=\frac{\pi^2}{\omega_0^2}f(t), \qquad \xi_2(t)=\frac{2}{\omega_0}g(t), \tag{9}$$

are the wideband stationary correlated stochastic noises with zero mean.

3. First-order stochastic averaging

The motion of system (8) can be described by an asymptotic solution in terms of the behavior of its amplitude and phase angle. The stationary response possesses an amplitude and phase which vary slowly around some average values. In order to use the method of stochastic averaging, the following transformation is applied

$$q(t)=a(t)\cos\Phi(t)\ , \quad \dot{q}(t)=-\omega_0\, a(t)\sin\Phi(t) \tag{10}$$

where $\Phi(t)=\omega_0 t+\varphi(t)$. From Eq. (10), one has

$$\dot{a}(t)\cos\Phi(t)-a(t)\dot{\varphi}(t)\sin\Phi(t)=0. \tag{11}$$

Substituting Eq. (10) into Eq. (8) yields

$$\dot{a}(t)\sin\Phi(t) + a(t)\dot{\varphi}(t)\cos\Phi(t) =$$
$$-2\varepsilon\alpha a(t)\sin\Phi(t) - \sqrt{\varepsilon}\omega_0\xi_1(t)a(t)\cos\Phi(t) - \sqrt{\varepsilon}\omega_0\xi_2(t)a(t)\sin\Phi(t). \qquad (12)$$

Letting $P = a^p$, it is easy to see that P is the *pth* norm of system (8). Thus, from Eqs. (11) and (12), $\dot{P}(t)$ and $\dot{\varphi}(t)$ can be solved and written in the standard form, as

$$\dot{P}(t) = \varepsilon F_1^{(1)}(P,\varphi,t) + \sqrt{\varepsilon}\left[F_{11}^{(1)}(P,\varphi,\xi_1,t) + F_{12}^{(1)}(P,\varphi,\xi_2,t)\right], \qquad (13)$$
$$\dot{\varphi}(t) = \varepsilon F_2^{(1)}(\varphi,t) + \sqrt{\varepsilon}\left[F_{21}^{(1)}(\varphi,\xi_1,t) + F_{22}^{(1)}(\varphi,\xi_2,t)\right],$$

where

$$F_1^{(1)}(P,\varphi,t) = -2\alpha pP\sin^2\Phi \qquad F_2^{(1)}(\varphi,t) = -\alpha\sin 2\Phi\,,$$
$$F_{11}^{(1)}(P,\varphi,\xi_1,t) = -\frac{\omega_0 pP}{2}\xi_1\sin 2\Phi\,, \qquad F_{21}^{(1)}(\varphi,\xi_1,t) = -\omega_0\xi_1\cos^2\Phi, \qquad (14)$$
$$F_{12}^{(1)}(P,\varphi,\xi_2,t) = -\omega_0 pP\xi_2\sin^2\Phi\,, \qquad F_{22}^{(1)}(\varphi,\xi_2,t) = -\frac{\omega_0}{2}\xi_2\sin 2\Phi.$$

Assume that $\xi_1(t)$, $\xi_2(t)$ are stationary wideband random correlated processes with zero means and with correlation matrix $\left[K_{ij}(t)\right]$, i,j=1,2. If the coefficients of Eq. (13) are sufficiently smooth, the processes $\xi_1(t)$, $\xi_2(t)$ have sufficiently good mixing properties and correlation matrix $\left[K_{ij}(t)\right]$ decreases sufficiently quickly when $\tau\to\infty$, as shown in Hijawi *et al*. (1997a). Then, there is a limit Markov diffusion process, as $\varepsilon\to 0$, which can be described by the well-known Itô stochastic differential equations

$$d\bar{P}(t) = \varepsilon\bar{m}_p + \sqrt{\varepsilon}\bar{\sigma}_{11}dB_1(t) + \sqrt{\varepsilon}\bar{\sigma}_{12}dB_2(t), \qquad (15)$$
$$d\bar{\varphi}(t) = \varepsilon\bar{m}_\varphi + \sqrt{\varepsilon}\bar{\sigma}_{21}dB_1(t) + \sqrt{\varepsilon}\bar{\sigma}_{22}dB_2(t),$$

where $B_1(t)$ and $B_2(t)$ are the two Brownian motion processes. When applying the averaging operation, $P(t)$ and $\varphi(t)$ are treated as unchanged, i.e. they are replaced by $\bar{P}(t)$ and $\bar{\varphi}(t)$ directly. The elements of drift vector $\bar{m}_p$ and $\bar{m}_\varphi$, and of the diffusion matrix $\boldsymbol{b}(\bar{P},\bar{\varphi}) = \boldsymbol{\sigma}(\bar{P},\bar{\varphi})\boldsymbol{\sigma}^T(\bar{P},\bar{\varphi})$ are given by the following expressions according to the Khasminskii limit theorem, Lin and Cai (1995)

$$\bar{m}_p = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{F_1^{(1)}(P,\varphi,t) + \sum_{k=1}^{2}\sum_{l=1}^{2}\int_{-\infty}^{0}\left[\frac{\partial F_{1k}^{(1)}(P,\varphi,\xi_k(t),t)}{\partial P}F_{1l}^{(1)}(P,\varphi,\xi_l(t+\tau),t+\tau)K_{kl}(\tau)\right.\right.$$
$$\left.\left. + \frac{\partial F_{1k}^{(1)}(P,\varphi,\xi_k(t),t)}{\partial\varphi}F_{2l}^{(1)}(P,\varphi,\xi_l(t+\tau),t+\tau)K_{kl}(\tau)\right]d\tau\right\}dt\,,$$

$$\bar{m}_\varphi = \lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{F_2^{(1)}(P,\varphi,t)\right.$$
$$+\sum_{k=1}^{2}\sum_{l=1}^{2}\int_{-\infty}^{0}\left[\frac{\partial F_{2k}^{(1)}(P,\varphi,\xi_k(t),t)}{\partial P}F_{1l}^{(1)}\big(P,\varphi,\xi_j(t+\tau),t+\tau\big)K_{kl}(\tau)\right.$$

$$+\frac{\partial F_{2k}^{(1)}(P,\varphi,\xi_k(t),t)}{\partial\varphi}F_{2l}^{(1)}(P,\varphi,\xi_l(t+\tau),t+\tau)K_{kl}(\tau)\Bigg]d\tau\Bigg\}dt, \tag{16}$$

$$b_{ij}=\lim_{T\to\infty}\frac{1}{T}\int_0^T\left\{\sum_{k=1}^{2}\sum_{l=1}^{2}F_{ik}^{(1)}(P,\varphi,\xi_k(t),t)F_{jl}^{(1)}(P,\varphi,\xi_l(t+\tau),t+\tau)K_{kj}(\tau)\right\}dt,$$

$$\boldsymbol{\sigma}(\bar{P},\bar{\varphi})=[\bar{\sigma}_{ij}],\qquad i,j=1,2.$$

Now, substituting Eq. (14) into system (16) and applying Khasminskii's limit theorem (1966), the following expressions for the drift and diffusion coefficients $\bar{m}_p$, $\bar{m}_\varphi$ and b_{11}, b_{12}, b_{21} and b_{22} are obtained

$$\bar{m}_p=-p\bar{P}\alpha+\bar{P}\omega_0^2\left\{\frac{p^2}{8}S_2(0)+\frac{p(p+2)}{16}[S_1(2\omega_0)+S_2(2\omega_0)-2\Psi_{12}(2\omega_0)]\right\},$$

$$\bar{m}_\varphi=\frac{\omega_0^2}{8}[\Psi_1(2\omega_0)+\Psi_2(2\omega_0)-2S_{12}(2\omega_0)],\qquad b_{12}=b_{21}=\frac{\bar{P}\omega_0^2p}{4}S_{12}(0), \tag{17}$$

$$b_{11}=\bar{P}^2p^2b_{22},\ b_{22}=\frac{\omega_0^2}{8}[S_1(2\omega_0)+S_2(2\omega_0)+2S_2(0)-2\Psi_{12}(2\omega_0)],$$

where S and Ψ are the cosine and sine power spectral density function of noises $\xi_1(t)$ and $\xi_2(t)$ are given by

$$S_1(\omega_0)=\int_{-\infty}^{\infty}K_{11}(\tau)\cos\omega_0\tau\,d\tau,\qquad S_2(\omega_0)=\int_{-\infty}^{\infty}K_{22}(\tau)\cos\omega_0\tau\,d\tau, \tag{18}$$

$$S_{12}(\omega_0)=\int_{-\infty}^{\infty}K_{12}(\tau)\cos\omega_0\tau\,d\tau,\qquad \Psi_{12}(\omega_0)=-\int_{-\infty}^{\infty}K_{12}(\tau)\sin\omega_0\tau\,d\tau.$$

It is clear that $\bar{P}$ does not depend on $\bar{\varphi}$, thus the first equation of system (15) can be solved independently. Considering the property of Brownian motion processes, it is clear that expectations of the second and third term are zero. Taking the expected value on both sides of the first equation of system (15) gives

$$dE[\bar{P}(t)]=\varepsilon p\left\langle-\alpha+\omega_0^2\left\{\frac{p}{8}S_2(0)+\frac{p+2}{16}[S_1(2\omega_0)+S_2(2\omega_0)-2\Psi_{12}(2\omega_0)]\right\}\right\rangle E[\bar{P}]dt. \tag{19}$$

From (19), according to (2), the moment Lyapunov exponents for the averaged system are

$$\Lambda(p)=\lim_{t\to\infty}\frac{\log E[\bar{P}(t)]}{t}$$
$$=\varepsilon p\left\langle-\alpha+\omega_0^2\left\{\frac{p}{8}S_2(0)+\frac{p+2}{16}[S_1(2\omega_0)+S_2(2\omega_0)-2\Psi_{12}(2\omega_0)]\right\}\right\rangle \tag{20}$$

and the Lyapunov exponent is given by

$$\lambda=\left.\frac{\mathrm{d}\Lambda(p)}{\mathrm{d}p}\right|_{p=0}=\varepsilon\left\{-\alpha+\frac{\omega_0^2}{8}[S_1(2\omega_0)+S_2(2\omega_0)-2\Psi_{12}(2\omega_0)]\right\}. \tag{21}$$

The boundaries for the almost-sure stability and the *pth* moment stability are determined by λ=0 and $\Lambda(p)$=0, respectively.

4. Second-order stochastic averaging

The first-order stochastic averaging may not be adequate in some apllications. Similar to deterministic systems, higher-order averaging may be applied to obtain better approximations. In this section, the second order averaging method from Hijawi *et al*. (1997a, b) is applied and the results are compared with those obtained using the first-order averaging. The terms in $F_1^{(1)}$ and $F_2^{(1)}$ contain products of sine and cosine functions with phase angle $\Phi(t)$. The functions with higher-order multiple phase angle represent rapid oscillations or higher harmonics in the solution for the slowly varying amplitude and phase shift. When considering only the system stationary response, the high-frequency oscillations have a localized effect and do not contribute significantly to the average behavior of the system over a long period of time. We can therefore eliminate the oscillatory effects and simplify the equations of motion by introducing the near-identity transformation

$$P(t) = \bar{P}(t) + \varepsilon P_1(\bar{P}, \bar{\varphi}, t), \tag{22}$$
$$\varphi(t) = \bar{\varphi}(t) + \varepsilon \varphi_1(\bar{P}, \bar{\varphi}, t),$$

where $\bar{P}(t)$ and $\bar{\varphi}(t)$ are the results of the first-order averaging. Differentiating Eq. (22) with respect to time t yields

$$\begin{Bmatrix} \dot{P}(t) \\ \dot{\varphi}(t) \end{Bmatrix} = \boldsymbol{A} \begin{Bmatrix} \dot{\bar{P}} \\ \dot{\bar{\varphi}} \end{Bmatrix} + \varepsilon \begin{Bmatrix} \dfrac{\partial P_1}{\partial t} \\ \dfrac{\partial \varphi_1}{\partial t} \end{Bmatrix}, \tag{23}$$

where

$$\boldsymbol{A} = \begin{bmatrix} 1 + \varepsilon \dfrac{\partial P_1}{\partial \bar{P}} & \varepsilon \dfrac{\partial P_1}{\partial \bar{\varphi}} \\ \varepsilon \dfrac{\partial \varphi_1}{\partial \bar{P}} & 1 + \varepsilon \dfrac{\partial \varphi_1}{\partial \bar{\varphi}} \end{bmatrix}. \tag{24}$$

It is easy to determine that

$$\boldsymbol{A}^{-1} = \begin{bmatrix} 1 - \varepsilon \dfrac{\partial P_1}{\partial \bar{P}} & -\varepsilon \dfrac{\partial P_1}{\partial \bar{\varphi}} \\ -\varepsilon \dfrac{\partial \varphi_1}{\partial \bar{P}} & 1 - \varepsilon \dfrac{\partial \varphi_1}{\partial \bar{\varphi}} \end{bmatrix}. \tag{25}$$

Substituting Eq. (13) in Eqs. (23) we have that

$$\begin{Bmatrix} \dot{\bar{P}} \\ \dot{\bar{\varphi}} \end{Bmatrix} = \varepsilon \boldsymbol{A}^{-1} \begin{Bmatrix} F_1^{(1)} - \dfrac{\partial P_1}{\partial t} \\ F_2^{(1)} - \dfrac{\partial \varphi_1}{\partial t} \end{Bmatrix} + \sqrt{\boldsymbol{\varepsilon}} \boldsymbol{A}^{-1} \begin{Bmatrix} F_{11}^{(1)} + F_{12}^{(1)} \\ F_{21}^{(1)} + F_{22}^{(1)} \end{Bmatrix}$$

$$
= \varepsilon \begin{Bmatrix} F_1^{(1)} - \dfrac{\partial P_1}{\partial t} \\ F_2^{(1)} - \dfrac{\partial \varphi_1}{\partial t} \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} -\dfrac{\partial P_1}{\partial \bar{P}}\left(F_1^{(1)} - \dfrac{\partial P_1}{\partial t}\right) - \dfrac{\partial P_1}{\partial \bar{\varphi}}\left(F_2^{(1)} - \dfrac{\partial \varphi_1}{\partial t}\right) \\ -\dfrac{\partial \varphi_1}{\partial \bar{P}}\left(F_1^{(1)} - \dfrac{\partial P_1}{\partial t}\right) - \dfrac{\partial \varphi_1}{\partial \bar{\varphi}}\left(F_2^{(1)} - \dfrac{\partial \varphi_1}{\partial t}\right) \end{Bmatrix} \tag{26}
$$

$$
+\sqrt{\varepsilon} \begin{Bmatrix} F_{11}^{(1)} + F_{12}^{(1)} \\ F_{21}^{(1)} + F_{22}^{(1)} \end{Bmatrix} + \varepsilon^{\frac{3}{2}} \begin{Bmatrix} -\dfrac{\partial P_1}{\partial \bar{P}}\left(F_{11}^{(1)} + F_{12}^{(1)}\right) - \dfrac{\partial P_1}{\partial \bar{\varphi}}\left(F_{21}^{(1)} + F_{22}^{(1)}\right) \\ -\dfrac{\partial \varphi_1}{\partial \bar{P}}\left(F_{11}^{(1)} + F_{12}^{(1)}\right) - \dfrac{\partial \varphi_1}{\partial \bar{\varphi}}\left(F_{21}^{(1)} + F_{22}^{(1)}\right) \end{Bmatrix}
$$

i.e.

$$
\begin{Bmatrix} \dot{\bar{P}} \\ \dot{\bar{\varphi}} \end{Bmatrix} = \varepsilon \begin{Bmatrix} F_1^* \\ F_2^* \end{Bmatrix} + \varepsilon^2 \begin{Bmatrix} F_1^{**} \\ F_2^{**} \end{Bmatrix} + \sqrt{\varepsilon} \begin{Bmatrix} F_{11}^* + F_{12}^* \\ F_{21}^* + F_{22}^* \end{Bmatrix} + \varepsilon^{\frac{3}{2}} \begin{Bmatrix} F_{11}^{**} + F_{12}^{**} \\ F_{21}^{**} + F_{22}^{**} \end{Bmatrix}, \tag{27}
$$

where

$$
F_1^* = -2\alpha p \bar{P} \sin^2 \bar{\Phi} - \frac{\partial P_1}{\partial t}, \quad F_2^* = -\alpha \sin 2\bar{\Phi} - \frac{\partial \varphi_1}{\partial t}, \quad \bar{\Phi}(t) = \omega_o t + \bar{\varphi}(t)
$$

$$
F_1^{**} = -\frac{\partial P_1}{\partial \bar{P}} F_1^* - \frac{\partial P_1}{\partial \bar{\varphi}} F_2^* - 2\alpha p \bar{P} \varphi_1 \sin 2\bar{\Phi} - 2\alpha p P_1 \sin^2 \bar{\Phi}
$$

$$
F_2^{**} = -\frac{\partial \varphi_1}{\partial \bar{P}} F_1^* - \frac{\partial \varphi_1}{\partial \bar{\varphi}} F_2^* - 2\alpha \varphi_1 \cos 2\bar{\Phi}
$$

$$
F_{11}^* = -\frac{\omega_0 p P}{2} \xi_1 \sin 2\bar{\Phi}\,, \qquad F_{12}^* = -\omega_0 p P \xi_2 \sin^2 \bar{\Phi}
$$

$$
F_{21}^* = -\omega_0 \xi_1 \cos^2 \bar{\Phi}\,, \qquad F_{22}^* = -\frac{\omega_0}{2} \xi_2 \sin 2\Phi \tag{28}
$$

$$
F_{11}^{**} = -\frac{\partial P_1}{\partial \bar{P}} F_{11}^* - \frac{\partial P_1}{\partial \bar{\varphi}} F_{21}^* - \left(\omega_0 p \bar{P} \varphi_1 \cos 2\bar{\Phi} + \frac{\omega_0 p P_1}{2} \sin 2\bar{\Phi}\right) \xi_1
$$

$$
F_{12}^{**} = -\frac{\partial P_1}{\partial \bar{P}} F_{12}^* - \frac{\partial P_1}{\partial \bar{\varphi}} F_{22}^* - (\omega_0 p \bar{P} \varphi_1 \sin 2\bar{\Phi} + \omega_0 p P_1 \sin^2 \bar{\Phi}) \xi_2
$$

$$
F_{21}^{**} = -\frac{\partial \varphi_1}{\partial \bar{P}} F_{11}^* - \frac{\partial \varphi_1}{\partial \bar{\varphi}} F_{21}^* + \omega_0 \varphi_1 \sin 2\bar{\Phi}\, \xi_1, \quad F_{22}^{**} = -\frac{\partial \varphi_1}{\partial \bar{P}} F_{12}^* - \frac{\partial \varphi_1}{\partial \bar{\varphi}} F_{22}^* - \omega_0 \varphi_1 \cos 2\bar{\Phi}\, \xi_2.
$$

The first-order term in the $\dot{\bar{P}}$ of Eq. (27) is given by F_1^*, which, after averaging, should be the same as the result of the first-order averaging. Setting F_1^* to the averaged result of the deterministic term in the P of (13)

$$
F_1^* = \lim_{T \to \infty} \frac{1}{T} \int_0^T F_1^{(1)} dt \tag{29}
$$

one obtains

$$
P_1 = \frac{\alpha p \bar{P}}{2\omega_0} \sin 2\bar{\Phi}. \tag{30}
$$

Similarly, it is clear that

$$\varphi_1 = \frac{\alpha}{2\omega_0}\cos 2\bar{\Phi}. \tag{31}$$

Substituting Eqs. (30) and (31) into (28), the above stochastic averaging method can be performed for Eq. (27). Following the same procedure as the first-order averaging, the averaged version of (27) is given by

$$\begin{aligned} d\bar{P}(t) &= \bar{m}_p^* + \bar{\sigma}_{11}^* dB_1(t) + \bar{\sigma}_{12}^* dB_2(t) \\ d\bar{\varphi}(t) &= \bar{m}_\varphi^* + \bar{\sigma}_{21}^* dB_1(t) + \bar{\sigma}_{22}^* dB_2(t) \end{aligned} \tag{32}$$

where higher-order terms are neglected and where

$$\begin{gathered} \bar{m}_p^* = \varepsilon\bar{m}_p + \varepsilon^2\frac{p(p+2)}{8}\bar{P}\alpha\omega_0 S_{12}(2\omega_0),\ \ \bar{m}_\varphi^* = \varepsilon\bar{m}_\varphi + \varepsilon^2\frac{\alpha\omega_0}{8}\Psi_{12}(2\omega_0) \\ b_{11}^* = \varepsilon b_{11} + \varepsilon^2\frac{p^2\bar{P}^2\alpha\omega_0}{4}S_{12}(2\omega_0),\quad b_{12}^* = b_{21}^* = \varepsilon b_{12} + \varepsilon^2\frac{p\bar{P}\alpha\omega_0}{8}S_2(0) \\ b_{22}^* = \varepsilon b_{22} + \varepsilon^2\frac{\alpha\omega_0}{4}[2S_{12}(0) + S_{12}(2\omega_0)] \\ \boldsymbol{b}^*(\bar{P},\bar{\varphi}) = \left[b_{ij}^*\right] = \boldsymbol{\sigma}^*(\bar{P},\bar{\varphi})\boldsymbol{\sigma}^{*T}(\bar{P},\bar{\varphi}),\ \ \boldsymbol{\sigma}^*(\bar{P},\bar{\varphi}) = \left[\bar{\sigma}_{ij}^*\right],\qquad i,j = 1,2. \end{gathered} \tag{33}$$

By taking the expected value on both sides of the first equation of system (32), one has

$$\begin{aligned} dE[\bar{P}(t)] = \langle -\varepsilon p\alpha + \varepsilon p\omega_0^2 &\left\{\frac{p}{8}S_2(0) + \frac{p+2}{16}[S_1(2\omega_0) + S_2(2\omega_0) - 2\Psi_{12}(2\omega_0)]\right\} \\ &+ \varepsilon^2\frac{p(p+2)}{8}\alpha\omega_0 S_{12}(2\omega_0)\rangle\, E[\bar{P}]dt. \end{aligned} \tag{34}$$

The *pth* moment Lyapunov exponents are

$$\begin{aligned} \Lambda(p) = \varepsilon p\,\langle -\alpha + \omega_0^2 &\left\{\frac{p}{8}S_2(0) + \frac{p+2}{16}[S_1(2\omega_0) + S_2(2\omega_0) - 2\Psi_{12}(2\omega_0)]\right\}\rangle \\ &+ \varepsilon^2\frac{p(p+2)}{8}\alpha\omega_0 S_{12}(2\omega_0) \end{aligned} \tag{35}$$

and the Lyapunov exponent for the second-order stochastic averaging is given by

$$\lambda = \left.\frac{d\Lambda(p)}{dp}\right|_{p=0} = \varepsilon\left\{-\alpha + \frac{\omega_0^2}{8}[S_1(2\omega_0) + S_2(2\omega_0) - 2\Psi_{12}(2\omega_0)]\right\} + \varepsilon^2\frac{\alpha\omega_0}{4}S_{12}(2\omega_0). \tag{36}$$

5. Stability regions

Paper Kozić *et al*. (2008) investigated the sample stability of the system (4) when parametric excitations $f(t)$ and $g(t)$ were wideband stochastic processes of constant spectral density. By using the transformation of Khasminskii, they converted Eq. (4) into an Itô equation and obtained the stability condition for a different constant axial force and damping constant using the perturbation method. In order to compare their results with the results presented in paper Kozić *et al*. (2008), we take $f(t)$ and $g(t)$ to be white noise processes with auto-correlation and cross-correlation function from Falsone and Settineri (2011) given by

$$R_{ff}(t_1,t_2)=E[f(t_1)f(t_2)]=4\varepsilon S_f\delta(t_2-t_1)=\varepsilon\beta\delta(t_2-t_1)=\sigma_f^2\delta(t_2-t_1)$$
$$R_{gg}(t_1,t_2)=E[g(t_1)g(t_2)]=\varepsilon\pi^4 S_g\delta(t_2-t_1)=\varepsilon\gamma\delta(t_2-t_1)=\sigma_g^2\delta(t_2-t_1). \quad (37)$$

The correlation functions of the processes $\xi_1(t)$ and $\xi_2(t)$ are

$$R_{\xi_1\xi_1}(t_1,t_2)=E[\xi_1(t_1)\xi_1(t_2)]=S_1\delta(t_2-t_1)$$
$$R_{\xi_2\xi_2}(t_1,t_2)=E[\xi_2(t_1)\xi_2(t_2)]=S_2\delta(t_2-t_1) \quad (38)$$
$$R_{\xi_1\xi_2}(t_1,t_2)=E[\xi_1(t_1)\xi_2(t_2)]=S_{12}\delta(t_2-t_1)=\rho\sqrt{S_1S_2}\delta(t_2-t_1)$$

where $0\le\rho\le1$ is the correlation coefficient. With respect to (9), cosine power spectral density functions of noises $\xi_1(t)$ and $\xi_2(t)$ are

$$S_1=\frac{\sigma_f^2}{\varepsilon\omega_0^4},\ S_2=\frac{\sigma_g^2}{\varepsilon\omega_0^2},\ S_{12}=\rho\frac{\sigma_f\sigma_g}{\varepsilon\omega_0^3}. \quad (39)$$

Using the above results for Lyapunov exponent in the first and second order stochastic averaging, the system is asymptotically stable only if λ is negative. Then, expression (21) for the Lyapunov exponent in the first order stochastic averaging is employed to determine the almost-sure stability boundary of the system (4)

$$\sigma_f<\omega_0\sqrt{8\zeta-\sigma_g^2} \quad (40)$$

where $\zeta=\varepsilon\alpha$. By the same procedure applied to expression (36), the Lyapunov exponents for the second-order averaging are employed to determine the almost-sure stability boundary of the system (4)

$$\sigma_f<-\zeta\rho\sigma_g+\sqrt{8\zeta\omega_0^2-\sigma_g^2\left(\omega_0^2-\zeta^2\rho^2\right)}. \quad (41)$$

It is clear that if ρ=0, then processes $f(t)$ and $g(t)$ are non-correlated, and condition (41) is reduced to (40). Then, the almost sure stability boundary determined on the basis of the obtained expressions from the first and second order stochastic averaging is the same. When $\rho\neq0$, then processes of $f(t)$ and $g(t)$ are correlated and the almost-sure stability boundary decreases (41), the smallest when ρ=1. These values for the almost-sure stability boundary will be compared with that of the paper Kozić *et al*. (2008) obtained by the perturbation method for the same system. Based on the results from references (Kozić *et al*. 2008, Eq. (45)), in which we replace $\omega^2=\omega_0^2-\zeta^2$, we found the almost-sure stability boundary of the system (4)

$$\sigma_f<\sqrt{8\zeta\omega_0^2-\omega_0^2\sigma_g^2-8\zeta^3}. \quad (42)$$

For the purpose of comparison the almost-sure stability boundaries in the first and second-order stochastic averaging Eqs. (40), (41) along with those obtained in Kozić *et al*. (2008) by the perturbation method in the first perturbation Eq. (42), for different values of ζ are shown in Fig. 1.

We can notice that the curves which define the stability boundaries are quite close to each other for small values of the damping coefficient ζ. Also, see that the condition for almost sure stability obtained by stochastic averaging in the second-order is the strictest.

In order to illustrate of the analytical results of the *pth* moment stability from the first and second-order stochastic averaging Eqs. (20), (35), and the results in Kozić *et al*. (2008, Eq. (48))

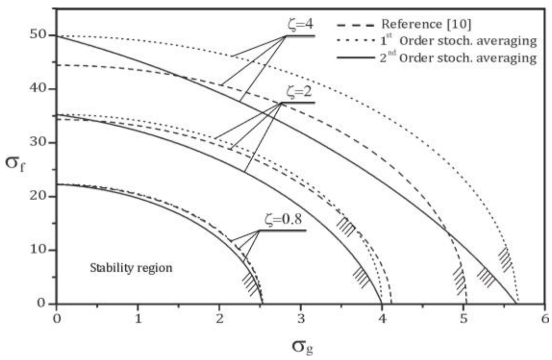


Fig. 1 Comparison of stability boundaries for the almost-sure stability for different values of ζ and for ρ=1

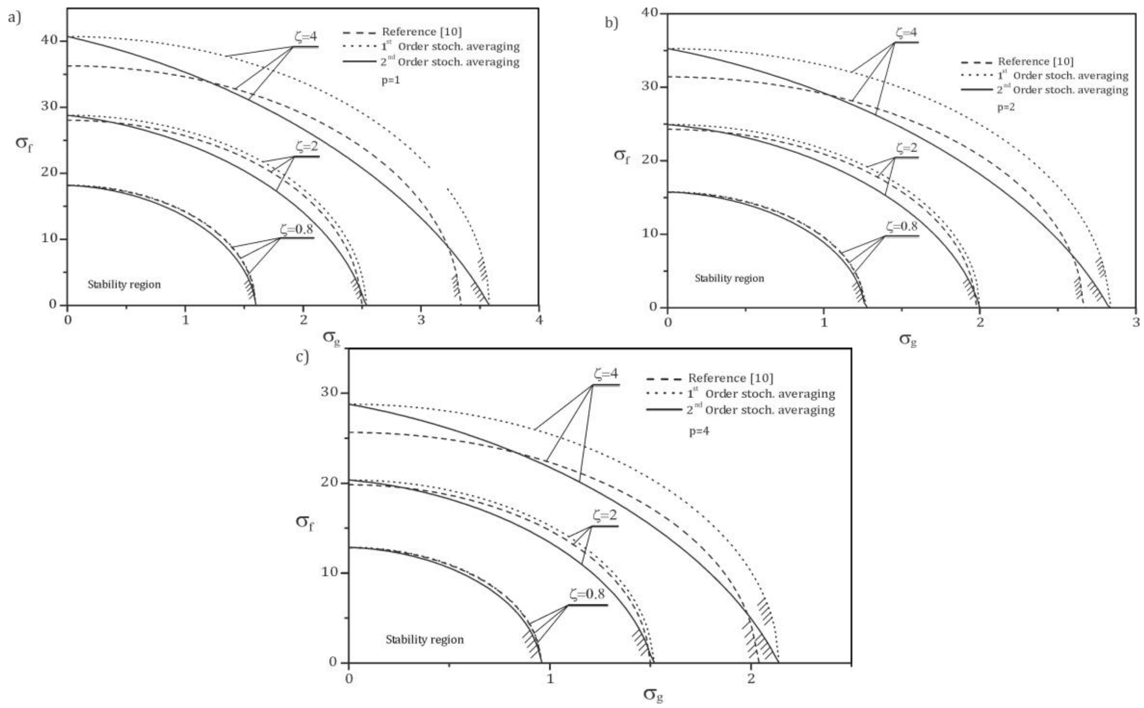


Fig. 2 Comparison of stability boundaries for the *pth* moment stability for different values of ζ and for ρ=1

for various values of p=1,2,4 are shown in Fig. 2.

By comparing stability regions shown in Fig. 2, it can be seen that the moment stability boundaries become narrower as p increases. Also indicates that the second-order averaging method does not improve the accuracy of approximation significantly. Therefore, the aproximate results from the first-order averaging are acceptable in engineering applications.

6. Numerical determination of the *pth* moment Lyapunov exponents

Numerical determination of the *pth* moment Lyapunov exponents is important in assessing the validity and the ranges of applicability of the approximate analytical results. In many engineering applications, the amplitudes of noise excitations are not small and the approximate analytical methods, such as the method of perturbation or the method of stochastic averaging, cannot be applied. Therefore, numerical approaches have to be employed to evaluate the moment Lyapunov exponents. The numerical approach is based on expanding the exact solution of the system of Itô stochastic differential equations in powers of the time increment h and the small parameter ε, as proposed in Milstein and Tret'Yakov (1997). The state vector of the system (8) is to be rewritten as a system of Itô stochastic differential equations with small noise in the form

$$dx_1 = \omega_0 x_2 dt$$

$$dx_2 = \left[-\omega_0 x_1 - \varepsilon\left(2\alpha - \frac{\gamma}{2}\right)\right]dt + \sqrt{\varepsilon}\frac{\sqrt{\beta}}{\omega_0}x_1 dw_1(t) - \sqrt{\varepsilon}\sqrt{\gamma}x_2 dw_2(t), \tag{43}$$

where $x_1 = \frac{1}{\omega_0}q(\mathrm{t})$, $x_2 = \frac{1}{\omega_0^2}\dot{q}(\mathrm{t})$ and $w_1(\mathrm{t})$, $w_2(\mathrm{t})$ are the standard Wiener processes; β and γ can be determined using the expression (37). For the numerical solutions of the stochastic differential equations, the Runge-Kutta approximation may be applied, also with error $R = O(h^4 + \varepsilon^4 h)$. The interval discretization is $[t_0, T] : \{t_k : k = 0, 1, 2, 3, \dots. N; t_0 < t_1 < t_2 < \cdots < t_N = T\}$ and the time increment is $h = t_{k+1} - t_k$. The following Runge-Kutta method is obtained for the *(k+1)th* iteration of the state vector $X = (x_1, x_2)$

$$\begin{aligned} x_1^{(k+1)} &= x_1^{(k)}\left(R_{11}^{(0)} + \sqrt{\varepsilon}R_{11}^{(1)} + \varepsilon R_{11}^{(2)}\right) + x_2^{(k)}\left(R_{12}^{(0)} + \sqrt{\varepsilon}R_{12}^{(1)} + \varepsilon R_{12}^{(2)}\right) \\ x_2^{(k+1)} &= x_1^{(k)}\left(R_{21}^{(0)} + \sqrt{\varepsilon}R_{21}^{(1)} + \varepsilon R_{21}^{(2)}\right) + x_2^{(k)}\left(R_{22}^{(0)} + \sqrt{\varepsilon}R_{22}^{(1)} + \varepsilon R_{22}^{(2)}\right) \end{aligned} \tag{44}$$

where

$$R_{11}^{(0)} = 1 - \frac{h^2\omega_0^2}{2} + \frac{h^4\omega_0^4}{24}, \quad R_{11}^{(1)} = h^{3/2}\left[\sqrt{\beta}\left(\eta + \frac{\xi}{2}\right) + \frac{h}{6}\sqrt{\gamma}\xi\omega_0^2\right], \quad R_{11}^{(2)} = \frac{h^3\omega_0^2}{3}\left(\alpha - \frac{\gamma}{4}\right)$$

$$R_{12}^{(0)} = h\omega_0\left(1 - \frac{h^2\omega_0^2}{6}\right), \qquad R_{12}^{(1)} = h^{3/2}\omega_0\left[\frac{h}{6}\sqrt{\beta}\xi - \sqrt{\gamma}\left(\eta + \frac{\xi}{2}\right)\right]$$

$$R_{12}^{(2)} = -h^2\omega_0\left(1 - \frac{h^2\omega_0^2}{9}\right)\left(\alpha - \frac{\gamma}{4}\right)$$

$$R_{21}^{(0)} = -R_{12}^{(0)}, \quad R_{21}^{(1)} = \frac{\sqrt{h}}{\omega_0}\left[\sqrt{\beta}\xi\left(1 - \frac{h^2\omega_0^2}{3}\right) + \omega_0^2 h\sqrt{\gamma}\left(-\eta + \frac{\xi}{2}\right)\right], \quad R_{21}^{(2)} = -R_{12}^{(2)} \tag{45}$$

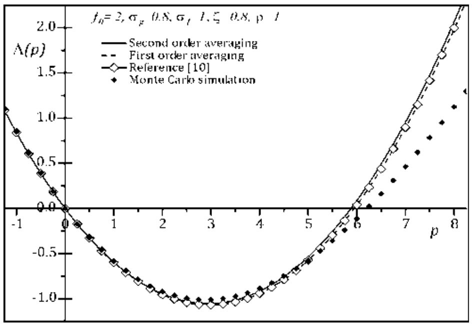
$$R_{22}^{(0)} = R_{11}^{(0)}, \qquad R_{22}^{(1)} = \sqrt{h}\left[h\sqrt{\beta}\left(-\eta + \frac{\xi}{2}\right) - \sqrt{\gamma}\xi\left(1 - \frac{h^2\omega_0^2}{3}\right)\right]$$

$$R_{22}^{(2)} = -h^2\omega_0\left(1 - \frac{h^2\omega_0^2}{9}\right)\left(\alpha - \frac{\gamma}{4}\right)$$

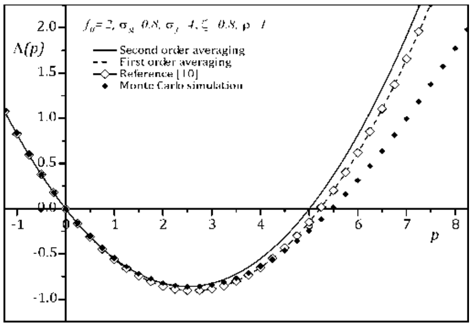
and random variables ξ and η are simulated as

$$P(\xi = -1) = P(\xi = 1) = \frac{1}{2}, P\left(\eta = \frac{-1}{\sqrt{12}}\right) = P\left(\eta = \frac{1}{\sqrt{12}}\right) = \frac{1}{2}. \tag{46}$$

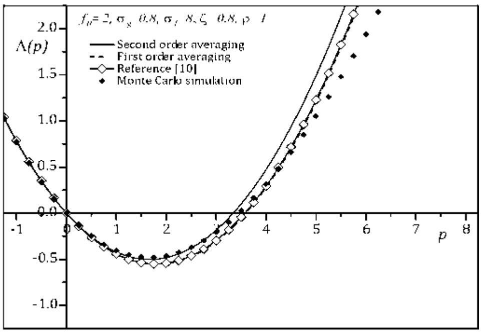
Having obtained L samples of the solutions of the stochastic differential Eq. (43), the *pth* moment can be determined as follows



(a)



(b)



(c)

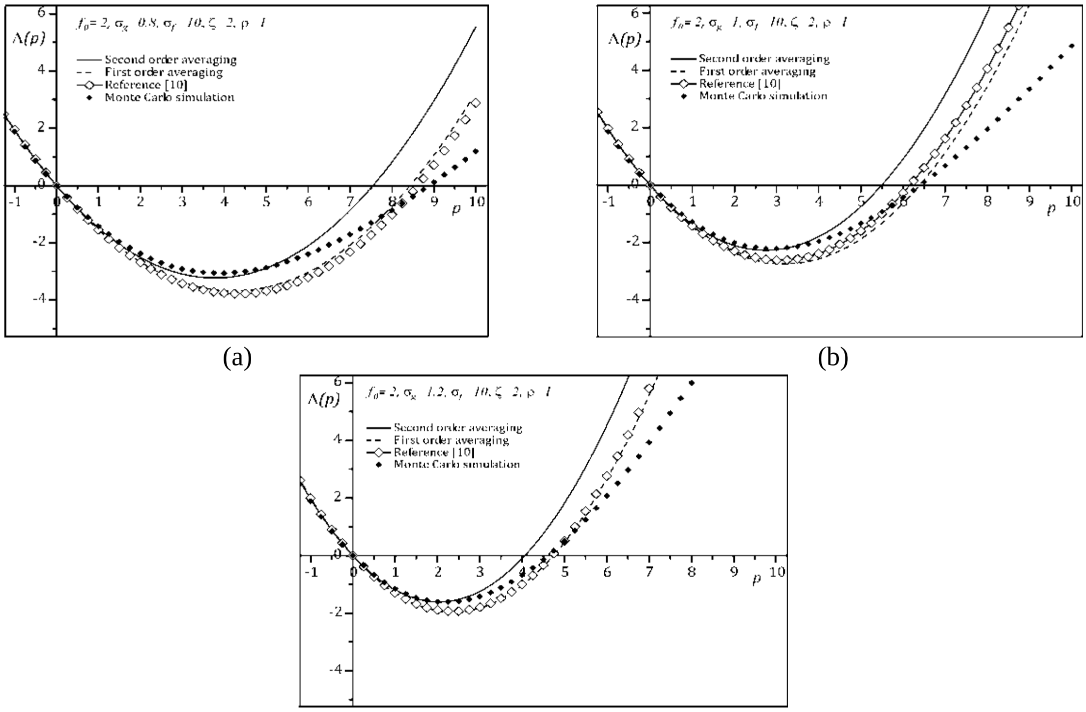
Fig. 3 Variation of the moment Lyapunov exponents, Λ*(p)* with *p*, for different values σ_f=1.0, 4.0 and 8.0

$$E[\|X(t_{k+1})\|^p] = \frac{1}{L}\sum_{s=1}^{L}\|X_s(t_{k+1})\|^p\,, \quad X_s(t_{k+1}) = \sqrt{[X_s(t_{k+1})]^T[X_s(t_{k+1})]}. \tag{47}$$

By the Monte-Carlo technique, we numerically calculate the *pth* moment Lyapunov exponents for all values of *p* of interest defined as

$$\Lambda(p) = \frac{1}{T}\log E[\|X(T)\|^p]. \tag{48}$$

Fig. 3 and Fig. 4 show the comparison of approximate analytical resulrs for moment Lyapunov exponents in the first and the second-order stochastic averaging given Eqs. (20), (35) and Monte Carlo simulation results for different values of $\sigma_g = 0.8, 1.0, 1.2$ and $\sigma_f = 1.0, 4.0, 8.0$. The analytical results for moment Lyapunov exponents in the first perturbation obtained in Kozić *et al.* (2008, Eq. (30)) are also included in this figures. In Monte Carlo simulation, the sample size for estimating the expected value is $L = 4000$, time step of integration is $t = 0{,}00005$ [s] and the total length of time for simulation is $T = 0{,}5$ [s]. It can be seen that the first-order stochastic averaging results agree with the simulation results very well when σ_g, σ_f and ζ are small, i.e., the intensity of noises is weak, as shown in Fig. 3(a). On the Fig. 3 and Fig. 4 also indicate that the second-order stochastic averaging method does not improve the accuracy of approximation significantly. Therefore, the approximate results from the first-order stochastic averaging are acceptable in engineering applications.



(a) (b)

(c)

Fig. 4 Variation of the moment Lyapunov exponents, $\Lambda(p)$ with p, for different values σ_q=0.8, 1.0 and 1.2

7. Conclusions

In this paper, the dynamic stability of a single degree-of-freedom system under the parametric excitations of correlated wideband noises is studied. The averaged Itô differential equations guverning the *pth* norm is established and then the method of stochastic averaging, both the first and second-order, is applied to obtain analitical results for the moment Lyapunov exponents and Lyapunov exponent in terms of small fluctuation parameter ε. Moment Lyapunov exponents are important characteristic numbers for describing the dynamic stability of a stochastic system. When the *pth* moment Lyapunov exponent is negative, the *pth* moment of the solution of the stochastic system is stable. For stochastic dynamical systems described by Itô differential equations, a Monte Carlo simulation algorithm used to determine the moment Lyapunov exponents. Monte Carlo simulation approaches complement approximate analytical method of stochastic averaging of the first-order and second-order in the determination of moment Lyapunov exponents and provides criteria an assessing the accuracy of approximate analitical results. It can be concluded, from the approximate analitycal results and the Monte Carlo simulation results of the moment Lyapunov exponents, that the increase of noise intensity σ_g, the stability region for $p > 0$ becomes decrease, which indicates decrease the stability of the system. Also, with the increase of noise intensity σ_f, the stability region of the *pth* moment for $p > 0$ dwindles away as expected. This results are useful in engineering applications.

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References

Ariaratnam, S.T. and Tam, D.S.F. (1979), "Random vibration ad stability of a linear parametrically excited oscillator", *ZAMM*, **59**, 79-84.

Ariaratnam, S.T. and Ly, B.L. (1989), "Almost-sure stability of some linear stochastic system", *ASME J. Appl. Mech*., **56**, 175-179.

Bai, Q.C. and Zhang, Y.H. (2012), "Almost sure asymptotic stability of a rotor systems subjected to stochastical axial loads", *Mech. Mach. Theory*, **58**, 192-201.

Falsone, G. and Settineri, D. (2011), "A method for the random analysis of linear systems subjected to non-stationary multi-correlated loads", *Probab. Engin. Mech.*, **26**(3), 447-453.

Hijawi, M., Moschuk, N. and Ibrahim, R.A. (1997a), "Unified second-order stochastic averaging approach", *ASME J. Appl. Mech.*, **64**, 281-291.

Hijawi, M., Ibrahim, R.A. and Moshchuk, N. (1997b), "Nonlinear random response of ocean structures using first-and second-order stochastic averaging", *Nonlin. Dyn*., **12**(2), 155-197.

Huang, Q. and Xie, W.C. (2008), "Stability of SDOF linear viscoelastic system under the excitation of wideband noise", *ASME J. Appl. Mech*., **75**(2), 021012-(1-9).

Kapitaniak, T. (1986), "Non-Markovian parametrical vibration", *Int. J. Engng. Sci*., **24**(8), 1335-1337.

Kozin, F. and Wu, C.M. (1973), "On the stability of linear stochastic differential equations", *ASME J. Appl. Mech*., **40**, 87-92.

Kozić, P., Pavlović, R. and Janevski, G. (2008), "Moment Lyapunov exponents of the stochastic parametrical Hill's equation", *Int. J. Solid. Struct*., **45**(24), 6056-6066.

Khasminskii, R.Z. (1966), "A limit theorem for solution of differential equations with random right-hand side", *Theor. Probab. Appl.*, **11**(2), 390-406.

Ku, C.J., Tamura, Y., Yoshida, A., Miyake, K. and Chou, L.S. (2013), " Output-only modal parameter identification for force-embedded acceleration data in the presence of harmonic and white noise excitations", *Wind Struct.*, **16**(2), 157-178.

Li, C., Xu, W., Feng, J. and Wang L. (2013), Response probability density functions of Duffing-Van der Pol vibro-impact system under correlated Gaussian white noise excitations", *Physica A*, **392**(6), 1269-1279.

Ling, Q., Jin, L.X. and Huang L.Z. (2011), "Response and stability of SDOF viscoelastic system under wideband noise excitations", *J. Franklin Inst*., **348**(8), 2026-2043.

Liu, D., Xu, W. and Xu, Y. (2012), "Dynamic responses of axially moving viscoelastic beam under a randomly disordered periodic excitation", *J. Sound Vib.*, **331**(17), 4045-4056.

Liu, W., Zhu, W. and Xu W. (2013), "Stochastic stability of quasi non-integrable Hamiltonian systems under parametric excitations of Gaussian and Poisson white noises", *Prob. Engin. Mech.*, **32**, 39-47.

Milstein, N.G. and Tret'Yakov, V.N. (1997), "Numerical methods in the weak sense for stochastic differential equations with small noise", *SIAM J. Numer. Anal*., **34**(6), 2142-2167.

Pavlović, R., Kozić, P. and Rajković, P. (2005), "Influence of randomly varying damping coefficient on the dynamic stability of continuous systems", *Europ. J. Mech. A/Solid.*, **24**, 81-87.

Rozycki, B. and Zembaty, Z. (2011), "On eigenvalue problem of bar structures with stochastic spatial stiffness variations", *Struct. Eng. Mech.*, **39**(4), 541-558.

Zhang, J., Xu, Y., Xia, Y. and Li, J. (2011), "Generalization of the statistical moment-based damage detection method", *Struct. Eng. Mech.*, **38**(6), 715-732.

Lin, Y.K. and Cai, G.Q. (1995), *Probabilistic Structural Dynamics, Advanced Theory and Applications*, McGraw-Hill, Inc., New York.

Stratonovich, R.L. (1963), *Topics in the Theory of Random Noise,* Vol. 1. Gordon and Breach, New York.
Xu, Y., Gu, R., Zhang, H., Xu, W. and Duan J. (2011), "Stochastic bifurcations in a bistable Duffing-Van der Pol oscillator with colored noise", *Phys. Rev. E,* **83**, 056215.
Xu, Y., Duan, J. and Xu, W. (2011), "An averaging principle for stochastic dynamical systems with Lévy noise", *Physica D*, **240**(17), 1395-1401.
Xu, Y., Wang, X., Zhang, H. and Xu, W. (2012), "Stochastic stability for nonlinear systems driven by Lévy noise", *Nonlin. Dyn.*, **68**, 7-15.
Xu, Y., Li, Y., Jia, W. and Huang, H. (2013), "Response of Duffing oscillator with fractional damping and random phase", *Nonlin. Dyn.*, **74**, 745-753.

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