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The analytical solution for buckling of curved sandwich beams with a transversely flexible core subjected to uniform load

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Abstract. In this paper, linear buckling analysis of a curved sandwich beam with a flexible core is investigated. Derivation of equations for face sheets is accomplished via the classical theory of curved beam, whereas for the flexible core, the elasticity equations in polar coordinates are implemented. Employing the von-Karman type geometrical non-linearity in strain-displacement relations, nonlinear governing equations are resulted. Linear pre-buckling analysis is performed neglecting the rotation effects in pre-buckling state. Stability equations are concluded based on the adjacent equilibrium criterion. Considering the movable simply supported type of boundary conditions, suitable trigonometric solutions are adopted which satisfy the assumed edge conditions. The critical uniform load of the beam is obtained as a closed-form expression. Numerical results cover the effects of various parameters on the critical buckling load of the curved beam. It is shown that, face thickness, core thickness, core module, fiber angle of faces, stacking sequence of faces and openin angle of the beam all affect greatly on the buckling pressure of the beam and its buckled shape.

Keywords: sandwich curved beam; buckling; flexible core; critical load

1. Introduction

When a structure is subjected to compressive in-plane loads, the buckling phenomenon may occur and is distinguished via a rapid change in displacements due to an increment in loading process. Buckling resistance is an important factor which should be taken into consideration for design purposes. For the cases when loads are below the yield limit, buckling phenomenon is of high importance and have to be studied in elastic range.

Sandwich structures with soft core made of foam or low-strength honeycomb like Aramid or Nomex are used in various industrial applications such as aerospace and civil engineering. The use of a foam or lowstrength honeycomb core rather than a metallic honeycomb is advantageous in terms of weight and manufacturing processes and resources. The major difference between a metallic honeycomb and a soft core is its flexibility in the vertical direction. This flexibility, significantly affects behavior, especially under localized loads, and yields quite different behaviors as compared to other structures that have a stiff honeycomb core. The general approach assumes that the global buckling of the beam and the local buckling of the skins are uncoupled. The global buckling is defined by the solution of an equivalent beam, which incorporates the shear stiffness of

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the core in the flexural rigidity of the beam. Local buckling is determined by considering the isolated skins as a beam resting on an elastic foundation provided by the core in the vertical direction.

A similar approach, which assumes that no interaction between the skins exists in the local buckling mode, was used by Bulson (1970) and by Brush and Almroth (1975). This approach is satisfactory as long as the core is incompressible in the vertical direction. However, when a compressible type of core is considered, an interaction between the global and local buckling modes exists, as well as collaboration between the two skins. Hence the critical mode can be shifted from the global mode to the local one and vice versa. They replaced the sandwich structure with a high-order shear deformable beam but were unable to determine the local buckling modes as well as the imperfection effect on overall behavior. Hunt and Da Silva (1990a, b) used a different approach, based on energy methods and superposition of symmetrical and nonsymmetrical buckling modes. This approach is limited to specific configurations and to specific boundary conditions. Frostig and Baruch (1990) and Frostig *et al*. (1991), analyzed the sandwich beams with soft core with the aid of a superposition approach. Frostig and Baruch (1993) presented the higher order buckling analysis of sandwich beams with transversely flexible core. Closed-form solutions are presented for simply supported beams with identical skins and only numerical results for other cases. Smith (1984) yielded a unified analysis method based on two-dimensional elasticity theory for evaluation of bending, buckling and vibration of multilayer orthotropic sandwich beams and panels. Cheng *et al*. (1995) presented a method of continuous analysis for predicting the local delamination buckling load of the face sheet of sandwich beams. In a research by Bozhevolnaya and Kildegaard (1998) a sandwich curved beam subjected to uniform loading is experimentally investigated. Wang and Shenoi (2001) performed an elasticity theory based approach for delamination and flexural strength of curved layered composite laminates and sandwich beams. They also performed the analysis of curved sandwich beams with a focus on debonding and buckling/wrinkling of the faces (Wang and Shenoi 2004). Lyckegaard and Thomsen (2005) formulated the buckling behaviour of straight sandwich beams joined with curved sandwich beams loaded in pure bending using two different models. A two-dimensional mechanical model is developed by Ji and Waas (2007) to predict the global and local buckling of a sandwich beam, using classical elasticity.

In this paper, the governing equilibrium equations of a three layered sandwich curved beam in the von-Karman sense are obtained. Two skins are formulated in the Euler-Bernoulli sense whereas the host layer is formulated by the two dimensional elasticity equations. The pre-buckling deformations of the arch are obtained under the linear membrane pre-buckling deformations. Adjacent equilibrium criterion is used to establish the stability equations. A closed form solution suitable for curved beams with both edges simply supported is developed which results in closed-form expression for the critical buckling pressures. Some numerical results are provided to study the effect of various involved parameters.

2. Geometry of problem and kinematic relations

A curved sandwich beam of the width b is considered. Other geometrical parameters of the model with the coordinate system are shown in Fig. (1).

In the following, indices t, b refer to the upper (top) and lower (bottom) faces of the beam, respectively. Each face has its own curvilinear coordinate system (z_i, s_i), where

$$s_i=r_i\varphi\,(i=t,b) \tag{1}$$

The local coordinate system (r,φ) for the core is polar and has its origin in the center of the beam curvature. The following assumptions form the basis of the presented model:

1. The length of the beam is of the order of its characteristic radii of curvature *L≤R*.

2. The faces may have a different thickness $d_t$and d_b that are small in comparison with the length of the beam and radii of curvature. The faces are treated as thin elastic panels that follow Bernoulli assumptions.

3. The core of thickness t_c is fully bonded with the faces. The core is considered to be a 2-D elastic medium with resistance to shear and radial stresses. In-plane (circumferential) stress in the core is neglected.

4. The kinematic relations of the core are those of small deformations and therefore they are linear. Note that, no priori assumptions on the deformation fields through the thickness of the core are made.

5. Different kinds of the boundary conditions may be implemented for the various faces at the same section.

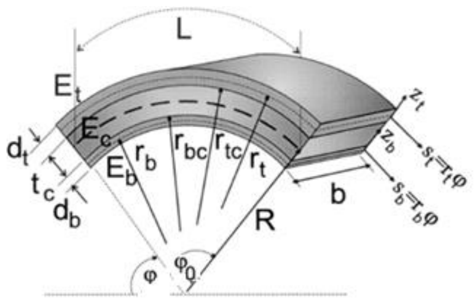

Fig. 1 Geometry of the mathematical model (Bozhevolnaya and Frostig 2001)

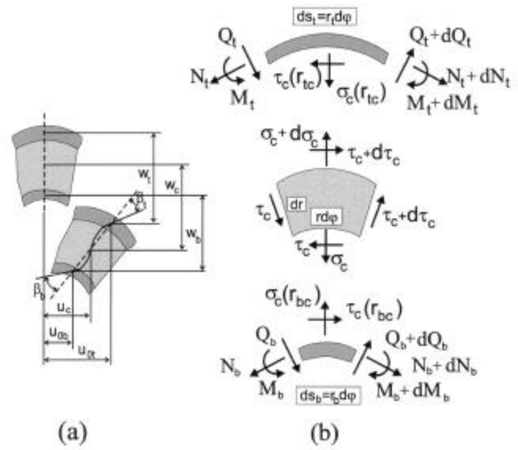

Fig. 2 Displacement in the element of the sandwich beam (a); internal resultants in the differential elements of the faces, stresses at the interfaces and stresses in the differential element of the core (b) (Bozhevolnaya and Frostig 2001)

Radial w_t, w_b and circumferential u_{0t}, u_{0b} displacements of the centroids of the face elements are shown in Fig. 2(a). In the polar coordinate system the kinematic relations for the faces read

$$u_i = u_{0i} + z_i \beta_i \tag{2}$$

$$\varepsilon_i = \varepsilon_{0i} + z_i \kappa_i + \frac{1}{2}\beta_i^2 \quad (i=t,b) \tag{3}$$

In Eqs. (2) and (3) u_i and ε_i are respectively, circumferential displacement and circumferential strain in the skins that can be measured upwardly from the center of each skin. Furthermore β_i is equal to

$$\beta_i = \frac{u_{0i} - w_i'}{r_i} \tag{4}$$

In Eq. (3) ε_{0i} and κ_i are the values of strain in the center of each skins and the curvature value of each skins respectively and are equal to

$$\varepsilon_{0i} = \frac{u_{oi}' + w_i}{r_i} \tag{5}$$

$$\kappa_i = \frac{u_{oi}' - w_i''}{r_i^2} \tag{6}$$

In the mentioned relation all derivations are considered based on angle φ. The appropriate kinematic relations for the core are

$$\varepsilon_{rr} = \frac{\partial w_c}{\partial r} \tag{7}$$

$$\gamma_{r\omega} = \gamma_{sr} = \frac{\partial u_c}{\partial r} - \frac{u_c}{r} + \frac{1}{r}\frac{\partial w_c}{\partial \varphi} \tag{8}$$

Compatibility conditions emerge from the conditions of the fully bonded faces and core.

At the upper interface

$$w_c|_{r=r_{tc}} = w_t|_{z=\frac{-d_t}{2}} \Rightarrow w_c|_{r=r_{tc}} = w_t \tag{9}$$

$$u_c|_{r=r_{tc}} = u_t|_{z=\frac{-d_t}{2}} \Rightarrow u_c|_{r=r_{tc}} = u_{0t} - \frac{d_t}{2}\beta_t \Rightarrow$$
$$u_c|_{r=r_{tc}} = u_{0t} - \frac{d_t}{2r_t}\left(u_{0t} - w_t'\right) \Rightarrow u_c|_{r=r_{tc}} = u_{0t}(1-k_t) + w_t' k_t \tag{10}$$

in the recent relation k_t is defined as follow

$$k_t = \frac{d_t}{2r_t} \tag{11}$$

For the lower interface

$$w_c|_{r=r_{bc}}=w_b|_{z=\frac{d_b}{2}}\Rightarrow w_c|_{r=r_{bc}}=w_b \tag{12}$$

$$\begin{gathered}u_c|_{r=r_{bc}}=u_b\,|_{z=\frac{d_b}{2}}\Rightarrow u_c|_{r=r_{bc}}=u_{0b}+\frac{d_b}{2}\beta_b\Rightarrow\\ u_c|_{r=r_{bc}}=u_{0b}+\frac{d_b}{2r_b}\left(u_{0b}-w_b'\right)\Rightarrow u_c|_{r=r_{bc}}=u_{0b}(1+k_b)-w_b'k_b\end{gathered} \tag{13}$$

in the recent relation k_b is defined as follow

$$k_b=\frac{d_b}{2r_b} \tag{14}$$

The stresses in the beam constituents are shown in Fig. 2(b). The constitutive relations for the faces and for the core are

$$\sigma_t^{(k)}=Q_{11t}^{(k)}\varepsilon_t\ ,\ \sigma_b^{(k)}=Q_{11b}^{(k)}\varepsilon_b\ ,\ \sigma_c=E_c\varepsilon_c\ ,\ \tau_c=G_c\gamma_{r\omega} \tag{15a-d}$$

Where

$$\begin{gathered}A_{11}=b\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}Q_{11}^{(k)}\,dz\\ D_{11}=b\sum_{k=1}^{N}\int_{z_{k-1}}^{z_k}Q_{11}^{(k)}\,z^2dz\end{gathered} \tag{16a-b}$$

In the relation (16), A_{11} and D_{11} are laminate membrane stiffness and flexural stiffness respectively. $Q_{11}^{(k)}$ is the stiffness of each layer based on laminate angle such that

$$Q_{11}^{(k)}=Q_{11}\cos^4\theta+Q_{22}\sin^4\theta+2\left(Q_{12}+2Q_{66}\right)\sin^2\theta\cos^2\theta \tag{17}$$

where in the above relation we have

$$\begin{gathered}Q_{11}=\frac{E_{11}}{1-\nu_{12}\nu_{21}}\quad,\quad Q_{22}=\frac{E_{22}}{1-\nu_{12}\nu_{21}}\\ Q_{12}=\nu_{21}Q_{11}\ ,\ Q_{66}=G_{12}\end{gathered} \tag{18a-d}$$

N_i and M_i respectively are axial force and bending moment for each skin that are defined as follow

$$N_i=\int_{\frac{-d_i}{2}}^{\frac{d_i}{2}}b\sigma_i\,dz\ ,\quad M_i=\int_{\frac{-d_i}{2}}^{\frac{d_i}{2}}b\sigma_i z\,dz(i=t,b) \tag{19}$$

Now with the aid of kinematic relations (2), (3), (7), (8) and (16)-(18) along with the simultaneous aid of constitutive relations (15), Eq. (19) reaches one to

$$N_i=A_{11}\left[\frac{u_{oi}'+w_i}{r_i}+\frac{1}{2}\left(\frac{u_{0i}-w_i'}{r_i}\right)^2\right]$$

$$M_i=D_{11}\left(\frac{u'_{oi}-w''_i}{r_i^2}\right) \qquad (i=t,b) \qquad (20a\text{-}b)$$

2.1 The governing equations

The mathematical formulation starts with the derivation of the field equations and the appropriate boundary and continuity conditions. After that, the solution of the stress and the core deformation field are obtained. The formulation ends with the governing equations expressed in terms of the unknowns and their solutions. The governing equations, the continuity requirements and the boundary conditions are derived via the variational principles, which minimize the total potential energy, as follows

$$\delta(U+V)=0 \qquad (21)$$

where in Eq. (21) U,V and δ respectively are internal and external energies and the variation operator. The internal energy reads

$$\delta U=\int_{v_{top}} \sigma_t\,\delta\varepsilon_t\, dv_t+\int_{v_{bot}} \sigma_b\,\delta\varepsilon_b\, dv_b+\int_{v_{core}} \left(\tau_c\delta\gamma_{r\omega}+\sigma_c\delta\varepsilon_c\right)\, dv_c \qquad (22)$$

where in Eq. (22) σ_t and ε_t are the longitudinal normal stresses and strains in the upper skins and σ_b and ε_b are the longitudinal normal stresses and strains in the lower skins; τ_c and $\gamma_{r\varphi}$ are the shear stresses and strains in the core; σ_c and ε_c are the vertical normal stresses and strains in the core;v_{top}, v_{bot}, and v_{core} are the volume of the upper and lower skins and the core, respectively; dv_t , dv_b and dv_c are the differential volume of the upper and lower skins and the core, respectively.

To obtain δU the value of strains should be set from Eqs. (3), (7), (8) into Eq. (22) which results in

$$\delta U=\int_{v_{top}} \sigma_t\,\delta\left(\varepsilon_{0t}+z_t\kappa_t+\frac{1}{2}\beta_t^2\right)\, dv_t+\int_{v_{bot}} \sigma_b\,\delta\left(\varepsilon_{0b}+z_b\kappa_b+\frac{1}{2}\beta_b^2\right)\, dv_b+$$

$$\int_{v_{core}}\left[\left(\tau_c\delta\left(\frac{\partial u_c}{\partial r}-\frac{u_c}{r}+\frac{1}{r}\frac{\partial w_c}{\partial \omega}\right)\right)+\sigma_c\delta\left(\frac{\partial w_c}{\partial r}\right)\right]dv_c \qquad (23)$$

The value of differential volume is equal to

$$dv_i=dA_i dx=bdz_i dx \qquad (24)$$

where in Eq. (24)

$$dx=r_i d\varphi \qquad (25)$$

The external energy reads

$$\delta V=-\left[\int_{s=0}^{1}\left(n_t\delta u_{0t}+q_t\delta w_t-m_t\delta\beta_t\right) ds_t-\int_{s=0}^{1}\left(n_b\delta u_{0b}+q_b\delta w_b-m_b\delta\beta_b\right) ds_b\right] \qquad (26)$$

where in Eq. (26) q_i, n_i,m_i (i=t, b) are the external distributed vertical, horizontal and bending moments, at upper and lower skins, u_i, w_i, β_i(i=t, b) are the horizontal displacements, vertical displacements and the rotation at upper and lower skins, respectively. The case of buckling in the curved beam may be analyzed for various status of loading. Even though researches in this field show that buckling of curved beam is analyzed in the presence of uniformly distributed transverse load. In this regard also in this paper the buckling of curved beam in the presence of uniformly distributed load is analyzed. To this end, other external loads that are introduced in relation (26) are considered to be equal to zero and consequently the work relation derived from external forces will be simplified as follow

$$\delta V = -\int q_t \delta w_t dA_t = -\int_0^{\varphi_0} q_t \delta w_t \, br_t d\varphi \tag{27}$$

In Eq. (27), dA=brdφ is the differential length in curvature line of the beam. Now by Substituting Eqs. (23) and (27) into (21) we will have

$$\delta(U+V)=0 \quad \Rightarrow \int_{v_{top}} \sigma_t \, \delta\left(\varepsilon_{0t}+z_t\kappa_t+\frac{1}{2}\beta_t^2\right) dv_t + \int_{v_{bot}} \sigma_b \, \delta\left(\varepsilon_{0b}+z_b\kappa_b+\frac{1}{2}\beta_b^2\right) dv_b +$$

$$\int_{v_{core}} \left[\left(\tau_c\delta\left(\frac{\partial u_c}{\partial r}-\frac{u_c}{r}+\frac{1}{r}\frac{\partial w_c}{\partial \omega}\right)\right)+\sigma_c\delta\left(\frac{\partial w_c}{\partial r}\right)\right] dv_c - \int_0^{\varphi_0} q_t \delta w_t \, br_t d\varphi = 0 \tag{28}$$

or

$$\int_0^{\varphi_0} -N_t'\delta u_{0t}\, d\varphi + N_t\delta u_{0t}\Big|_0^{\varphi_0} + \int_0^{\varphi_0} N_t\delta w_t\, d\varphi_0 + \int_0^{\varphi_0}\frac{N_t}{r_t}(u_{0t}-w_t)\,\delta u_{0t}d\varphi + \int_0^{\varphi_0}\left(\frac{N_t}{r_t}(u_{0t}-w_t)'\right)'\delta w_t d\varphi$$

$$-\frac{N_t}{r_t}(u_{0t}-w_t)'\delta w_t\Big|_0^{\varphi_0} - \int_0^{\varphi_0}\frac{M_t'}{r_t}\delta u_{0t}d\varphi + \frac{M_t}{r_t}\delta u_{0t}\Big|_0^{\varphi_0} - \int_0^{\alpha_0}\frac{M_t''}{r_t}\delta w_t d\varphi + \frac{M_t'}{r_t}\delta w_t\Big|_0^{\varphi_0} - \frac{M_t}{r_t}\delta w_t'\Big|_0^{\varphi_0} +$$

$$\int_0^{\varphi_0} -N_b'\delta u_{0b}\, d\varphi + N_b\delta u_{0b}\Big|_0^{\varphi_0} + \int_0^{\varphi_0} N_b\delta w_b\, d\varphi + \int_0^{\varphi_0}\frac{N_b}{r_b}(u_{0b}-w_b)\,\delta u_{0b}d\varphi + \int_0^{\varphi_0}\left(\frac{N_b}{r_b}(u_{0b}-w_b)'\right)'\delta w_b d\varphi$$

$$+\int_0^{\varphi_0}\left(\frac{N_b}{r_b}(u_{0b}-w_b)'\right)'\delta w_b d\varphi - \frac{N_b}{r_b}(u_{0b}-w_b)'\delta w_b\Big|_0^{\varphi_0} - \int_0^{\varphi_0}\frac{M_b'}{r_b}\delta u_{0b}d\varphi + \frac{M_b}{r_b}\delta u_{0b}\Big|_0^{\varphi_0} - \int_0^{\varphi_0}\frac{M_b''}{r_b}\delta w_b d\varphi$$

$$+\frac{M_b'}{r_b}\delta w_b\Big|_0^{\varphi_0} - \frac{M_b}{r_b}\delta w_b'\Big|_0^{\varphi_0} + \int_0^{\varphi_0} br_{tc}\,(1-k_t)\tau_c|_{r_{tc}}\delta u_{0t}\, d\varphi - \int_0^{\varphi_0} br_{tc}\, k_t\tau_c'\Big|_{r_{tc}}\delta w_t\, d\varphi + br_{tc}k_t\tau_c|_{r_{tc}}\delta w_t\Big|_0^{\varphi_0}$$

$$-\int_0^{\varphi_0} br_{bc}\,(1+k_b)\tau_c|_{r_{bc}}\delta u_{0b}\, d\varphi - \int_0^{\varphi_0} br_{bc}\, k_b\tau_c'\Big|_{r_{bc}}\delta w_b\, d\varphi + br_{bc}k_b\tau_c|_{r_{bc}}\delta w_b\Big|_0^{\varphi_0}$$

$$-\frac{\partial}{\partial r}\int_\varphi\int_r \frac{\partial}{\partial r}(br\tau_c)\delta u_c dr d\varphi - \int_\varphi\int_r b\tau_c\,\delta u_c dr d\varphi + \int_\varphi\int_r -b\frac{\partial\delta\tau_c}{\partial\omega}\partial w_c dr d\varphi + \int_r b\tau_c\,\partial\delta w_c\Big|_0^{\varphi_0} dr$$

$$+\int_0^{\varphi_0} br_{tc}\,\sigma_c|_{r_{tc}}\delta w_t d\varphi - \int_0^{\varphi_0} br_{bc}\,\sigma_c|_{r_{bc}}\delta w_b d\varphi - b\int_{\varphi}\int_{r}\frac{\partial}{\partial r}(r\sigma_c)\delta w_c dr d\varphi - \int_0^{\varphi_0} q_t \delta w_t\, r_t d\varphi = 0 \tag{29}$$

To obtain the equilibrium equations relation (29) might be established. By equaling the coefficients of $\delta u_{0t}, \delta u_{0b}, \delta w_t, \delta w_b, \delta w_c, \delta u_c$ to zero the following equilibrium equations will be obtained.

$$\delta u_{0t}:\quad -N_t' - \frac{M_t'}{r_t} + \frac{N_t}{r_t}\left(u_{0t}-w_t'\right) + br_{tc}(1-k_t)\tau_c|_{r_{tc}} = 0$$

$$\delta u_{0b}:\quad -N_b' - \frac{M_b'}{r_b} + \frac{N_b}{r_b}\left(u_{0b}-w_b'\right) - br_{bc}(1+k_b)\tau_c|_{r_{bc}} = 0$$

$$\delta w_t\quad : N_t - \frac{M_t''}{r_t} + \frac{1}{r_t}\left[N_t\left(u_{0t}-w_t'\right)\right]' + br_{tc}\sigma_c|_{r_{tc}} - br_{tc}k_t\tau_c'\Big|_{r_{tc}} - br_t q_t = 0$$

$$\delta w_b\quad : N_b - \frac{M_b''}{r_b} + \frac{1}{r_b}\left[N_b\left(u_{0b}-w_b'\right)\right]' - br_{bc}\sigma_c|_{r_{bc}} - br_{bc}k_b\tau_c'\Big|_{r_{bc}} = 0$$

$$\delta u_c\quad : -\frac{\partial}{\partial r}(r\tau_c) - \tau_c = 0 \qquad \Rightarrow\quad r\frac{\partial \tau_c}{\partial r} + 2\tau_c = 0$$

$$\delta w_c\quad : -\frac{\partial \tau_c}{\partial \varphi} - \frac{\partial}{\partial r}(r\sigma_c) = 0 \quad \Rightarrow \frac{\partial \sigma_c}{\partial r} + \sigma_c + \frac{\partial \tau_c}{\partial \varphi} = 0 \tag{30a-f}$$

According to Eq. (29), the boundary conditions in each end are as follows.

$$\delta u_{0b} = 0 \qquad \text{or} N_b - \frac{1}{r_b}M_b = 0$$

$$\delta u_{0t} = 0 \qquad \text{or} \qquad N_t - \frac{1}{r_t}M_t = 0$$

$$\delta w_b = 0 \quad \text{or} \frac{M_b'}{r_b} - \frac{N_b}{r_b}\left(u_{0b}-w_b'\right) + br_{bc}k_b\tau_c|_{r_{bc}} = 0$$

$$\delta w_t = 0 \quad \text{or} \frac{M_t'}{r_t} - \frac{N_t}{r_t}\left(u_{0t}-w_t'\right) + br_{tc}k_t\tau_c|_{r_{tc}} = 0$$

$$\delta w_b' = 0 \qquad \text{or} M_b = 0$$

$$\delta w_b' = 0 \qquad \text{or} M_t = 0$$

$$\delta w_c = 0 \qquad \text{or} \tau_c = 0 \tag{31a-g}$$

One can compute the displacement field in the core precisely. The last two governing equations in the relation (30) are related to equilibrium equations of the core. Of the first one we have

$$r\frac{\partial \tau_c}{\partial r} + 2\tau_c = 0 \qquad \Rightarrow 2\tau_c r + r^2\frac{\partial \tau_c}{\partial r} = 0 \qquad \Rightarrow$$

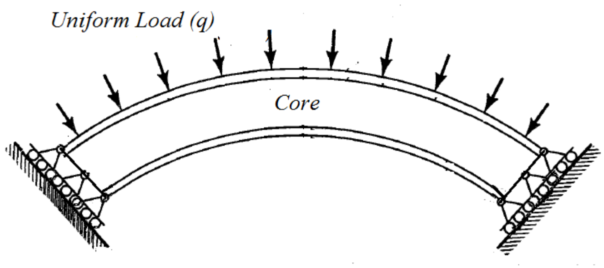

Fig. 3 Curved sandwich beam with simply supported boundary condition subjected to uniform pressure

$$\frac{\partial}{\partial r}\left(r^2\tau_c\right)=0 \qquad \Rightarrow r^2\tau_c=\tau(\varphi)\Rightarrow\tau_c=\frac{\tau}{r^2} \tag{32}$$

where in the recent relation τ is only function of φ. By substituting τ_c from the last equation into Eq. (30) we have

$$\frac{\partial}{\partial r}(r\sigma_c)+\frac{\partial\tau_c}{\partial\varphi}=0 \quad \Rightarrow\frac{\partial}{\partial r}(r\sigma_c)+\frac{1}{r^2}\tau'=0 \quad \Rightarrow \quad r\sigma_c-\frac{\tau'}{r}=C_1 \tag{33}$$

In Eq. (33), C_1 is a constant value of integration. Using the core constitutive relations (15) we may write

$$rE_c\frac{\partial w_c}{\partial r}-\frac{\tau'}{r}=C_1\Rightarrow\frac{\partial w_c}{\partial r}=\frac{1}{E_c}\left(\frac{C_1}{r}+\frac{\tau'}{r^2}\right)\Rightarrow w_c=\frac{1}{E_c}\left(C_2+C_1\ln r-\frac{\tau'}{r}\right) \tag{34}$$

To obtain constant values C_1 and C_2 we use the matching condition in the upper and lower points of core. According to the compatibility condition in these two points we can write

at $r=r_{tc}$

$$w_c=w_t\Rightarrow w_t=\frac{1}{E_c}\left(C_2+C_1\ln r_{tc}-\frac{\tau'}{r_{tc}}\right) \tag{35}$$

at $r=r_{bc}$

$$w_c=w_b\Rightarrow w_b=\frac{1}{E_c}\left(C_2+C_1\ln r_{bc}-\frac{\tau'}{r_{bc}}\right) \tag{36}$$

By accomplishing the two equations simultaneously, the constants of C_1 and C_2 will be achieved as follow.

$$C_1=\frac{E_c(w_b-w_t)+\tau'\left(\frac{1}{r_{bc}}-\frac{1}{r_{tc}}\right)}{\ln\frac{r_{bc}}{r_{tc}}}$$

$$C_2=E_cw_b+\frac{\tau'}{r_{bc}}-\frac{\ln r_{bc}}{\ln\frac{r_{bc}}{r_{tc}}}\left[E_c(w_b-w_t)+\tau'\left(\frac{1}{r_{bc}}-\frac{1}{r_{tc}}\right)\right] \tag{37a-b}$$

By defining parameter of $k_0=\frac{r_{tc}-r_{bc}}{r_{bc}r_{tc}}$ and setting the constants of C_1 and C_2 in the previous equation, the function w_c of core will be

$$w_c=w_b+\frac{\tau'}{E_c}\left(\frac{1}{r_{bc}}-\frac{1}{r}+\frac{\ln\frac{r}{r_{bc}}}{\ln\frac{r_{bc}}{r_{tc}}}k_0\right)+\frac{\ln\frac{r}{r_{bc}}}{\ln\frac{r_{bc}}{r_{tc}}}(w_b-w_t) \tag{38}$$

By certifying distribution of w_c we can obtain the distribution of u_c in the core. Using the core constitutive relations (15) we have.

$$\tau_c=G_c\gamma_{r\varphi}\Rightarrow\gamma_{r\varphi}=\frac{\tau_c}{G_c}=\frac{1}{r^2}\frac{\tau}{G_c}\Rightarrow$$
$$\frac{\partial u_c}{\partial r}-\frac{u_c}{r}+\frac{1}{r}\frac{\partial w_c}{\partial\varphi}=\frac{1}{r^2}\frac{\tau}{G_c}\Rightarrow r\frac{\partial}{\partial r}\left(\frac{u_c}{r}\right)+\frac{1}{r}\frac{\partial w_c}{\partial\varphi}=\frac{1}{r^2}\frac{\tau}{G_c} \tag{39}$$

Substituting Eqs. (38) into (39) we have

$$\frac{\partial}{\partial r}\left(\frac{u_c}{r}\right)=\frac{1}{r^3}\frac{\tau}{G_c}-\left\{\frac{w_b'}{r^2}+\frac{\tau''}{E_c}\left[\left(\frac{1}{r^2 r_{bc}}-\frac{1}{r^3}\right)+k_0\frac{\ln r-\ln r_{bc}}{r^2\ln\frac{r_{bc}}{r_{tc}}}\right]+\left(w_b'-w_t'\right)\left(\frac{\ln r-\ln r_{bc}}{r^2\ln\frac{r_{bc}}{r_{tc}}}\right)\right\} \tag{40}$$

By integration from relation (40) related to r in the interval $[r,r_{tc}]$ we will have.

$$\frac{u_{tc}}{r_{tc}}-\frac{u_c}{r}=\frac{-\tau}{2G_c}\left(\frac{1}{r_{tc}^2}-\frac{1}{r^2}\right)+w_b'\left(\frac{1}{r_{tc}}-\frac{1}{r}\right)+\frac{\tau''}{r_{bc}E_c}\left(\frac{1}{r_{tc}}-\frac{1}{r}\right)\frac{\tau''}{2E_c}\left(\frac{1}{r_{tc}^2}-\frac{1}{r^2}\right)+\frac{k_0}{E_c}\tau''-\frac{1}{\ln\frac{r_{bc}}{r_{tc}}}$$
$$\left(\frac{\ln r_{tc}}{r_{tc}}-\frac{\ln r}{r}+\frac{1}{r_{tc}}-\frac{1}{r}\right)-\frac{\tau''}{E_c}\frac{\ln r_{bc}}{r_{bc}}\left(\frac{1}{r_{tc}}-\frac{1}{r}\right)+\frac{\left(w_b'-w_t'\right)}{\ln\frac{r_{bc}}{r_{tc}}}\left(\frac{\ln r_{tc}}{r_{tc}}-\frac{\ln r}{r}+\frac{1}{r_{tc}}-\frac{1}{r}\right)-\frac{\left(w_b'-w_t'\right)}{\ln\frac{r_{bc}}{r_{tc}}}\left(\frac{1}{r_{tc}}-\frac{1}{r}\right) \tag{41}$$

Solving Eq. (41) for u_c, its distribution in the core will be as follow

$$u_c=\frac{r}{r_{tc}}(1-k_t)u_{0t}+\frac{r}{r_{tc}}w_t'+\left(1-\frac{r}{r_{tc}}\right)w_b'+\left(w_b'-w_t'\right)\left(\frac{r_{tc}-r}{r_{tc}\ln\frac{r_{bc}}{r_{tc}}}+\frac{\ln\frac{r}{r_{bc}}}{\ln\frac{r_{bc}}{r_{tc}}}+\frac{r}{r_{tc}}\right)+\frac{\tau''}{E_c}$$
$$\left[\left(\frac{r_{tc}-r}{r_{tc}\ln\frac{r_{bc}}{r_{tc}}}+\frac{\ln\frac{r}{r_{bc}}}{\ln\frac{r_{bc}}{r_{tc}}}+\frac{r}{r_{tc}}\right)k_0\frac{r_{tc}-r}{r_{tc}r_{bc}}-\frac{1}{2}\left(\frac{r_{tc}^2-r^2}{r\,r_{tc}^2}\right)\right]-\frac{\tau}{2G_c}\left(\frac{r_{tc}^2-r^2}{r\,r_{tc}^2}\right) \tag{42}$$

It should be mentioned that by solving the related equations to the core all parameters have been related to variable $\tau(\varphi)$. Therefore a new equation should be replaced by two equilibrium equations of the core. To obtain this equation we use the compatibility condition (36). At $r=r_{bc}$we have.

$$u_c|_{r=r_{bc}}=u_{0b}(1+k_b)-k_b w_b' \tag{43}$$

By equaling Eqs. (42) and (43) we have.

$$u_{0b}(1+k_b)-k_b w_b^{'}=\frac{r_{bc}}{r_{tc}}(1-k_t)u_{0t}+\frac{r_{bc}}{r_{tc}}w_t^{'}+\left(1-\frac{r_{bc}}{r_{tc}}\right)w_b^{'}+\left(w_b^{'}-w_t^{'}\right)\left(\frac{r_{tc}-r_{bc}}{r_{tc}\ln\frac{r_{bc}}{r_{tc}}}+\frac{r_{bc}}{r_{tc}}\right)$$

$$+\frac{\tau^{''}}{E_c}\left[\left(\frac{r_{tc}-r_{bc}}{r_{tc}\ln\frac{r_{bc}}{r_{tc}}}+\frac{r_{bc}}{r_{tc}}\right)k_0+\frac{1}{r_{bc}}-\frac{1}{r_{tc}}-\frac{1}{2r_{bc}}+\frac{r_{bc}}{2r_{tc}^2}-\frac{\tau}{2G_c}\left(\frac{r_{tc}^2-r_{bc}^2}{r_{bc}r_{tc}^2}\right)=0\right] \tag{44}$$

And after simplifications, relation (44) takes the form

$$(1-k_t)u_{0t}-\frac{r_{tc}}{r_{bc}}(1+k_b)u_{0b}-\left[(1-k_t)+\frac{k_0 r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]w_t^{'}+\frac{r_{tc}}{r_{bc}}\left[1+k_b+\frac{k_0 r_{bc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]w_b^{'}-\frac{k_0}{2G_c}\tau\left(\frac{r_{tc}+r_{bc}}{r_{bc}}\right)+\frac{k_0}{2E_c}\tau^{''}$$

$$\left(2k_0\frac{r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}+\frac{r_{tc}+r_{bc}}{r_{bc}}\right)=0 \tag{45}$$

Eq. (45) along with the four first equilibrium equations, will be the governing equations.

3. Pre-buckling analysis

This study analyses the buckling of curved sandwich beam with flexible core which its upper skin is subjected to uniform load of intensity q_t. In the pre-buckling analysis of curved beam, von-Karman non-linear terms can be disregards. Deformation in the beam is not so great, consequently rotations of β_t and β_b are not so great and therefore the values of β_t^2 and β_b^2 may be ignored. In other words in the case of pre-buckling only linear analysis is sufficient. Also in this case it is supposed that the beam contracts uniformly and therefore the components of displacement u_{0t},u_{0b} and u_c in the pre-buckling case can be disregarded.

According to the above assumptions and using superscript of zero for pre-buckling case, the equilibrium equations in this case would be as follow (It should be considered that because of the uniform contraction in beam all derivatives would be ignored (Hodges and Simitses 2006))

$$\delta u_{0t} \quad : \quad -br_{tc}(1+k_t)\tau_c^0\big|_{r_{tc}}=0$$

$$\delta u_{0b} \quad : \quad -br_{bc}(1+k_b)\tau_c^0\big|_{r_{bc}}=0$$

$$\delta w_t \quad : \quad N_t^0+br_{tc}\sigma_c^0\big|_{r_{tc}}-br_{tc}k_t\tau_c^{'0}\big|_{r_{tc}}-br_t q_t=0$$

$$\delta w_b \quad : \quad N_b^0-br_{bc}\sigma_c^0\big|_{r_{bc}}-br_{bc}k_b\tau_c^{'0}\big|_{r_{bc}}=0$$

$$\delta\tau \quad : \quad \tau^0=0 \tag{46}$$

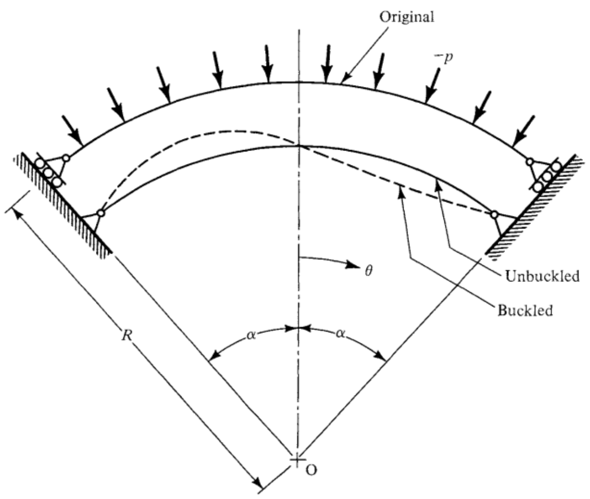

Fig. 4 Schematic of pre-buckling path and buckled form in a homogeneous curved beam with simply supported boundary conditions subjected to uniform load

Based on the last relation (46) we conclude that distribution of shear in the pre-buckling state will be equal to zero. Therefore, first equations would be satisfied and third and fourth equations would be simplified as follow

$$N_t^0+br_{tc}\sigma_c^0\big|_{r_{tc}}=br_tq_t$$
$$N_b^0-br_{bc}\sigma_c^0\big|_{r_{bc}}=0 \tag{47a-b}$$

To estimate the value of stress of σ_c^0 we use the relation $\sigma_c^0=E_c\varepsilon_{rr}^0$. According to the resulted distribution for w_c we can write

$$w_c^0=w_b+\frac{\ln\frac{r}{r_{bc}}}{\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^0-w_t^0\right)\Rightarrow\varepsilon_{rr}^0=\frac{1}{r\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^0-w_t^0\right)\Rightarrow\sigma_c^0=\frac{E_c}{r\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^0-w_t^0\right) \tag{48}$$

And consequently pre-buckling equations will be as follows

$$N_b^0=\frac{bE_c}{\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^0-w_t^0\right)$$
$$N_t^0=\frac{-bE_c}{\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^0-w_t^0\right)+br_tq_t \tag{49a-b}$$

On the other hand according to the stress resultants and regardless of β_t and β_b in the pre-bucking case we will have

$$N_b^0=A_{11b}\frac{w_b^0}{r_b}$$

$$N_t^0=A_{11t}\frac{w_t^0}{r_t} \tag{50a-b}$$

By equaling relations of (49) and (50), the following equations for $w_t^0$and w_b^0 will be achieved.

$$\begin{gathered} A_{11b}\frac{w_b^0}{r_b}=\frac{bE_c}{\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^0-w_t^0\right) \\ A_{11t}\frac{w_t^0}{r_t}=\frac{-bE_c}{\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^0-w_t^0\right)+br_tq_t \end{gathered} \tag{51a-b}$$

By solving two equations and two unknowns in the relation (51) for w_t^0 and w_b^0 we will have

$$\begin{gathered} w_b^0=\frac{br_tq_t}{\frac{A_{11t}}{r_t}\left(1-\frac{A_{11b}}{E_cbr_b}.\ \ln\frac{r_{bc}}{r_{tc}}\right)+\frac{A_{11b}}{r_b}} \\ w_t^0=\frac{br_tq_t\left(1-\frac{A_{11b}}{E_cbr_b}.\ \ln\frac{r_{bc}}{r_{tc}}\right)}{\frac{A_{11t}}{r_t}\left(1-\frac{A_{11b}}{E_cbr_b}.\ \ln\frac{r_{bc}}{r_{tc}}\right)+\frac{A_{11b}}{r_b}} \end{gathered} \tag{52a-b}$$

and pre-bucking forces are equal to.

$$\begin{gathered} N_t^0=\frac{br_tq_t\left(1-\frac{A_{11b}}{E_cbr_b}.\ \ln\frac{r_{bc}}{r_{tc}}\right)}{1-\frac{A_{11b}}{E_cbr_b}.\ \ln\frac{r_{bc}}{r_{tc}}+\frac{r_t}{r_b}.\frac{A_{11b}}{A_tE_t}}=q_tn_t^0 \\ N_b^0=\frac{br_tq_t}{1+\frac{A_{11t}}{A_{11b}}.\frac{r_b}{r_t}\left(1-\frac{A_{11b}}{E_cbr_b}.\ \ln\frac{r_{bc}}{r_{tc}}\right)}=q_tn_b^0 \end{gathered} \tag{53a-b}$$

4. Stability equations

For derivation of stability equations from the primary path the concept of adjacent equilibrium criterion is used. According to this criterion a bucking state on pre-bucking path is considered that is shown by superscript zero. This state is perturbed. The amount of this perturbation is nonzero. Because, if it is equal to zero the structure will remain on its initial path. If we show the amount of development by superscript one, we will have a new equilibrium path that its components will be as follow (Hodges and Simitses 2006)

$$\begin{gathered} u_{0b}=u_{0b}^0+u_{0b}^1 \\ u_{0t}=u_{0t}^0+u_{0t}^1 \\ w_b=w_b^0+w_b^1 \\ w_t=w_t^0+w_t^1 \\ \tau=\tau^0+\tau^1 \end{gathered} \tag{54a-e}$$

Due to the increment in the displacement components, the stress resultants will perturb too. It should be considered that the values with superscripts one are very small and the second order of them will be disregarded. Accordingly, stability equations for curved sandwich beam will be as follow

$$-N_b^{1'}-\frac{1}{r_b}M_b^{1'}+\frac{N_b^0}{r_b}\left(u_{0b}^1-w_b^{1'}\right)-br_{bc}(1+k_b)\tau_c^1\big|_{r_{bc}}=0$$

$$-N_t^{1'}-\frac{1}{r_t}M_t^{1'}+\frac{N_t^0}{r_t}\left(u_{0t}^1-w_t^{1'}\right)+br_{tc}(1-k_t)\tau_c^1\big|_{r_{tc}}=0$$

$$N_b^1\ -\frac{1}{r_b}M_b^{1''}+\frac{N_b^0}{r_b}\left(u_{0b}^{1'}-w_b^{1''}\right)-br_{bc}\sigma_c^1\big|_{r_{bc}}-br_{bc}\,k_b\tau_c^{1'}\big|_{r_{bc}}=0$$

$$N_t^1\ -\frac{1}{r_t}M_t^{1''}+\frac{N_t^0}{r_t}\left(u_{0t}^{1'}-w_t^{1''}\right)+br_{tc}\sigma_c^1\big|_{r_{bc}}-br_{tc}\,k_t\tau_c^{1'}\big|_{r_{tc}}=0$$

$$(1-k_t)u_{0t}^1-\frac{r_{tc}}{r_{bc}}(1+k_b)u_{0b}^1-\left[(1-k_t)+\frac{k_0 r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]w_t^{1'}+\frac{r_{tc}}{r_{bc}}\left[1+k_b+\frac{k_0 r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]w_b^{1'}-\frac{k_0}{2G_c}\tau^1\left(\frac{r_{tc}+r_{bc}}{r_{bc}}\right)$$
$$+\frac{k_0}{2E_c}\tau^{1''}\left(2k_0\frac{r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}+\frac{r_{tc}+r_{bc}}{r_{bc}}\right)=0 \tag{55a-e}$$

The presented equations in relation (55) should be like an eigenvalue system with unknown coefficients N_t^0 and N_b^0. It should be stated that N_t^1, M_t^1, N_b^1 and M_b^1 are the value of developments of stress resultants that are calculated as follow

$$N_b^1=A_{11b}\left(\frac{u_{0b}^{1'}+w_b^1}{r_b}\right)$$

$$N_t^1=A_{11t}\left(\frac{u_{0t}^{1'}+w_t^1}{r_t}\right)$$

$$M_b^1=D_{11b}\left(\frac{u_{0b}^{1'}-w_b^{1''}}{r_b^2}\right)$$

$$N_t^1=D_{11t}\left(\frac{u_{0t}^{1'}+w_t^1}{r_t^2}\right)$$

$$\sigma_c^1=\frac{1}{r^2}\tau^{1'}+\frac{1}{r}\left(\frac{E_c}{\ln\frac{r_{bc}}{r_{tc}}}\left(w_b^1-w_t^1\right)+\frac{\tau^{1'}k_0}{\ln\frac{r_{bc}}{r_{tc}}}\right) \tag{56a-e}$$

Boundary conditions in this analysis are considered to be simply supported on two edges. For two edges of $\varphi=0$, α_0 these conditions are

$$N_b^1+\frac{1}{r}M_b^1=0$$

$$N_t^1+\frac{1}{r}M_t^1=0$$

$$w_b^1=0$$

$$w_t^1=0$$

$$M_b^1=0$$

$$M_t^1=0$$

$$w_c^1=0 \qquad (57a\text{-}g)$$

According to obtained result for w_c and according to the fact that w_b and w_t are equal to zero in two edges, the boundary conditions will be simplified as follow

$$N_b^1=0$$

$$N_t^1=0$$

$$w_b^1=0$$

$$w_t^1=0$$

$$M_b^1=0$$

$$M_t^1=0$$

$$\tau^{1'}=0 \qquad (58a\text{-}g)$$

According to the definitions of N_i^1 and M_i^1 that are presented in relation (56), the above boundary conditions can be offered as follow

$$u_{0b}^{1'}=0$$

$$u_{0t}^{1'}=0$$

$$w_b^1=0$$

$$w_t^1=0$$

$$w_b^{1''}=0$$

$$w_t^{1''}=0$$

$$\tau^{1'}=0 \qquad (59a\text{-}g)$$

To satisfy the boundary conditions (59), functions u_{0b}^1, u_{0t}^1, w_b^1,w_t^1 and τ^1 will be considered as follow

$$u_{0b}^1=U_b\cos\frac{n\pi\varphi}{\varphi_0}$$

$$u_{0t}^{1}=U_t\cos\frac{n\pi\varphi}{\varphi_0}$$
$$w_b^{1}=W_b\sin\frac{n\pi\varphi}{\varphi_0}$$
$$w_t^{1}=W_t\sin\frac{n\pi\varphi}{\varphi_0}$$
$$\tau^{1}=T_t\cos\frac{n\pi\varphi}{\varphi_0} \tag{60a-e}$$

The assumed mode shapes (60) are set in the stability equations. It is to mention that stability equations based on displacement components will be written as follow that for easy usage the superscript one is ignored.

$$-\frac{A_{11b}}{r_b}\left(u_{0b}''+w_b'\right)-\frac{D_{11b}}{r_b^3}\left(u_{0b}''-w_b'''\right)+\frac{N_b^0}{r_b}\left(u_{0b}-w_b'\right)-b\tau\frac{(1+k_b)}{r_{bc}}=0$$
$$-\frac{A_{11t}}{r_t}\left(u_{0t}''+w_t'\right)-\frac{D_{11t}}{r_t^3}\left(u_{0t}''-w_t'''\right)+\frac{N_t^0}{r_t}\left(u_{0t}-w_t'\right)+b\tau\frac{(1-k_t)}{r_{tc}}=0$$
$$\frac{A_{11b}}{r_b}\left(u_{0b}'+w_b\right)-\frac{D_{11b}}{r_b^3}\left(u_{0b}'''-w_b''''\right)+\frac{N_b^0}{r_b}\left(u_{0b}-w_b''\right)-br_{bc}\left\{\frac{\tau'}{r_{bc}^2}+\frac{E_c}{r_{bc}\ln\frac{r_{bc}}{r_{tc}}}\left(w_b-w_t\right)+\frac{\tau' k_0}{\ln\frac{r_{bc}}{r_{tc}}}\right\}-\frac{bk_b}{r_{bc}}\tau'=0$$
$$\frac{A_{11t}}{r_t}\left(u_{0t}'+w_t\right)-\frac{D_{11t}}{r_t^3}\left(u_{0t}'''-w_t''''\right)+\frac{N_t^0}{r_t}\left(u_{0t}-w_t''\right)-br_{tc}\left\{\frac{\tau'}{r_{tc}^2}+\frac{E_c}{r_{tc}\ln\frac{r_{bc}}{r_{tc}}}\left(w_b-w_t\right)+\frac{\tau' k_0}{\ln\frac{r_{bc}}{r_{tc}}}\right\}-\frac{bk_t}{r_{tc}}\tau'=0$$
$$(1-k_t)u_{0t}-\frac{r_{tc}}{r_{bc}}(1+k_b)u_{0b}-\left[(1-k_t)+\frac{k_0}{\ln\frac{r_{bc}}{r_{tc}}}\right]w_t'+\frac{r_{tc}}{r_{bc}}\left[1+k_b+\frac{k_0 r_{bc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]w_b'-\frac{k_0}{2G_c}\left(1+\frac{r_{tc}}{r_{bc}}\right)\tau+\frac{k_0}{2E_c}\tau''$$
$$\times\left(2k_0\frac{r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}+1+\frac{r_{tc}}{r_{bc}}\right)=0 \tag{61a-e}$$

Therefore if we set the substitute the solutions (60) in the relation (61) the problem will be written as follow

$$\left([K]_e-q[K]_g\right)\begin{Bmatrix}W_t\\W_b\\U_t\\U_b\\T\end{Bmatrix}=\begin{Bmatrix}0\\0\\0\\0\\0\end{Bmatrix} \tag{62}$$

Where in relation (62), $[K]_e$ and $[K]_g$ respectively show elastic and geometric stiffnesses. By definitions of $\varphi_n=\frac{n\pi}{\varphi_0}$ elements of each of these two matrices can be written as follow

$$K_e^{11}=\frac{D_{11t}}{r_t^3}\varphi_n^4+\frac{A_{11t}}{r_t}-\frac{bE_c}{\ln\frac{r_{bc}}{r_{tc}}},\quad K_e^{12}=\frac{bE_c}{\ln\frac{r_{bc}}{r_{tc}}}$$

$$K_e^{13}=-\frac{D_{11t}}{r_t^3}\varphi_n^3-\frac{A_{11t}}{r_t}\varphi_n\quad,\quad K_e^{14}=0\quad,\quad K_e^{15}=\frac{-b}{r_{tc}}\left[(1-k_t)+\frac{k_0 r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]\varphi_n$$

$$K_e^{21}=\frac{bE_c}{\ln\frac{r_{bc}}{r_{tc}}},\quad K_e^{22}=\frac{D_{11b}}{r_b^3}\varphi_n^4+\frac{A_{11b}}{r_b}-\frac{bE_c}{\ln\frac{r_{bc}}{r_{tc}}}$$

$$K_e^{23}=0\quad,\quad K_e^{24}=-\frac{D_{11b}}{r_b^3}\varphi_n^3-\frac{A_{11b}}{r_b}\varphi_n\,,\quad K_e^{25}=\frac{b}{r_{bc}}\left[(1+k_b)+\frac{k_0 r_{bc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]\varphi_n$$

$$K_e^{31}=-\frac{D_{11t}}{r_t^3}\varphi_n^3-\frac{A_{11t}}{r_t}\varphi_n,\quad K_e^{32}=0$$

$$K_e^{33}=\left(\frac{D_{11t}}{r_t^3}+\frac{A_{11t}}{r_t}\right)\varphi_n^2\,,\quad K_e^{34}=0\quad,\quad K_e^{35}=\frac{b}{r_{tc}}(1-k_t)$$

$$K_e^{41}=0,\quad K_e^{42}=-\frac{D_{11b}}{r_b^3}\varphi_n^3-\frac{A_{11b}}{r_b}\varphi_n$$

$$K_e^{43}=0,\quad K_e^{44}=\left(\frac{D_{11b}}{r_b^3}+\frac{A_{11b}}{r_b}\right)\varphi_n^2\quad,\qquad K_e^{45}=\frac{-b}{r_{bc}}(1+k_b)$$

$$K_e^{51}=\frac{-b}{r_{tc}}\left[(1-k_t)+\frac{k_0 r_{tc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]\varphi_n,\qquad K_e^{52}=\frac{b}{r_{bc}}\left[(1+k_b)+\frac{k_0 r_{bc}}{\ln\frac{r_{bc}}{r_{tc}}}\right]\varphi_n$$

$$K_e^{53}=\frac{b}{r_{tc}}(1-k_t)\quad,\quad K_e^{54}=\frac{-b}{r_{bc}}(1+k_b)$$

$$K_e^{55}=\frac{-b}{2G_c}\left(\frac{r_{bc}+r_{tc}}{r_{bc}r_{tc}}\right)-\frac{bk_0}{2E_c}\left(\frac{r_{bc}+r_{tc}}{r_{bc}r_{tc}}+\frac{2k_0}{\ln\frac{r_{bc}}{r_{tc}}}\right)\varphi_n^2 \tag{63a-y}$$

where in derivation, the existent relations in the Eq. (63), the last stability equation is multiplied in $\frac{b}{r_{bc}}$in order to set the membersof the elastic matrix symmetric. Accordingly members of K_g are.

$$K_g^{11}=-\frac{n_t^0}{r_t}\varphi_n^2,\quad K_g^{12}=0$$

$$K_g^{13}=-\frac{n_t^0}{r_t}\varphi_n\,,\quad K_g^{14}=K_g^{15}=0$$

$$K_g^{21}=0,\quad K_g^{22}=-\frac{n_b^0}{r_b}\varphi_n^2$$

$$K_g^{23}=0,\quad K_g^{24}=\frac{n_b^0}{r_b}\varphi_n \quad,\quad K_g^{25}=0$$

$$K_g^{31}=\frac{n_t^0}{r_t}\varphi_n \quad,\quad K_g^{32}=0$$

$$K_g^{33}=-\frac{n_t^0}{r_t},\quad K_g^{34}=K_g^{35}=0$$

$$K_g^{41}=0 \quad,\quad K_g^{42}=\frac{n_b^0}{r_b}\varphi_n$$

$$K_g^{43}=0,\quad K_g^{44}=-\frac{n_b^0}{r_b},\quad K_g^{45}=0$$

$$K_g^{51}=K_g^{52}=K_g^{53}=K_g^{54}=K_g^{55}=0 \tag{64a-y}$$

By equaling determinant of the coefficient matrix of the problem there would be an eigenvalue problem in which critical pressure is an eigenvalue and the buckled shape of the beam will be like an eigenvector. As in the analysis of eigenvalue, the eigenvector is not specified uniquely, the values of lateral rising and longitudinal displacement at the state of bucking are not determined uniquely. For this reason the bucking shapes only show the schematic at the state of bucking.

5. Results

In this section, using Matlab software and the theory offered in the previous sections, bucking of curved sandwich beam with simply supported boundary conditions is analyzed. This analysis is established for orthotropic and isotropic face sheets.

5.1 Numerical examples

Example 1. In this example buckling analysis of curved sandwich beams with a transversely flexible core is done. Skins are composite in the cross-ply form. The type of skins is of graphite-epoxy A54/3501 that has the following mechanical and geometrical properties. Numerical results are shown in Figs. (5) to (8).

E_{11}=144.8 Gpa , E_{22}= 9.65 Gpa , G_{12}= G_{13}= 4.14 Gpa ,G_{23}=3.45Gpa

ρ=1389.23 kg/m^3, ν_{12}= 0.3

r_{bc}= 0.7 m , r_{tc}=0.9 m , r_t=0.92 m , r_b=0.68 m , b=0.1 m

In Fig. (5) for three values of elasticity module of core E_c=200, 400, 500 Mpa the critical pressure is obtained. The lay-up of the two skins is considered as $[0,90]_s$. The results of the amounts of critical pressure is offered against the beam angle. As it is expected, keeping higher the elasticity module of core leads to the increasing of critical load.

In Fig. (6) the effects of entire thickness of laminate on critical load is studied. Three below cases are considered

Table 1 Critical buckling load P_{cr} (MN/m) of curved symmetric sandwich beam subjected to uniform load

φ_0	E_c (Mpa)	P_{cr} (MN/m)
30	200	79.04
	400	89.20
	500	90.08
60	200	79.00
	400	87.28
	500	90.08
90	200	77.68
	400	83.92
	500	90.08
120	200	79.00
	400	83.92
	500	88.80
150	200	77.60
	400	84.32
	500	88.40
180	200	77.68
	400	83.92
	500	88.32

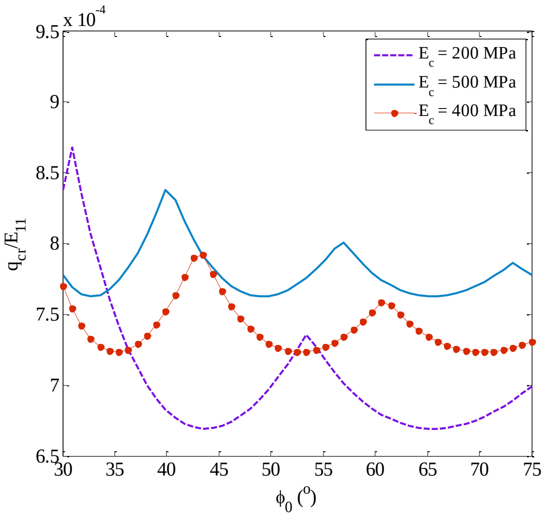

Fig. 5 Variation of critical buckling pressure versus curved beam angle

$$d_t=d_b=\begin{cases}0.025\ \text{m}\\ 0.02\ \text{m}\\ 0.015\ \text{m}\end{cases}$$

Critical load is offered versus angle φ_0. Numerical results are given for three states of skin thickness. Core properties are assumed as E_c= 200 Mpa ν_c= 0.3. Stacking sequence is assumed as $[0,90]_s$. As it is expected, as the thickness of skins increases, the critical load increasing. In each state, the amount of pile angle indicates the change in the bucking mode.

Table 2 The critical buckling load of curved sandwich beam subjected to uniform load

φ_0	d_b=d_t (m)	P_{cr} (MN/m)
30	0.015 0.02 0.025	46.45 79.07 127.70
60	0.015 0.02 0.025	46.45 79.01 127.70
90	0.015 0.02 0.025	44.41 77.68 127.70
120	0.015 0.02 0.025	44.24 79.01 124.28
150	0.015 0.02 0.025	44.38 77.60 122.69
180	0.015 0.02 0.025	5.62 77.68 122.18

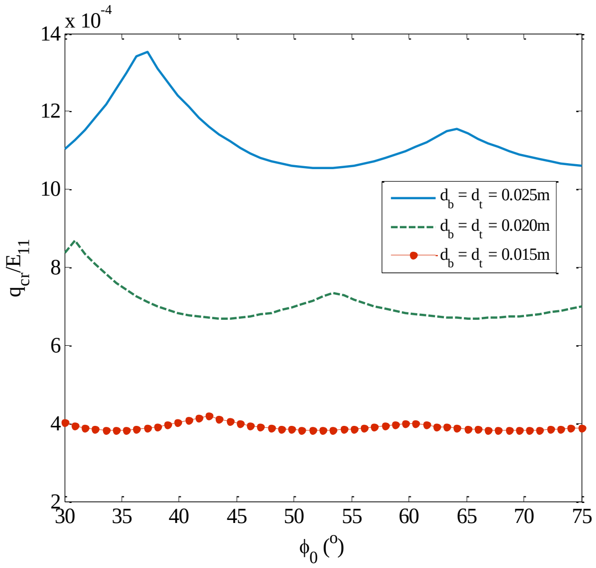

Fig. 6 Variation of critical buckling pressure versus curved beam angle

In Fig. (7) lay-up influence for two cross-ply laminates is analyzed. Two different arrays of $[0,90]_s$ and $[90,0]_s$ are studied. The thickness of the two cases is equal to 0.015 m and each case consists of 4 layers with equal thicknesses. Core module is equal to E_c=200 Mpa. According to Fig. (7) in the studied area the amount of critical load for case $[0,90]_s$ is more than the state in which skins have $[90,0]_s$ layup.

Table 3 The critical buckling load of curved sandwich beam subjected to uniform load

φ_0	Lay-up	P_{cr} (MN/m)
30	$[0,90]_s$ $[90,0]_s$	46.45 79.07
60	$[0,90]_s$ $[90,0]_s$	46.45 79.01
90	$[0,90]_s$ $[90,0]_s$	44.41 77.68
120	$[0,90]_s$ $[90,0]_s$	44.24 79.01
150	$[0,90]_s$ $[90,0]_s$	44.38 77.60
180	$[0,90]_s$ $[90,0]_s$	5.62 77.68

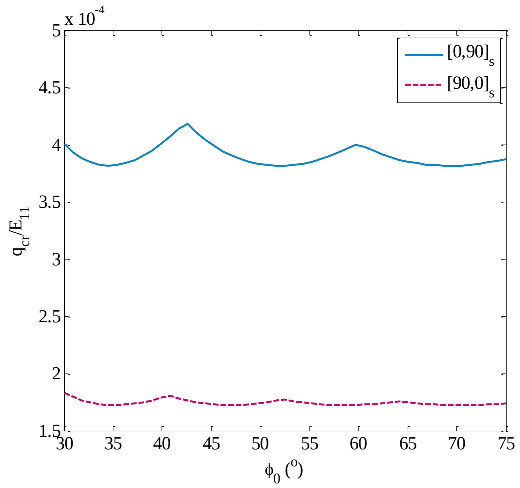

Fig. 7 Variation of critical buckling pressure versus curved beam angle

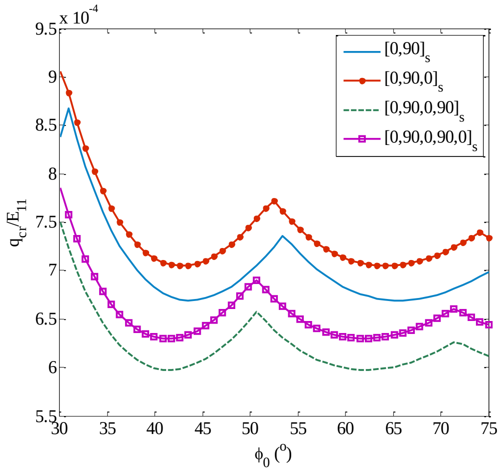

Fig. 8 Variation of critical buckling pressure versus curved beam angle

In Fig. (8) the layering influence and the number of layers on critical pressure is studied. The thickness of the two cases is considered as 0.02 m. Core module is equal to $E_c = 200$ Mpa. Four different layups are considered that are

1:$[0,90]_s$
2:$[0,90,0]_s$
3:$[0,90,0,90]_s$
4:$[0,90,0,90,0]_s$

According to Fig. (8), by adding two layers of 0° to the skins, the critical load of the structure increases. Critical load of pile with stacking (2) is much more than (1) and with the layup (4) is more than (3). Also adding two layers 90° to arrays leads to decreasing the critical load, because as it is obvious critical load in the state (3) is less than state (2).

Example 2. In this example we study the critical load of composite curve sandwich beam with

Table 4 The critical buckling load of curved sandwich beam subjected to uniform load

φ_0	Lay-up	P_{cr} (MN/m)
30	$[0,90]_s$	79.07
	$[0,90,0]_s$	104.88
	$[0,90,0,90]_s$	86.88
	$[0,90,0,90,0]_s$	90.95
60	$[0,90]_s$	79.01
	$[0,90,0]_s$	82.50
	$[0,90,0,90]_s$	69.41
	$[0,90,0,90,0]_s$	73.11
90	$[0,90]_s$	77.68
	$[0,90,0]_s$	82.14
	$[0,90,0,90]_s$	70.36
	$[0,90,0,90,0]_s$	74.23
120	$[0,90]_s$	79.01
	$[0,90,0]_s$	82.50
	$[0,90,0,90]_s$	69.41
	$[0,90,0,90,0]_s$	73.11
150	$[0,90]_s$	77.60
	$[0,90,0]_s$	81.63
	$[0,90,0,90]_s$	69.40
	$[0,90,0,90,0]_s$	73.18
180	$[0,90]_s$	77.68
	$[0,90,0]_s$	82.14
	$[0,90,0,90]_s$	69.40
	$[0,90,0,90,0]_s$	73.11

Table 5 The critical buckling load of curved sandwich beam subjected to uniform load

φ_0	Lay-up	P_{cr} (MN/m)
30	$[30,-30]_s$	158.10
	$[45,-45]_s$	93.16
	$[60,-60]_s$	40.02
60	$[30,-30]_s$	158.10
	$[45,-45]_s$	80.45
	$[60,-60]_s$	40.02
90	$[30,-30]_s$	158.10
	$[45,-45]_s$	77.50
	$[60,-60]_s$	38.30
120	$[30,-30]_s$	158.10
	$[45,-45]_s$	78.17
	$[60,-60]_s$	38.11
150	$[30,-30]_s$	158.10
	$[45,-45]_s$	78.08
	$[60,-60]_s$	38.22
180	$[30,-30]_s$	157.70
	$[45,-45]_s$	77.50
	$[60,-60]_s$	38.30

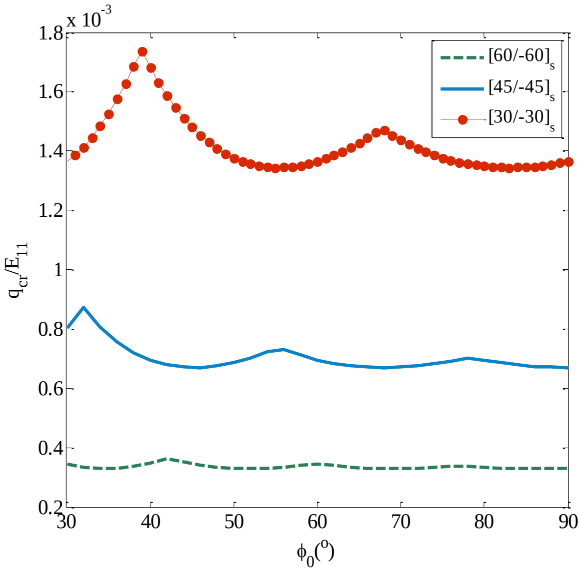

Fig. 9 Variation of critical buckling pressure versus curved beam angle

Table 4 The critical buckling load of curved sandwich beam subjected to uniform load

φ_0	Lay-up	P_{cr} (MN/m)
30	$[0,90]_s$	79.07
	$[0,90,0]_s$	104.88
	$[0,90,0,90]_s$	86.88
	$[0,90,0,90,0]_s$	90.95
60	$[0,90]_s$	79.01
	$[0,90,0]_s$	82.50
	$[0,90,0,90]_s$	69.41
	$[0,90,0,90,0]_s$	73.11
90	$[0,90]_s$	77.68
	$[0,90,0]_s$	82.14
	$[0,90,0,90]_s$	70.36
	$[0,90,0,90,0]_s$	74.23
120	$[0,90]_s$	79.01
	$[0,90,0]_s$	82.50
	$[0,90,0,90]_s$	69.41
	$[0,90,0,90,0]_s$	73.11
150	$[0,90]_s$	77.60
	$[0,90,0]_s$	81.63
	$[0,90,0,90]_s$	69.40
	$[0,90,0,90,0]_s$	73.18
180	$[0,90]_s$	77.68
	$[0,90,0]_s$	82.14
	$[0,90,0,90]_s$	69.40
	$[0,90,0,90,0]_s$	73.11

Table 6 The critical buckling load of curved sandwich beam subjected to uniform load

φ_0	d_b=d_t (m)	P_{cr} (MN/m)
30	0.01	16.97
	0.015	35.52
	0.02	75.34
	0.025	112.86
60	0.01	16.46
	0.015	35.36
	0.02	62.83
	0.025	107.98
90	0.01	16.21
	0.015	35.62
	0.02	64.46
	0.025	101.21
120	0.01	16.39
	0.015	35.45
	0.02	62.57
	0.025	100.33
150	0.01	16.28
	0.015	35.17
	0.02	63.43
	0.025	100.76
180	0.01	16.33
	0.015	35.30
	0.02	62.74
	0.025	101.10

soft core and composite skins in the form of Angle-ply. The core is flexible and skins follow the classical theory. The type of skins is of graphite-epoxy A54/3501that has the following mechanical and geometrical properties. Numerical results are shown in the Figs. (9) and (12).

E_{11}=144.8 Gpa , E_{22}= 9.65 Gpa , G_{12}= G_{13}= 4.14 Gpa ,G_{23}=3.45Gpa

ρ=1389.23 kg/m^3, ν_{12}= 0.3

r_{bc}= 0.7 m , r_{tc}=0.9 m , r_t=0.92 m , r_b=0.68 m , b=0.1 m

In Fig. (9), the influence of layup skins on critical load is studied. Both of the two skins have the array of $[\theta/-\theta]_s$. The thickness of both skins is 0.03 m and the radii of curvature are equal to 0.7 m , 0.9 m. Core elasticity module and the Poisson ratio are respectively 200 Mpa and 0.3. In each laminate, the thicknesses of layers are the same. The numerical results for three various angles of layering φ_0=30,45 and 60 are offered. The results show that in the three studied state by increasing layering angle, the critical load will decreases. Also layup angle and beam head angle, are influential parameters on the value of critical load and the critical shape of beam.

In Fig. (10) the effect of skins thickness is analyzed. Both skins have the same thickness and have the arrays of $[30/-30]_s$. The value of module and Possion's ratio of the core and geometric parameters of the core according to the mentioned cases are chosen in the Fig. (9). Four different values are considered for skins thicknesses. As we expected by increasing the thickness of skins

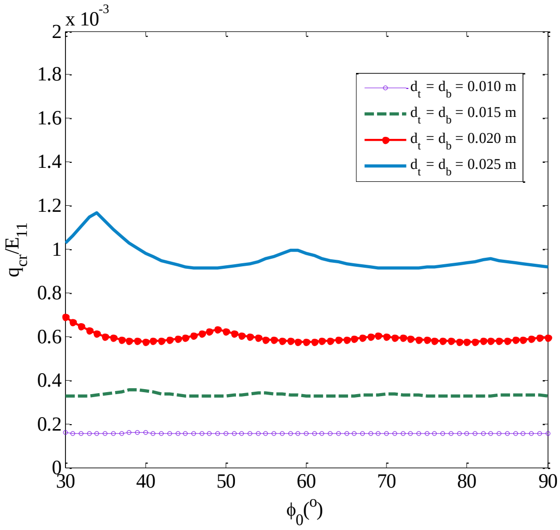

Fig. 10 Variation of critical buckling pressure versus curved beam angle

Table 7 The critical buckling load of curved sandwich beam subjected to uniform load

φ_0	r_{bc} (m)	P_{cr} (MN/m)
30	0.6	46.76
	0.65	40.64
	0.7	38.11
60	0.6	38.22
	0.65	38.64
	0.7	38.11
90	0.6	39.17
	0.65	37.62
	0.7	38.11
120	0.6	40.77
	0.65	37.83
	0.7	38.11
150	0.6	38.48
	0.65	37.77
	0.7	38.11
180	0.6	40.77
	0.65	37.62
	0.7	38.11

the critical load will increase because beam elastic stiffness will increase by increasing the thickness. In each state the thickness of skin, curve has several relative maximum points that shape of the structure at the onset of bucking changes.

In Fig. (11) the effect of core thickness is analyzed. The values of core elasticity module and the Poisson ratio are respectively 200 Mpa and 0.3. Thickness of skins are equal to 0.015 m and the skins follow the array of $[30/\text{-}30]_s$. The radius of upper curve of core is equal to r_{tc}=0.9 m.

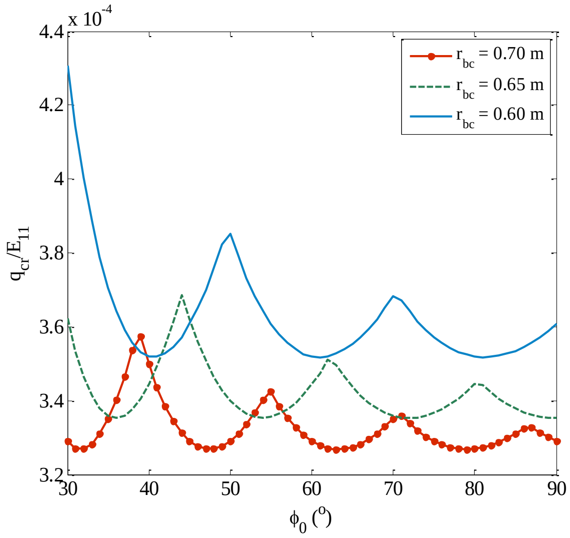

Fig. 11 Variation of critical buckling pressure versus curved beam angle

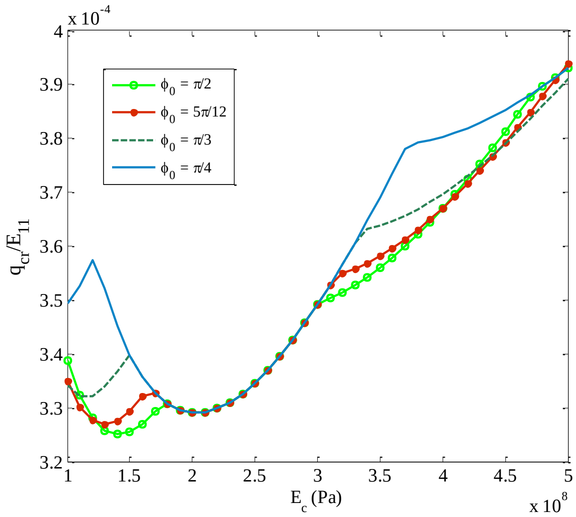

Fig. 12 Variation of critical buckling pressure versus core stiffness

Three different thicknesses are chosen for the core that are 0.2 m, 0.25mand 0.3 m. Therefore the internal radial curve is equal to r_{bc}=0.6 m, r_{bc}=0.65 m and r_{bc}=0.7 m. Results show that by increasing the thickness of the core the critical load increases unless the bucking mode changing occur (like what is observed in the angle φ_0=40°). The buckling pressure intensively depends on beam angle and thickness of the core.

In Fig. (12), the effect of core module for four curved beams is analyzed. All of the applied parameters are similar to Fig. (11). Thickness of skins are equal to 0.015 m. The results show that by increasing the stiffness of core for the state of E_c>200 Mpa the critical load will be enhanced. Even though, in the area 100Mpa>E_c>200 Mpa increasing the stiffness of the core lead to decreasing the critical load.

6. Conclusions

In this study, the stability behavior of a sandwich arch with flexible core and composite laminated face sheets is studied analytically. Displacement field in the core is solved via the compatible elasticity equations, while for the two skins classical laminate theory is adopted. Various lay-ups and types of laminations are used for the faces. The resulted governing equations are established in general form via the virtual displacements principle. The case of an arch under uniform lateral pressure is analyzed. The pre-buckling solution is accomplished with proper linearizations and the stability equations are obtained via the adjacent equilibrium criterion. An exact solution is obtained for the case of a beam with both edges simply supported. Analytical closed form phrase is presented to deduce the critical buckling load of the arch. As concluded, the stiffness of the core, thickness of the core, curved beam angle and face sheets lamination have influential effects on critical buckling loads of the arch. Furthermore, buckling shape of the arch is highly dependent to the above mentioned parameters.

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