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Random vibration analysis of structures by a time-domain explicit formulation method

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Abstract. Non-stationary random vibration of linear structures with uncertain parameters is investigated in this paper. A time-domain explicit formulation method is first presented for dynamic response analysis of deterministic structures subjected to non-stationary random excitations. The method is then employed to predict the random responses of a structure with given values of structural parameters, which are used to fit the conditional expectations of responses with relation to the structural random parameters by the response surface technique. Based on the total expectation theorem, the known conditional expectations are averaged to yield the random responses of stochastic structures as the total expectations. A numerical example involving a frame structure is investigated to illustrate the effectiveness of the present approach by comparison with the power spectrum method and the Monte Carlo simulation method. The proposed method is also applied to non-stationary random seismic analysis of a practical arch bridge with structural uncertainties, indicating the feasibility of the present approach for analysis of complex structures.

Keywords: non-stationary random vibration; stochastic structures; time-domain; explicit formulation

1. Introduction

Uncertainties in structures such as physical and/or geometric parameters as well as uncertainties in external excitations are inevitably encountered in practical engineering. The propagation of such uncertainties to the response of interest provides the basis for a realistic prediction. Therefore it is more reasonable to consider the variability of both structural parameters and external excitations in dynamic response analysis of structures. After several decades of extensive analytical and numerical investigations, some approaches, including the sampling and non-sampling approaches, have been proposed to attain the statistics of dynamic responses of stochastic structures.

The well-known Monte Carlo simulation (MCS) (Astill *et al*. 1972, Shinozuka 1972) can be categorized as the sampling methodology, which is robust and provides a universal means of

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solving various stochastic problems. To enhance the efficiency, two basic strategies can be followed. One strategy aims at reducing computational time of single sample analysis by exploiting the fact that the same structure with similar properties has to be analyzed many times. The Neumann expansion MCS (Chakraborty and Dey 1998, Zhao and Chen 2000) is atypical example for such consideration. Along with the development of computer techniques, the other option is to apply parallel processing (Papadrakakis and Kotsopulos 1999, Szèkely and Schuëller 2001, Kanapady and Tamma 2003, Shioya and Yagawa 2003) for different sample analyses. However, it should be noted that the structure of interest for engineering practice is generally of large-scale, usually including tens of thousands of degrees of freedom (DOFs). In this sense, one still has to face the challenge of computational effort if MCS is to be used.

Meanwhile, to meet the practical engineering requirements, the alternative non-sampling approaches such as the random perturbation method (RPM) and the orthogonal polynomial expansion method (OPEM) are also investigated. RPM has been developed based on the early stochastic finite element method (SFEM) (Contreras 1980). This method was initially applied to random eigenvalue problems, random static problems and random buckling problems (Zhu and Wu 1991, Kleiber and Hien 1992). Afterwards, the method was extended for solving dynamic problems of stochastic structures with or without random excitations (Liu *et al*. 1986, Wall and Bucher 1987, Benaroya and Rehak 1988, Ghanem and Spanos 1991, Benfratello and Muscolino 1998, Zhao and Chen 1998, Nieuwenhof and Coyette 2003, Barbato and Conte 2007, Śniady *et al*. 2008, Wang *et al*. 2010). RPM follows all the steps of a deterministic analysis and is therefore applicable for arbitrarily large structures. It seems to be the most computationally efficient and feasible when used in industry. However, the method sometimes yields inaccurate results in the case of large variation of structural parameters. To compensate for this defect, a second- or higher-order perturbation expansion is employed (Choi and Noh 2000), but accordingly, the computational burden is exponentially increased. In addition, RPM encounters great difficulty in dynamic response analysis due to the so-called secular term problem (Liu *et al*. 1988). The random dynamic response has also been studied for special types of linear structures, mostly truss structures, by the random factor method (RFM) (Gao *et al*. 2003, 2004, 2005, 2009) closely related to RPM. Since this method is tailored for truss structures and hence is not applicable for general structures. The idea of orthogonal polynomial expansion was first used to study a particular class of random differential equations with random coefficients, and the structural response was expanded as a set of orthogonal series and the corresponding numerical characteristics were given as analytical solutions (Sun 1979). The approach was termed as OPEM and was further extended for the response analysis of structures with uncertain parameters under deterministic or random dynamic excitations (Jensen and Iwan 1991, 1992, Iwan and Jensen 1993). An expanded order system method (EOSM) for the same problem has also been developed based on the idea of subspace orthogonal decomposition of the response of structures (Li and Liao 2001). Unlike RPM, EOSM does not require the assumption that the variation of structural parameters is small. However, the size of the extended system derived in this method is much larger than the counterpart of the original system, which means an exponential growth of computational efforts with the extended order will occur.

The aforementioned methods are available to obtain the statistical characteristics of structural responses, i.e., mean values and variances, but are powerless for obtaining accurate probabilistic information on structural performance. The probability density evolution method (PDEM) provides feasible ways for capturing the instantaneous probability density function and its evolution of the response of structures. However, in this method, it is required to solve the

Dostupov-Pugachev probability density evolution equation (Dostupov and Pugachev 1957), a multi-dimensional partial differential equation, which is difficult to solve for multi-DOF systems. Recently, a generalized PDEM has been proposed and verified to be applicable to linear and nonlinear dynamical systems (Li and Chen 2005, 2009, Chen and Li 2005, 2009), in which the multi-dimensional probability density evolution equation is reduced to a series of one-dimensional probability density evolution equations and could be solved more efficiently. Even so, the application of this approach to practical engineering is still heavily restrained by the number of random variables involved in the description of structural properties and external excitations. In other words, it suffers from the so-called "curse of dimensionality", i.e. an exponential growth of efforts with the increase of the number of random variables involved.

The purpose of the current study is to present an efficient approach to analyze the non-stationary random vibration of linear structures with random parameters. In recent years, a time-domain explicit formulation method (TDEFM) (Su and Xu 2010, Su *et al*. 2011) has been developed for random vibration analysis and dynamic reliability analysis of deterministic structures subjected to non-stationary random excitations. The method has been shown to be more efficient than the traditional power spectrum method (PSM) which requires a large amount of numerical integrals in both time domain and frequency domain when non-stationary random excitations are involved. Combined with the total expectation theorem in probability theory and the response surface techniques as well, the TDEFM originally developed for deterministic structures is extended to random response analysis of stochastic structures in this paper. Two examples, including a frame structure and a practical arch bridge, are investigated to illustrate the accuracy and efficiency of the proposed method and its feasibility for analysis of complex structures.

2. Non-stationary random vibration analysis of deterministic structures

In this section, random response analysis of deterministic structures, rather than stochastic structures, will be first investigated. It is well-known that PSM is a major approach to solve the problem of structural random vibration, in which the power spectra of structural responses are obtained in the frequency domain by using the given power spectra of excitations. In what follows, an outline of PSM will be first given so as to better present TDEFM for non-stationary random vibration analysis.

2.1 The PSM

A structural dynamic finite element problem can be expressed by the following general form of the equation of motion as

$$\boldsymbol{M}\ddot{\boldsymbol{Y}} + \boldsymbol{C}\dot{\boldsymbol{Y}} + \boldsymbol{K}\boldsymbol{Y} = \boldsymbol{\Psi}\boldsymbol{F}(t) \tag{1}$$

where $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are respectively the global mass, damping and stiffness matrixes of the discretized structure; $\boldsymbol{Y}$, $\dot{\boldsymbol{Y}}$ and $\ddot{\boldsymbol{Y}}$ denote the nodal displacement, velocity and acceleration vectors of the structure, respectively; $\boldsymbol{\Psi}$ is the position matrix of the random excitation vector $\boldsymbol{F}(t)=[F_1(t)\ F_2(t)\ \ldots\ F_m(t)]^{\mathrm{T}}$, where $\boldsymbol{F}_i(t)$ is the ith excitation among the m excitations and the superscript T denotes matrix transposition.

When $\boldsymbol{F}(t)$ is a stationary random process vector, the power spectrum density function matrix

$\boldsymbol{S}_{\boldsymbol{YY}}(\omega)$ of displacement vector $\boldsymbol{Y}(t)$ can be expressed as (Li and Chen 2009)

$$\boldsymbol{S}_{\boldsymbol{YY}}(\omega)=\boldsymbol{H}(\omega)\boldsymbol{S}_{\boldsymbol{FF}}(\omega)\boldsymbol{H}(\omega)^{*\mathrm{T}} \tag{2}$$

where $\boldsymbol{S}_{\boldsymbol{FF}}(\omega)$ is the power spectrum density function matrix of $\boldsymbol{F}(t)$; $\boldsymbol{H}(\omega)$ is the frequency response function matrix of the structure, and the superscript * denotes complex conjugation.

It can be seen from Eq. (2) that, for the stationary solution of a structure, simple algebraic relations exist between the power spectra of the responses and those of the random excitations, and therefore PSM is of high efficiency in this sense. But for the non-stationary case, PSM is no longer as efficient as in the case of stationary problems.

Assume that $\boldsymbol{F}(t)$ consists of non-stationary excitations taken to be the widely-used uniformly modulated form as

$$\boldsymbol{F}(t)=\boldsymbol{G}(t)\boldsymbol{f}(t) \tag{3}$$

where $\boldsymbol{G}(t)=\mathrm{diag}[g_1(t)\ g_2(t)\ \ldots\ g_m(t)]$ and $\boldsymbol{f}(t)=[f_1(t)\ f_2(t)\ \ldots\ f_m(t)]^{\mathrm{T}}$, in which $g_i(t)$ and $f_i(t)$ (i=1,2,…,m) are respectively the modulation function and the corresponding stationary random process for the ith excitation. The time-variant power spectrum density function matrix of displacement vector $\boldsymbol{Y}(t)$ can be derived as (Lin *et al*. 1997, 2001)

$$\boldsymbol{S}_{\boldsymbol{YY}}(\omega,t)=\boldsymbol{I}(\omega,t)\boldsymbol{S}_{\boldsymbol{ff}}(\omega)\boldsymbol{I}(\omega,t)^{*\mathrm{T}} \tag{4}$$

where $\boldsymbol{S}_{\boldsymbol{ff}}(\omega)$ is the power spectrum density function matrix of $\boldsymbol{f}(t)$. Let $\boldsymbol{I}_i(\omega,t)$ and $\boldsymbol{G}_i(t)$ (i=1,2,…,m) be the i th column vector of $\boldsymbol{I}(\omega,t)$ and $\boldsymbol{G}(t)$, respectively. Then $\boldsymbol{I}_i(\omega,t)$ represents the time-domain solution of the following equation of motion

$$\boldsymbol{M}\ddot{\boldsymbol{I}}_i(\omega,t)+\boldsymbol{C}\dot{\boldsymbol{I}}_i(\omega,t)+\boldsymbol{K}\boldsymbol{I}_i(\omega,t)=\boldsymbol{\Psi}\boldsymbol{G}_i(t)\mathrm{e}^{\mathrm{i}\omega t}\quad (i=1,2,\cdots,m) \tag{5}$$

The covariance function matrix of the response vector $\boldsymbol{Y}(t)$ can be obtained by integrating the power spectrum density function matrix $\boldsymbol{S}_{\boldsymbol{YY}}(\omega,t)$ in the frequency domain, that is

$$\mathrm{cov}[\boldsymbol{Y}(t),\boldsymbol{Y}(t)]=\mathrm{E}[\boldsymbol{Y}(t)\boldsymbol{Y}(t)^{\mathrm{T}}]-\boldsymbol{\mu}_{\boldsymbol{Y}}\boldsymbol{\mu}_{\boldsymbol{Y}}^{\mathrm{T}}=\int_{-\infty}^{+\infty}\boldsymbol{S}_{\boldsymbol{YY}}(\omega,t)\mathrm{d}\omega-\boldsymbol{\mu}_{\boldsymbol{Y}}\boldsymbol{\mu}_{\boldsymbol{Y}}^{\mathrm{T}}=\sum_{j=1}^{p}\boldsymbol{S}_{\boldsymbol{YY}}(\omega_j,t)\Delta\omega_j-\boldsymbol{\mu}_{\boldsymbol{Y}}\boldsymbol{\mu}_{\boldsymbol{Y}}^{\mathrm{T}} \tag{6}$$

where the symbols cov and E denote the covariance and the mathematical expectation, respectively; $\boldsymbol{\mu}_{\boldsymbol{Y}}=\boldsymbol{\mu}_{\boldsymbol{Y}}(t)$ is the mean function vector of $\boldsymbol{Y}(t)$; p is the number of the frequency intervals; ω_j and $\Delta\omega_j$ are respectively the frequency and the frequency step corresponding to the jth frequency interval.

It can be seen from Eq. (6) that, in order to obtain the covariance of the response, a process of numerical integration in the frequency domain must be operated on the power spectrum of the response, and hence a large number of spectral values must be computed at regular intervals within the range of frequency concerned, usually up to dozens to hundreds of intervals. Further more, to obtain the time-variant spectral values of the response at each frequency interval, time-history integrals must also be conducted according to Eqs. (4) and (5). Apparently, numerical integrals in both time and frequency domains are required in PSM when non-stationary random excitations are involved. In particular, for m excitations and p frequency intervals, a total of $2mp$ times of time-history integrals are required for non-stationary random vibration analysis. Such computational cost is huge for a structure with a large number of DOFs. Actually, for non-stationary analysis, the PSM is now a mixed approach in both time domain and frequency domain rather than a pure frequency-domain method, which is the major reason for the huge computational cost involved.

2.2 The TDEFM

As an alternative to the spectral approach, the problem of non-stationary random vibration could be solved only in the time domain without any mixed-domain integrals.

By defining the following state vector as

$$\boldsymbol{V} = \begin{Bmatrix} \boldsymbol{Y} \\ \dot{\boldsymbol{Y}} \end{Bmatrix} \tag{7}$$

Eq. (1) can be recast into the form of state equation as

$$\dot{\boldsymbol{V}} = \boldsymbol{H}\boldsymbol{V} + \boldsymbol{P}(t) \tag{8}$$

where

$$\boldsymbol{H} = \begin{bmatrix} \mathbf{0} & \mathbf{I} \\ -\boldsymbol{M}^{-1}\boldsymbol{K} & -\boldsymbol{M}^{-1}\boldsymbol{C} \end{bmatrix}, \ \boldsymbol{P}(t) = \boldsymbol{W}\boldsymbol{F}(t), \ \boldsymbol{W} = \begin{bmatrix} \mathbf{0} \\ \boldsymbol{M}^{-1}\boldsymbol{\Psi} \end{bmatrix} \tag{9}$$

in which **0** and **I** are the zero matrix and the unit matrix, respectively.

The general solution of Eq. (8) is

$$\boldsymbol{V}(t) = \mathrm{e}^{\boldsymbol{H}t}\boldsymbol{V}(0) + \int_0^t \mathrm{e}^{\boldsymbol{H}(t-\tau)}\boldsymbol{P}(\tau)\mathrm{d}\tau \tag{10}$$

From the above equation, the response $\boldsymbol{V}_i$ at instant t_i can be expressed by the response $\boldsymbol{V}_{i-1}$ at the preceding instant t_{i-1} as

$$\boldsymbol{V}_i = \boldsymbol{T}\boldsymbol{V}_{i-1} + \int_{t_{i-1}}^{t_i} \mathrm{e}^{\boldsymbol{H}(t_i-\tau)}\boldsymbol{P}(\tau)\mathrm{d}\tau = \boldsymbol{T}\boldsymbol{V}_{i-1} + \int_{t_{i-1}}^{t_i} \mathrm{e}^{\boldsymbol{H}(t_i-\tau)}\boldsymbol{W}\boldsymbol{F}(\tau)\mathrm{d}\tau \qquad (i = 1, 2, \cdots, l) \tag{11}$$

where $\boldsymbol{T}=\mathrm{e}^{\boldsymbol{H}\Delta t}$ with Δt being the time step, and $t_i=i\Delta t$ (i=1,2,…,l) with l being the number of time steps for the time-history analysis. The precise algorithm for computation of the exponential matrix $\boldsymbol{T}$ can be found in references (Moler and Loan 2003, Zhong 2004).

In the following formulation, the random excitation vector $\boldsymbol{F}(t)$ is discretized and characterized by a series of random vectors $\boldsymbol{F}_0$, $\boldsymbol{F}_1$, …, $\boldsymbol{F}_l$, where $\boldsymbol{F}_i=\boldsymbol{F}(t_i)$ (i=0,1,…,l). With the assumption that $\boldsymbol{F}(t)$ changes linearly with time within each time step Δt, Eq. (11) can be converted into the following form as

$$\boldsymbol{V}_i = \boldsymbol{T}\boldsymbol{V}_{i-1} - \boldsymbol{H}^{-1}[\boldsymbol{H}^{-1}\boldsymbol{W}(\boldsymbol{F}_i - \boldsymbol{F}_{i-1})/\Delta t + \boldsymbol{W}\boldsymbol{F}_i] + \boldsymbol{T}\boldsymbol{H}^{-1}[\boldsymbol{H}^{-1}\boldsymbol{W}(\boldsymbol{F}_i - \boldsymbol{F}_{i-1})/\Delta t + \boldsymbol{W}\boldsymbol{F}_{i-1}]$$
$$(i = 1, 2, \cdots, l) \tag{12}$$

where the calculation of $\boldsymbol{H}^{-1}$ can be transformed into the calculation of $\boldsymbol{K}^{-1}$ as shown below

$$\boldsymbol{H}^{-1} = \begin{bmatrix} -\boldsymbol{K}^{-1}\boldsymbol{C} & -\boldsymbol{K}^{-1}\boldsymbol{M} \\ \mathbf{I} & \mathbf{0} \end{bmatrix} \tag{13}$$

Let

$$\left.\begin{aligned} \boldsymbol{Q}_1 &= (\mathbf{I}-\boldsymbol{T})\boldsymbol{H}^{-2}\boldsymbol{W}/\Delta t + \boldsymbol{T}\boldsymbol{H}^{-1}\boldsymbol{W} \\ \boldsymbol{Q}_2 &= (\boldsymbol{T}-\mathbf{I})\boldsymbol{H}^{-2}\boldsymbol{W}/\Delta t - \boldsymbol{H}^{-1}\boldsymbol{W} \end{aligned}\right\} \tag{14}$$

where $\boldsymbol{H}^{-2}=\boldsymbol{H}^{-1}\boldsymbol{H}^{-1}$. Then Eq. (12) can be written as

$$\boldsymbol{V}_i = \boldsymbol{T}\boldsymbol{V}_{i-1} + \boldsymbol{Q}_1\boldsymbol{F}_{i-1} + \boldsymbol{Q}_2\boldsymbol{F}_i \ (i=1,2,\cdots,l) \tag{15}$$

When $\boldsymbol{V}_0=\mathbf{0}$, the expression of the response $\boldsymbol{V}_i$ can be deduced by the above recurrence relation as

$$\left.\begin{aligned} \boldsymbol{V}_1 &= \boldsymbol{Q}_1\boldsymbol{F}_0 + \boldsymbol{Q}_2\boldsymbol{F}_1 \\ \boldsymbol{V}_i &= \boldsymbol{T}^{i-1}\boldsymbol{Q}_1\boldsymbol{F}_0 + \boldsymbol{T}^{i-2}\boldsymbol{Q}_3\boldsymbol{F}_1 + \cdots + \boldsymbol{T}^{0}\boldsymbol{Q}_3\boldsymbol{F}_{i-1} + \boldsymbol{Q}_2\boldsymbol{F}_i \ (2 \le i \le l) \end{aligned}\right\} \tag{16}$$

where

$$\boldsymbol{Q}_3 = \boldsymbol{T}\boldsymbol{Q}_2 + \boldsymbol{Q}_1 \tag{17}$$

If we use $\boldsymbol{A}_{i,0}$, $\boldsymbol{A}_{i,1}$,…, $\boldsymbol{A}_{i,i}$ to denote the coefficient matrixes of $\boldsymbol{F}_0$, $\boldsymbol{F}_1$,…, $\boldsymbol{F}_i$, Eq. (16) can be expressed in an explicit form as

$$\boldsymbol{V}_i = \boldsymbol{A}_{i,0}\boldsymbol{F}_0 + \boldsymbol{A}_{i,1}\boldsymbol{F}_1 + \cdots + \boldsymbol{A}_{i,i}\boldsymbol{F}_i \qquad (i=1,2,\cdots,l) \tag{18}$$

where $\boldsymbol{A}_{i,0}$, $\boldsymbol{A}_{i,1}$,…, $\boldsymbol{A}_{i,i}$ are only associated with structural parameters and reflect the influence of structural parameters on random responses. They can be expressed as

$$\left.\begin{aligned} &\boldsymbol{A}_{1,0} = \boldsymbol{Q}_1, \quad \boldsymbol{A}_{i,0} = \boldsymbol{T}\boldsymbol{A}_{i-1,0} \ (2 \le i \le l) \\ &\boldsymbol{A}_{1,1} = \boldsymbol{Q}_2,\ \boldsymbol{A}_{2,1} = \boldsymbol{Q}_3,\ \boldsymbol{A}_{i,1} = \boldsymbol{T}\boldsymbol{A}_{i-1,1} \ (3 \le i \le l) \\ &\boldsymbol{A}_{i,j} = \boldsymbol{A}_{i-1,j-1} \ (2 \le j \le i \le l) \end{aligned}\right\} \tag{19}$$

According to the above recurrence formula, the coefficient matrixes for structural responses at each instant can be arranged in the form shown in Table 1.

As can be seen from Eq. (19) or from Table 1, only the coefficient matrixes $\boldsymbol{A}_{i,0}$ and $\boldsymbol{A}_{i,1}$ (i=1,2,…,l), which are the matrixes in the first two columns in Table 1, need to be calculated and stored. They represent the influence of $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ on the structural responses at each instant, and the computational cost for such coefficient matrixes is only equivalent to that required by a

Table 1 Coefficient matrixes for structural responses at each instant

Instant of response	Coefficient matrix							
	$\boldsymbol{F}_0$	$\boldsymbol{F}_1$	$\boldsymbol{F}_2$	$\boldsymbol{F}_3$	$\cdots$	$\boldsymbol{F}_{l-2}$	$\boldsymbol{F}_{l-1}$	$\boldsymbol{F}_l$
t_1	$\boldsymbol{A}_{1,0}$	$\boldsymbol{A}_{1,1}$						
t_2	$\boldsymbol{A}_{2,0}$	$\boldsymbol{A}_{2,1}$	$\boldsymbol{A}_{1,1}$					
t_3	$\boldsymbol{A}_{3,0}$	$\boldsymbol{A}_{3,1}$	$\boldsymbol{A}_{2,1}$	$\boldsymbol{A}_{1,1}$				
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$			
t_{l-2}	$\boldsymbol{A}_{l-2,0}$	$\boldsymbol{A}_{l-2,1}$	$\boldsymbol{A}_{l-3,1}$	$\boldsymbol{A}_{l-4,1}$	$\cdots$	$\boldsymbol{A}_{1,1}$		
t_{l-1}	$\boldsymbol{A}_{l-1,0}$	$\boldsymbol{A}_{l-1,1}$	$\boldsymbol{A}_{l-2,1}$	$\boldsymbol{A}_{l-3,1}$	$\cdots$	$\boldsymbol{A}_{2,1}$	$\boldsymbol{A}_{1,1}$	
t_l	$\boldsymbol{A}_{l,0}$	$\boldsymbol{A}_{l,1}$	$\boldsymbol{A}_{l-1,1}$	$\boldsymbol{A}_{l-2,1}$	$\cdots$	$\boldsymbol{A}_{3,1}$	$\boldsymbol{A}_{2,1}$	$\boldsymbol{A}_{1,1}$

particular deterministic time-history analysis of the structure subjected to only $\boldsymbol{F}_0$ and $\boldsymbol{F}_1$ without any excitations at the other instants. Therefore, the calculation and storage of the coefficient matrixes in the explicit expression can be easily accomplished without any extra effort.

Let

$$\left.\begin{aligned} \boldsymbol{B}_i &= [\boldsymbol{A}_{i,0} \;\; \boldsymbol{A}_{i,1} \;\; \cdots \;\; \boldsymbol{A}_{i,i}] \\ \boldsymbol{R}_i &= [\boldsymbol{F}_0^{\mathrm{T}} \;\; \boldsymbol{F}_1^{\mathrm{T}} \cdots \; \boldsymbol{F}_i^{\mathrm{T}}]^{\mathrm{T}} \end{aligned}\right\} \qquad (i=1,2,\cdots,l) \tag{20}$$

Then Eq. (18) can be further expressed in a compact form as

$$\boldsymbol{V}_i = \boldsymbol{B}_i \boldsymbol{R}_i \;\; (i=1,2,\cdots,l) \tag{21}$$

Evidently, any individual displacement response or velocity response at instant t_i, say v_i, can now be expressed as

$$v_i = \boldsymbol{b}_i \boldsymbol{R}_i \;\; (i=1,2,\cdots,l) \tag{22}$$

where $\boldsymbol{b}_i$ is a row extracted from $\boldsymbol{B}_i$. Similarly, the other response of the structure, such as the stress, strain or internal force, etc., can also be deduced in a closed-form manner like Eq. (22) based on the displacement and velocity responses. Therefore, in what follows, the symbol v_i is used to denote any individual response of the structure at instant t_i.

The mean and variance of v_i can be obtained directly from Eq. (22) according to the operation rules of the first and second moments of a random variable. They are respectively

$$\mu_{v_i} = \mathrm{E}(v_i) = \boldsymbol{b}_i \mathrm{E}(\boldsymbol{R}_i) \qquad (i=1,2,\cdots,l) \tag{23}$$

and

$$\sigma_{v_i}^2 = \mathrm{cov}(v_i, v_i) = \boldsymbol{b}_i \, \mathrm{cov}(\boldsymbol{R}_i, \boldsymbol{R}_i) \boldsymbol{b}_i^{\mathrm{T}} \qquad (i=1,2,\cdots,l) \tag{24}$$

where $\mathrm{E}(\boldsymbol{R}_i)$ and $\mathrm{cov}(\boldsymbol{R}_i, \boldsymbol{R}_i)$ are respectively

$$\mathrm{E}(\boldsymbol{R}_i) = [\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_0) \;\; \boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_1) \;\; \cdots \;\; \boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_i)]^{\mathrm{T}} \tag{25}$$

and

$$\mathrm{cov}(\boldsymbol{R}_i, \boldsymbol{R}_i) = \begin{bmatrix} \boldsymbol{R}_{\boldsymbol{FF}}(t_0,t_0) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_0)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_0) & \boldsymbol{R}_{\boldsymbol{FF}}(t_0,t_1) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_0)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_1) & \cdots & \boldsymbol{R}_{\boldsymbol{FF}}(t_0,t_i) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_0)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_i) \\ \boldsymbol{R}_{\boldsymbol{FF}}(t_1,t_0) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_1)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_0) & \boldsymbol{R}_{\boldsymbol{FF}}(t_1,t_1) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_1)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_1) & \cdots & \boldsymbol{R}_{\boldsymbol{FF}}(t_1,t_i) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_1)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_i) \\ \vdots & \vdots & \ddots & \vdots \\ \boldsymbol{R}_{\boldsymbol{FF}}(t_i,t_0) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_i)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_0) & \boldsymbol{R}_{\boldsymbol{FF}}(t_i,t_1) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_i)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_1) & \cdots & \boldsymbol{R}_{\boldsymbol{FF}}(t_i,t_i) - \boldsymbol{\mu}_{\boldsymbol{F}}(t_i)\boldsymbol{\mu}_{\boldsymbol{F}}^{\mathrm{T}}(t_i) \end{bmatrix} \tag{26}$$

In the above equations, $\boldsymbol{\mu}_{\boldsymbol{F}}(t)$ and $\boldsymbol{R}_{\boldsymbol{FF}}(t,\tau)$ are the mean function vector and the cross-correlation function matrix of the non-stationary random excitation vector $\boldsymbol{F}(t)$, respectively.

Thus far, the structural random responses have been obtained based on the closed-form solutions of dynamic responses. By comparison with the traditional PSM based on the time-variant power spectra of responses as shown in Eq. (6), the present approach, termed as TDEFM, is of high efficiency since the time-variant mean and variance responses are obtained directly from the explicit expression of dynamic responses using Eqs. (23) and (24). Several advantages of the present method are worth noting. The first one is that once the closed-form representation is obtained, one can only focus on any particular random responses of interest. In fact, for random vibration analysis of a structure, usually more attention is paid to the random responses at certain critical locations. Therefore, it is unnecessary to solve for the statistic characteristics of all the responses according to Eq. (21). The second advantage lies in the fact that when the variances of random responses are slowly time-variant functions, it is unnecessary to acquire their values at each instant $t_i(i=1,2,\ldots,l)$. In other words, one can calculate random responses at a larger time interval, for example, several times of Δt, by using Eq. (24), which can further improve the computational efficiency. The third advantage of the present approach is that it does not rely on any power spectrum models, e.g., the evolutionary power spectrum model (Priestley 1965, 1967) commonly used in the traditional PSM. In this sense, the method can be applicable to any kind of non-stationary random excitations provided that the cross-correlation functions of the excitations are given.

2.3 Closed-form representation of dynamic responses based on mode decomposition

As can be seen from Eqs. (23) and (24), the most important step in TDEFM is to obtain the closed-form representation of dynamic responses as shown in Eqs. (18) and (19). These equations are derived in the preceding section using the precise time-integral approach in consideration of the high accuracy of the algorithm as compared with the other numerical time-integral approaches. However, this incorporates the calculation of the exponential matrix $\boldsymbol{T}=\mathrm{e}^{\boldsymbol{H}\Delta t}$ as shown in Eq. (11). The size of the exponential matrix $\boldsymbol{T}$ is $2N\times 2N$, where N is the number of DOFs of the structure. When N is large, say $N>10^4$, the solution of exponential matrix $\boldsymbol{T}=\mathrm{e}^{\boldsymbol{H}\Delta t}$ is much more time-consuming. In this case, the mode decomposition technique can be applied to derive the explicit expression of dynamic responses in conjunction with the precise time-integral approach.

For decoupling of the equation of motion as shown Eq. (1), the mode decomposition can be performed as follows

$$\boldsymbol{Y}(t)=\sum_{k=1}^{q}\boldsymbol{\varphi}_k u_k(t) \tag{27}$$

where $\boldsymbol{\varphi}_k$ and $u_k(t)$ $(k=1,2,\ldots,q)$ are the kth mode shape of the structure and the corresponding generalized coordinate, respectively; q denotes the truncation number of the mode shapes and is usually much smaller than N. With the assumption that $\boldsymbol{C}$ is the orthogonal damping matrix, substituting Eq. (27) into Eq. (1) yields the following decoupled single-DOF equation of motion as

$$\ddot{u}_k(t)+2\varsigma_k\omega_k\dot{u}_k(t)+\omega_k^2 u_k(t)=\boldsymbol{\varphi}_k^{\mathrm{T}}\boldsymbol{\Psi}\boldsymbol{F}(t)\ \ (k=1,2,\cdots,q) \tag{28}$$

where ω_k and ς_k are the angular frequency and the damping ratio of the kth mode, respectively.

Following exactly the same process as adopted in solving the multi-DOF equation of motion as shown in Eq. (1), we can obtain the closed-form solution of Eq. (28), just like Eq. (18), as

$$\begin{Bmatrix} u_{k,i} \\ \dot{u}_{k,i} \end{Bmatrix}=\boldsymbol{\alpha}_{i,0}^{k}\boldsymbol{F}_0+\boldsymbol{\alpha}_{i,1}^{k}\boldsymbol{F}_1+\cdots+\boldsymbol{\alpha}_{i,i}^{k}\boldsymbol{F}_i \qquad (i=1,2,\cdots,l;\ k=1,2,\cdots,q) \tag{29}$$

where $u_{k,i}$ and $\dot{u}_{k,i}$ are respectively displacement and velocity response of Eq. (28) at instant t_i; $\boldsymbol{\alpha}_{i,0}^{k},\boldsymbol{\alpha}_{i,1}^{k},\cdots,\boldsymbol{\alpha}_{i,i}^{k}$ are the corresponding coefficient matrixes, which can be determined using Eq. (19).

Note that the exponential matrix $\boldsymbol{T}=\mathrm{e}^{\boldsymbol{H}\Delta t}$ in Eq. (19) is now a matrix with a size of 2×2, in which

$$\boldsymbol{H}=\begin{bmatrix} 0 & 1 \\ -\omega_k^2 & -2\varsigma_k\omega_k \end{bmatrix} \quad (k=1,2,\cdots,q) \tag{30}$$

Apparently, the exponential matrix $\boldsymbol{T}$ can now be calculated without any difficulties.

The coefficient matrixes in Eq. (29) can be expanded in the form as

$$\boldsymbol{\alpha}_{i,j}^k=\begin{bmatrix} \boldsymbol{c}_{i,j}^k \\ \boldsymbol{d}_{i,j}^k \end{bmatrix} \quad (i=1,2,\cdots,l;\ j=0,1,\cdots,i;\ k=1,2,\cdots,q\ ) \tag{31}$$

where $\boldsymbol{c}_{i,j}^k$ and $\boldsymbol{d}_{i,j}^k$ represent the first row and second row of $\boldsymbol{\alpha}_{i,j}^k$, respectively. Then $u_{k,i}$ and $\dot{u}_{k,i}$ in Eq. (29) can be expressed as

$$\left.\begin{aligned} u_{k,i}&=\boldsymbol{c}_{i,0}^k\boldsymbol{F}_0+\boldsymbol{c}_{i,1}^k\boldsymbol{F}_1+\cdots+\boldsymbol{c}_{i,i}^k\boldsymbol{F}_i \\ \dot{u}_{k,i}&=\boldsymbol{d}_{i,0}^k\boldsymbol{F}_0+\boldsymbol{d}_{i,1}^k\boldsymbol{F}_1+\cdots+\boldsymbol{d}_{i,i}^k\boldsymbol{F}_i \end{aligned}\right\} \quad (i=1,2,\cdots,l;\ \ k=1,2,\cdots,q) \tag{32}$$

Substitution of Eq. (32) into Eq. (27) yields the displacement and velocity responses as follows

$$\left.\begin{aligned} \boldsymbol{Y}_i&=\sum_{k=1}^q\boldsymbol{\varphi}_k u_{k,i}=\sum_{k=1}^q\boldsymbol{\varphi}_k(\boldsymbol{c}_{i,0}^k\boldsymbol{F}_0+\boldsymbol{c}_{i,1}^k\boldsymbol{F}_1+\cdots+\boldsymbol{c}_{i,i}^k\boldsymbol{F}_i) \\ \dot{\boldsymbol{Y}}_i&=\sum_{k=1}^q\boldsymbol{\varphi}_k \dot{u}_{k,i}=\sum_{k=1}^q\boldsymbol{\varphi}_k(\boldsymbol{d}_{i,0}^k\boldsymbol{F}_0+\boldsymbol{d}_{i,1}^k\boldsymbol{F}_1+\cdots+\boldsymbol{d}_{i,i}^k\boldsymbol{F}_i) \end{aligned}\right\} \quad (i=1,2,\cdots,l) \tag{33}$$

Let

$$\boldsymbol{\Phi}=[\boldsymbol{\varphi}_1\ \boldsymbol{\varphi}_2\ \cdots\ \boldsymbol{\varphi}_q], \qquad \boldsymbol{C}_{i,j}=\begin{bmatrix} \boldsymbol{c}_{i,j}^1 \\ \boldsymbol{c}_{i,j}^2 \\ \vdots \\ \boldsymbol{c}_{i,j}^q \end{bmatrix}, \quad \boldsymbol{D}_{i,j}=\begin{bmatrix} \boldsymbol{d}_{i,j}^1 \\ \boldsymbol{d}_{i,j}^2 \\ \vdots \\ \boldsymbol{d}_{i,j}^q \end{bmatrix} \quad (i=1,2,\cdots,l;\ \ j=0,1,\cdots,i) \tag{34}$$

Then Eq. (33) can be further expressed in the closed-form as Eq. (18), that is

$$\boldsymbol{V}_i=\begin{Bmatrix} \boldsymbol{Y}_i \\ \dot{\boldsymbol{Y}}_i \end{Bmatrix}=\begin{bmatrix} \boldsymbol{\Phi C}_{i,0} \\ \boldsymbol{\Phi D}_{i,0} \end{bmatrix}\boldsymbol{F}_0+\begin{bmatrix} \boldsymbol{\Phi C}_{i,1} \\ \boldsymbol{\Phi D}_{i,1} \end{bmatrix}\boldsymbol{F}_1+\cdots+\begin{bmatrix} \boldsymbol{\Phi C}_{i,i} \\ \boldsymbol{\Phi D}_{i,i} \end{bmatrix}\boldsymbol{F}_i \qquad (i=1,2,\cdots,l) \tag{35}$$

By comparison between Eq. (18) and Eq. (35), it can be seen that the coefficient matrixes $\boldsymbol{A}_{i,j}$ in Eq. (18) are now redefined as

$$\boldsymbol{A}_{i,0}=\begin{bmatrix} \boldsymbol{\Phi C}_{i,0} \\ \boldsymbol{\Phi D}_{i,0} \end{bmatrix}, \quad \boldsymbol{A}_{i,1}=\begin{bmatrix} \boldsymbol{\Phi C}_{i,1} \\ \boldsymbol{\Phi D}_{i,1} \end{bmatrix}, \ \boldsymbol{A}_{i,j}=\boldsymbol{A}_{i-1,j-1} \quad (2\le j\le i\le l) \tag{36}$$

Since the mode decomposition technique is incorporated in the above derivation, the coefficient matrixes in the closed-form representation of dynamic responses can be calculated more efficiently using Eq. (36) than using Eq. (19).

3. Non-stationary random vibration analysis of stochastic structures

In this section, not only random excitations but also structural uncertainties are taken into consideration. In general stochastic dynamic systems, due to the influence of different physical mechanism, structural random parameters and random excitations are usually regarded as mutually independent. Therefore, the strategy of first considering the effects of the random excitations and then those of the structural random parameters is applied to random response analysis of stochastic structures. The total expectation theorem provides a mathematical tool to solve the problem based on the above two-step strategy. According to the probability theory, the total expectation theorem can be expressed as (Nowak and Collins 2000)

$$\mathrm{E}(\eta)=\int_{-\infty}^{+\infty}\mathrm{E}(\eta\,|\,\xi=x)f_{\xi}(x)\mathrm{d}x \tag{37}$$

where η is a random variable dependent on another random variable ξ; $\mathrm{E}(\eta)$ is the total expectation of η; $\mathrm{E}(\eta|\xi=x)$ denotes the conditional expectation of η when $\xi=x$; $f_\xi(x)$ is the probability density function of ξ.

For a stochastic structure subjected to random excitations, suppose that the structural random parameters are denoted as a n-dimensional random vector $\boldsymbol{\Theta}=[\Theta_1\ \Theta_2\ \dots\ \Theta_n]^{\mathrm{T}}$. Then the structural mass matrix $\boldsymbol{M}$, damping matrix $\boldsymbol{C}$ and stiffness matrix $\boldsymbol{K}$ are stochastic matrixes containing the structural random parameters in $\boldsymbol{\Theta}$. In this case, the structural responses are related to the random vector $\boldsymbol{\Theta}$. Assume that $v(t;\boldsymbol{\Theta})$ is one of the responses of the structure. On the basis of Eq. (37), the mean of $v(t;\boldsymbol{\Theta})$ and $v^2(t;\boldsymbol{\Theta})$ at instant t_i are respectively written as

$$\mu_{v_i}=\mathrm{E}(v(t_i;\boldsymbol{\Theta}))=\int_{-\infty}^{+\infty}\mu_{v_i|\boldsymbol{\theta}}f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}\qquad(i=1,2,\cdots,l) \tag{38}$$

and

$$\mu_{v_i^2}=\mathrm{E}(v^2(t_i;\boldsymbol{\Theta}))=\int_{-\infty}^{+\infty}\mu_{v_i^2|\boldsymbol{\theta}}f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})\mathrm{d}\boldsymbol{\theta}\qquad(i=1,2,\cdots,l) \tag{39}$$

where $\mu_{v_i|\boldsymbol{\theta}}=\mathrm{E}(v(t_i;\boldsymbol{\Theta})\,|\,\boldsymbol{\Theta}=\boldsymbol{\theta})$ and $\mu_{v_i^2|\boldsymbol{\theta}}=\mathrm{E}(v^2(t_i;\boldsymbol{\Theta})\,|\,\boldsymbol{\Theta}=\boldsymbol{\theta})$ are the conditional expectation of $v(t;\boldsymbol{\Theta})$ and $v^2(t;\boldsymbol{\Theta})$ when $\boldsymbol{\Theta}=\boldsymbol{\theta}$, respectively, and $f_{\boldsymbol{\Theta}}(\boldsymbol{\theta})$ is the joint probability density function of the random vector $\boldsymbol{\Theta}$.

It can be seen from Eqs. (38) and (39) that, in order to obtain μ_{v_i} and $\mu_{v_i^2}$, one should determine $\mu_{v_i|\boldsymbol{\theta}}$ and $\mu_{v_i^2|\boldsymbol{\theta}}$ first. Evidently, the solutions to $\mu_{v_i|\boldsymbol{\theta}}$ and $\mu_{v_i^2|\boldsymbol{\theta}}$ belong to the random vibration problem of a deterministic structure with the structural parameter vector given as $\boldsymbol{\theta}=[\theta_1\ \theta_2\ \dots\ \theta_n]^{\mathrm{T}}$, and thus they can be directly obtained by the approach presented in Section 2.2. In general, it is difficult to get a closed-form solutions to $\mu_{v_i|\boldsymbol{\theta}}$ and $\mu_{v_i^2|\boldsymbol{\theta}}$ with respect to $\boldsymbol{\theta}$, and certain numerical methods are usually employed. In the present study, the idea of response surface method (Bucher and Bourgund 1990) is applied to obtaining the approximate expressions of $\mu_{v_i|\boldsymbol{\theta}}$ and $\mu_{v_i^2|\boldsymbol{\theta}}$. They can be expressed in quadratic polynomial forms with respect to the structural parameters as

$$\mu_{v_i|\boldsymbol{\theta}}=a_i+\sum_{j=1}^{n}b_{ij}\theta_j+\sum_{j=1}^{n}c_{ij}\theta_j^2\qquad(i=1,2,\cdots,l) \tag{40}$$

and

$$\mu_{v_i^2|\boldsymbol{\theta}} = d_i + \sum_{j=1}^{n} e_{ij}\theta_j + \sum_{j=1}^{n} f_{ij}\theta_j^2 \qquad (i=1,2,\cdots,l) \tag{41}$$

where a_i, b_{ij}, c_{ij} and d_i, e_{ij}, f_{ij} (j=1,2,…,n; i=1,2,…,l) are unknown coefficients. As can be seen from Eqs. (40) and (41), for $\mu_{v_i|\boldsymbol{\theta}}$ or $\mu_{v_i^2|\boldsymbol{\theta}}$, there are a total of $2n$+1 undetermined coefficients. Therefore, $2n$+1 experimental points are needed for fitting Eqs. (40) and (41), which means that $2n$+1 times of random response analyses are required corresponding to different deterministic structural parameters given. The specific steps of fitting Eqs. (40) and (41) are as follows:

(1) Select $2n$+1 numerical experimental points according to the experimental design method suggested by Bucher and Bourgund (1990). Normally, they can be taken to be the mean point (μ_1, μ_2,…, μ_n) and the $2n$ axial points (μ_1,…, $\mu_j \pm f\sigma_j$,…, μ_n) (j=1,2,…,n), where μ_j and σ_j are the mean and standard deviation of the structural random parameter Θ_j, respectively. The factor f is generally taken to be 0.5~2, which means that the probability of the event $\mu_j - f\sigma_j \leq \Theta_j \leq \mu_j + f\sigma_j$ ranges from 38.30% to 95.44% when Θ_j is a normal variable. For convenience, the above $2n+1$ numerical experimental points are denoted as $\boldsymbol{\theta}_k$ (k=1,2,…,$2n$+1).

(2) Calculate the mass matrix $\boldsymbol{M}(\boldsymbol{\theta}_k)$, damping matrix $\boldsymbol{C}(\boldsymbol{\theta}_k)$ and stiffness matrix $\boldsymbol{K}(\boldsymbol{\theta}_k)$ corresponding to each numerical experimental point $\boldsymbol{\theta}_k$ (k=1,2,…,$2n$+1). Then the conditional expectations $\mu_{v_i|\boldsymbol{\theta}_k}$ and $\mu_{v_i^2|\boldsymbol{\theta}_k}$ (i=1,2,…,l; k=1,2,…,$2n$+1) can be obtained by Eqs. (23) and (24) in Section 2.2.

(3) Solve for the unknown coefficients in Eqs. (40) and (41) by the known conditional expectations $\mu_{v_i|\boldsymbol{\theta}_k}$ and $\mu_{v_i^2|\boldsymbol{\theta}_k}$ (i=1,2,…,l; k=1,2,…,$2n$+1), and acquire the general expressions of $\mu_{v_i|\boldsymbol{\theta}}$ and $\mu_{v_i^2|\boldsymbol{\theta}}$.

Once the general expressions of $\mu_{v_i|\boldsymbol{\theta}}$ and $\mu_{v_i^2|\boldsymbol{\theta}}$ as shown in Eqs. (40) and (41) are obtained, they are substituted into Eqs. (38) and (39), and the mean and mean square value of the response at instant t_i are derived as

$$\mu_{v_i} = \int_{-\infty}^{+\infty} (a_i + \sum_{j=1}^{n} b_{ij}\theta_j + \sum_{j=1}^{n} c_{ij}\theta_j^2) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} = a_i + \sum_{j=1}^{n} b_{ij}\mu_j + \sum_{j=1}^{n} c_{ij}(\sigma_j^2 + \mu_j^2) \qquad (i=1,2,\cdots,l) \tag{42}$$

and

$$\mu_{v_i^2} = \int_{-\infty}^{+\infty} (d_i + \sum_{j=1}^{n} e_{ij}\theta_j + \sum_{j=1}^{n} f_{ij}\theta_j^2) f_{\boldsymbol{\Theta}}(\boldsymbol{\theta}) \mathrm{d}\boldsymbol{\theta} = d_i + \sum_{j=1}^{n} e_{ij}\mu_j + \sum_{j=1}^{n} f_{ij}(\sigma_j^2 + \mu_j^2) \qquad (i=1,2,\cdots,l) \tag{43}$$

respectively. According to the operation rules of moments, the variance of v_i can be obtained by using the μ_{v_i} and $\mu_{v_i^2}$ as follows

$$\sigma_{v_i}^2 = \mu_{v_i^2} - (\mu_{v_i})^2 \qquad (i=1,2,\cdots,l) \tag{44}$$

As can be seen from the above formulation, the random response analysis of deterministic structures is the basis for the random response analysis of stochastic structures. The total computation effort for analysis of a stochastic structure is approximately equivalent to that required by $2n$+1 times of response analyses of a deterministic structure.

4. Numerical examples

In this section, two examples are investigated. For the first example, random response analysis of a stochastic frame subjected to non-stationary ground motion is given to demonstrate the accuracy and efficiency of the present method. The influence of parameter f on the results is discussed in terms of different coefficients of variation (COVs) of the random parameters. In the second example, non-stationary random seismic analysis of a practical arch bridge with random parameters is presented to verify the feasibility of the present approach for analysis of complex structures.

4.1 Example 1: a frame structure

A frame structure subjected to non-stationary ground motion is shown in Fig. 1. The member numbering is also shown in the figure. All the members have the same Young's modulus E and the mass density ρ. The cross-section areas of members 1-3 are denoted as A_i(i=1,2,3), respectively. The structure is assumed to have the same damping ratio ξ for all mode shapes. E, ρ, A_i(i=1,2,3) and ξ are considered as basic random variables, and their means, COVs and distribution types are listed in Table 2. The relationship between the cross-section area and the moment of inertia is defined as $I_i = \alpha_i A_i^2$, where $\alpha_i (i=1,2,3)$ are 0.08333, 0.08333 and 0.2, respectively. Clearly, the moment of inertia I_i, bending stiffness EI_i, tensile stiffness EA_i, and the mass of each member are also random variables, but are dependent on the basic random variables A_i(i=1,2,3), E and ρ. The mass of the frame is lumped at the beam-column nodes, and each member of the structure is taken as a beam element, yielding 144 DOFs in total. To increase the DOFs of the structure, each member of the frame is also discretized into 10 beam elements, leading to a total of 2412 DOFs. Since lumped mass at beam-column nodes has been adopted in the calculation, the results are identical for the two finite element models with different DOFs.

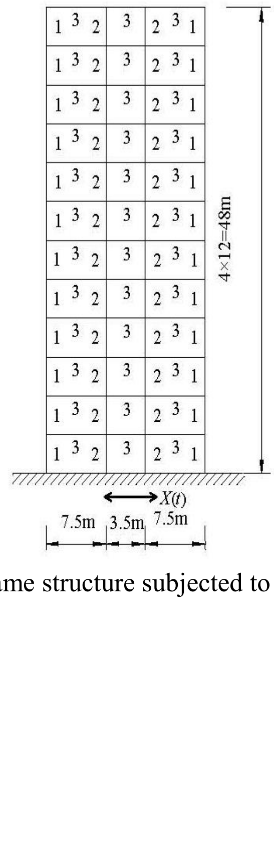


Fig. 1 A frame structure subjected to ground motion

Table 2 Statistical characteristics of structural random parameters for the frame

Random parameter	Mean	COV	Distribution type
E	2.0 GPa	0.1~0.2	normal
ρ	2500 kg.m^{-3}	0.1~0.2	normal
A_1	0.36 m^2	0.1~0.2	lognormal
A_2	0.49 m^2	0.1~0.2	lognormal
A_3	0.15 m^2	0.1~0.2	lognormal
ξ	0.05	0.15~0.3	lognormal

The ground acceleration $X(t)$ is assumed to be a uniformly modulated non-stationary random process expressed as $X(t)=g(t)x(t)$, in which $g(t)$ is the modulation function and $x(t)$ is a stationary random process with zero mean. The modulation function is set to be

$$g(t)=\begin{cases}(t/t_1)^2 & 0\le t\le t_1\\ 1 & t_1\le t\le t_2\\ \mathrm{e}^{-a(t-t_2)} & t_2\le t\end{cases} \tag{45}$$

where t_1=8s, t_2=20s and a=0.3. The power spectrum of $x(t)$ is taken to be the Kanai-Tajimi spectrum (Kanai 1957) as follows

$$S_{xx}(\omega)=\frac{\omega_{\mathrm{g}}^4+4\varsigma_{\mathrm{g}}^2\omega_{\mathrm{g}}^2\omega^2}{(\omega_{\mathrm{g}}^2-\omega^2)^2+4\varsigma_{\mathrm{g}}^2\omega_{\mathrm{g}}^2\omega^2}S_0 \tag{46}$$

where ω_{g}=14 rad/s, ς_{g}=0.6 and $S_0=6\times10^{-4}$ m^{2}s^{-3}. The correlation function of $x(t)$ is

$$R_{xx}(\tau)=\frac{\pi S_0}{2}\mathrm{e}^{-\varsigma_{\mathrm{g}}\omega_{\mathrm{g}}|\tau|}(\eta_1\cos\omega_{\mathrm{d}}\tau+\eta_2\sin\omega_{\mathrm{d}}|\tau|) \tag{47}$$

where

$$\omega_{\mathrm{d}}=\omega_{\mathrm{g}}\sqrt{1-\varsigma_{\mathrm{g}}^2}\quad;\quad \eta_1=\frac{\omega_{\mathrm{g}}(1+4\varsigma_{\mathrm{g}}^2)}{\varsigma_{\mathrm{g}}}\quad;\quad \eta_2=\frac{\omega_{\mathrm{g}}(1-4\varsigma_{\mathrm{g}}^2)}{\sqrt{1-\varsigma_{\mathrm{g}}^2}} \tag{48}$$

Accordingly, the correlation function of $X(t)$ can be expressed as

$$R_{XX}(t,t+\tau)=g(t)g(t+\tau)R_{xx}(\tau) \tag{49}$$

In this example, the PSM and the TDEFM for analysis of deterministic structures, the latter being denoted as TDEFM_DS, are first adopted for obtaining the standard deviations of horizontal displacements of the frame without considering structural uncertainties. The duration of ground motion is set to be T=30s with the time step being Δt=0.02s and the number of integral steps being $l=T/\Delta t$=1500. The range of the frequency domain considered in PSM is taken to be [0, 60] rad/s, and the frequency interval is set to be $\Delta\omega$=0.2 rad/s. Thus, a total of k=60/0.2+1=301 times of time-history integrals are required in PSM.

The standard deviations of the horizontal displacements of the frame are shown in Fig. 2. It can be seen from the figures that the results of the two methods are in good agreement, showing that

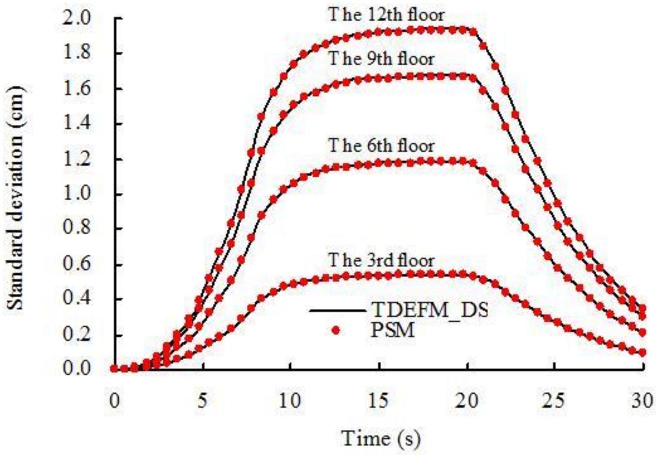


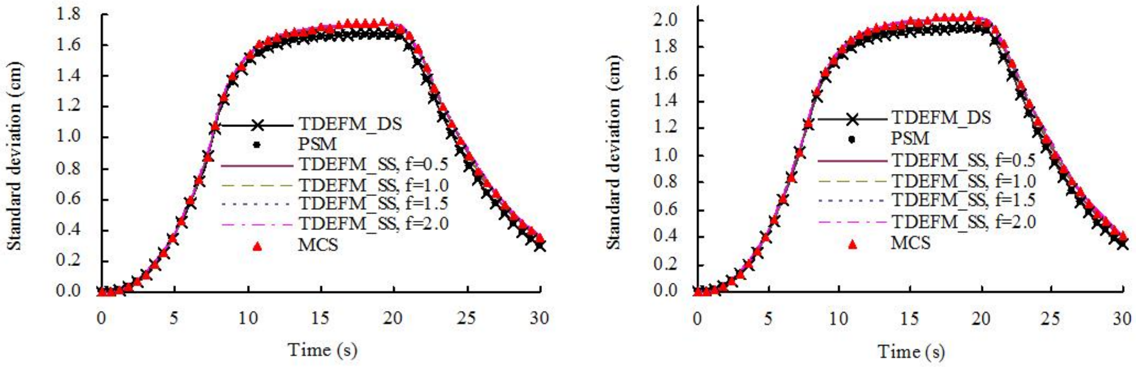
Fig. 2 Standard deviations of horizontal displacements of the frame without consideration of structural uncertainties

Table 3 CPU time elapsed by different methods for analysis of the frame (s)

Method	Deterministic structure		Stochastic structure	
	DOF=144	DOF=2412	DOF=144	DOF=2412
TDEFM_DS	3	35	—	—
PSM	112	1037	—	—
TDEFM_SS	—	—	13	484
MCS	—	—	2279	376961

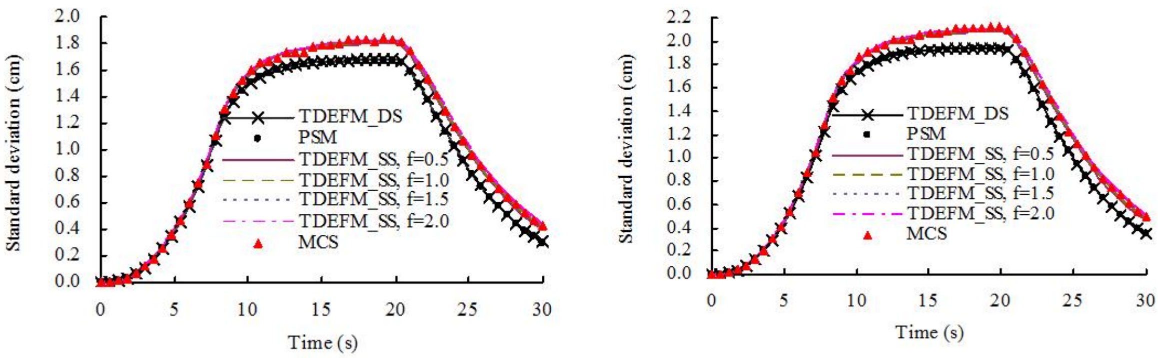
these two methods have the same accuracy. As for the computational effort, the time elapsed by TDEFM_DS and PSM are shown in Table 3. The computational cost of TDEFM_DS is only 2.7% and 3.4% as much as that of PSM when DOF=144 and 2412, respectively, indicating that the present method is far more efficient than the traditional PSM when non-stationary random vibration analysis is involved.

Now consider the random vibration of the frame with structural uncertainties. The COVs of the random variables E, ρ and A_i(i=1,2,3) are first taken as 0.1, and the COV of the damping ratio ξ is assumed to be 0.15. The MCS with 10^4 samples and the proposed TDEFM for stochastic structures, denoted as TDEFM_SS, are employed for obtaining the standard deviations of horizontal displacements of the structure. The Newmark-β scheme is applied to solve the equations of motion in MCS. To verify the robustness of the present method, the influence of the parameter f on the results is discussed by taking f=0.5, 1.0, 1.5 and 2.0, respectively. The standard deviations of horizontal displacements of the 9th and 12th floor are shown in Fig. 3. For the purpose of comparison, the results corresponding to the deterministic structural parameters are also presented in Fig. 3. In order to more fully study the feasibility of the proposed method, another two cases with the COVs of the random parameters being respectively 1.5 and 2 times of the COVs in the first case are also considered in this example. Figs. 4 and 5 show the corresponding results of random responses. The maximum standard deviations of the horizontal displacements for different cases are summarized in Tables 4 and 5.



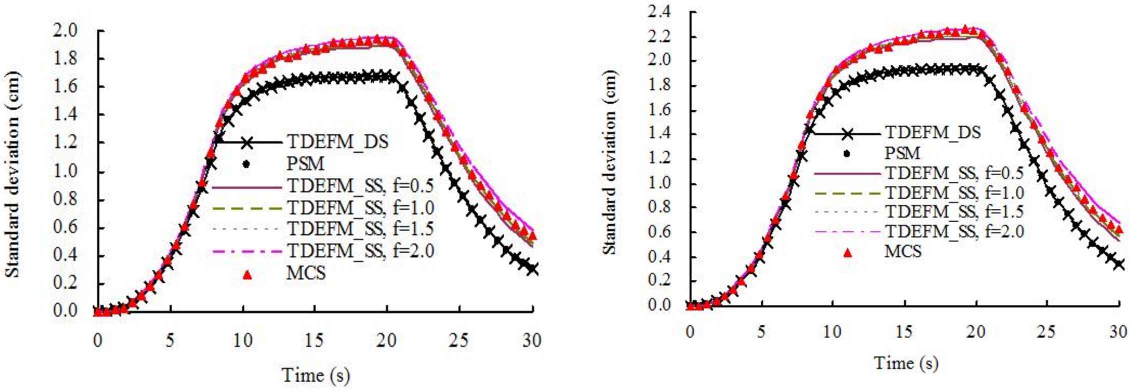
(a) The 9th floor (b) The 12th floor

Fig. 3 Standard deviations of horizontal displacements of the 9th and 12th floor of the frame (COV=0.1 for E, ρ, A_i(i=1,2,3), and COV=0.15 for ξ)



(a) The 9th floor (b) The 12th floor

Fig. 4 Standard deviations of horizontal displacements of the 9th and 12th floor of the frame (COV=0.15 for E, ρ, A_i(i=1,2,3), and COV=0.225 for ξ)



(a) The 9th floor (b) The 12th floor

Fig. 5 Standard deviations of horizontal displacements of the 9th and 12th floor of the frame (COV=0.2 for E, ρ, A_i(i=1,2,3), and COV=0.3 for ξ)

Table 4 Maximum standard deviations of horizontal displacements of the 9th floor of the frame (cm)

COV		MCS	TDEFM_ SS (f=0.5)	Relative error (%)	TDEFM_ SS (f=1.0)	Relative error (%)	TDEFM _SS (f=1.5)	Relative error (%)	TDEFM_ SS (f=20.)	Relative error (%)
E, ρ, A_i(i=1,2,3)	ξ									
0.1	0.15	1.730	1.733	0.15	1.733	0.18	1.735	0.24	1.736	0.34
0.15	0.225	1.831	1.800	1.70	1.803	1.53	1.809	1.22	1.817	0.75
0.2	0.3	2.244	2.190	2.43	2.201	1.91	2.223	0.94	2.259	0.65

Table 5 Maximum standard deviations of horizontal displacements of the 12th floor of the frame (cm)

COV		MCS	TDEFM_ SS (f=0.5)	Relative error (%)	TDEFM_ SS (f=1.0)	Relative error (%)	TDEFM _SS (f=1.5)	Relative error (%)	TDEFM_ SS (f=20.)	Relative error (%)
E, ρ, A_i(i=1,2,3)	ξ									
0.1	0.15	2.002	2.006	0.18	2.006	0.21	2.008	0.28	2.010	0.37
0.15	0.225	2.120	2.084	1.72	2.088	1.54	2.094	1.23	2.105	0.74
0.2	0.3	2.244	2.190	2.43	2.201	1.91	2.223	0.94	2.259	0.65

As can be seen from Figs. 3-5, the standard deviations of the horizontal displacements of the frame become larger when structural uncertainties are also taken into consideration, and increase as the COVs of the structural parameters increase. It can be found from the above figures that the value of *f* has little effect on the results when *f* lies in the range of 0.5 to 2.0, and the results of TDEFM_SS are in good agreement with those obtained with MCS, indicating that the proposed method has a good computational stability and accuracy. However, from Tables 4 and 5, it can be seen that the discrepancies of the results with different values of *f* may become somewhat larger with the increase of the COVs of the random variables, and it seems that, when *f*=2.0, the results of TDEFM_SS agree the best with those of MCS for different cases.

As far as the computational efficiency is concerned, the computational effort of TDEFM_SS and MCS is also shown in the preceding Table 3. It can be found from Table 3 that the CPU time elapsed by TDEFM for analysis of the stochastic frame is even much less than that elapsed by PSM for analysis of the deterministic frame, showing the high efficiency of the proposed method.

4.2 Example 2: a practical arch bridge

The elevation and finite element model of the Xinguang Bridge, a steel arch bridge across the Pearl River in Guangzhou, China, are shown in Figs. 6 and 7, respectively. The bridge span combination is 177 m+428 m+177 m. The finite element model consists of 2705 beam and bar elements and 608 shell elements, resulting in approximately 9200 DOFs. A total of 500 mode shapes of the bridge have been adopted in the following dynamic analysis using the mode decomposition technique. The selected structural random parameters and their COVs and distribution types are listed in Table 6. The means of these random variables are taken from the design documents, and the mean of the damping ratio of the bridge is assumed to be 0.03 for all modes. The COVs of the Young's modulus and the mass of the steel arch ribs are taken to be 0.08 in this study, while for the triangular concrete piers, the COVs of the corresponding random parameters are assumed to be 0.12, considering the fact that the uncertainties of the concrete material are in general more noticeable than those of the steel material. The COV for the mass of

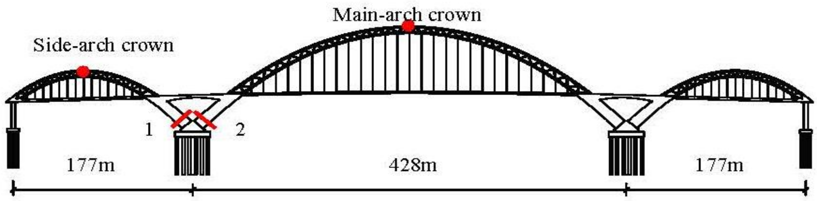


Fig. 6 Elevation of the Xinguang Bridge

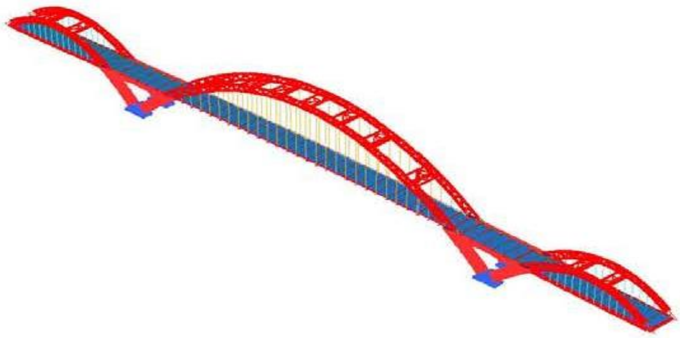


Fig. 7 Finite element model of the Xinguang Bridge

Table 6 Statistical characteristic of structural random parameters for the Xinguang Bridge

Random parameter	COV	Distribution type
Young's modulus of the steel arch ribs	0.08	Normal
Mass of the steel arch ribs	0.08	Lognormal
Young's modulus of the triangular concrete piers	0.12	Normal
Mass of the triangular concrete piers	0.12	Lognormal
Mass of the bridge deck	0.15	Lognormal
Rigidity of the steel-concrete connections	0.20	Normal
Damping ratio	0.25	Lognormal

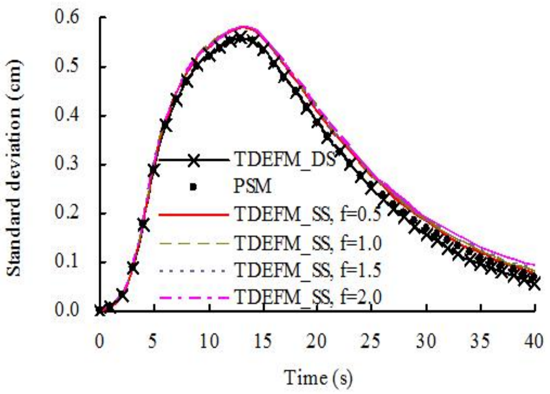
the bridge deck is taken as 0.15 due to the somewhat larger variation of the mass of the pavement and the auxiliary devices attached to the bridge deck. The steel arch ribs are supported at the top of the triangular concrete piers, as shown in Fig. 6, and the rigidity of the supports exhibits considerably large variation due to the inherent uncertainties of the steel-concrete connections. Therefore, the COV for the rigidity of such connections is assumed to be 0.2 in the current study. In addition, it has been found that the role played by damping is important to the structural responses, and the COV of the damping ratio might vary up to 0.4 (Davenport and Larose 1989). Thus, in this investigation, the COV of the damping ratio is assumed to take the value of 0.25.

The structure is excited longitudinally by seismic excitation $X(t)=g(t)x(t)$, in which $g(t)$ has the same form as that in Example 1. The parameters t_1=4.55s, t_2=12.37s and a=0.14 have been used. The power spectrum of $x(t)$ is taken to be the Clough-Penzien spectrum (Clough and Penzien 1993) as follows

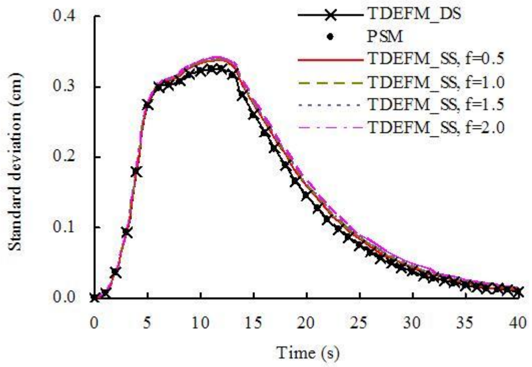
$$S_x(\omega)=\frac{\omega_g^4+4\varsigma_g^2\omega_g^2\omega^2}{(\omega_g^2-\omega^2)^2+4\varsigma_g^2\omega_g^2\omega^2}\frac{\omega^4}{(\omega_f^2-\omega^2)^2+4\varsigma_f^2\omega_f^2\omega^2}S_0 \tag{50}$$

where ω_g=18.21 rad/s, ς_f= ς_g=0.72, ω_f=|0.1−0.2ω_g|=3.64 rad/s, and S_0=1.4×10^{-3} m^{2}s^{-3}.

The proposed method, TDEFM_SS, is used to analyze the random seismic responses of the bridge. In the analysis, the time step is set to be Δt=0.02s with a duration of T=40s, and the parameter f is taken within the range of 0.5 to 2.0. The standard deviations of the longitudinal displacements of the main-and side-arch crown are shown in Fig. 8, and the standard deviations of the in-plane shear forces, the axial forces, and the in-plane bending moments of cross-sections 1 and 2, as shown in Fig. 6, are presented in Figs. 9, 10 and 11, respectively. For comparison, the random seismic response analysis of the bridge without considering structural uncertainties is also conducted using both TDEFM_DS and PSM, and the results are also shown in Figs. 8-11. Note that, in PSM, the range of the frequency domain considered is taken to be [0,100] rad/s, and the frequency interval is assumed to be $\Delta\omega$=0.25 rad/srad/s.

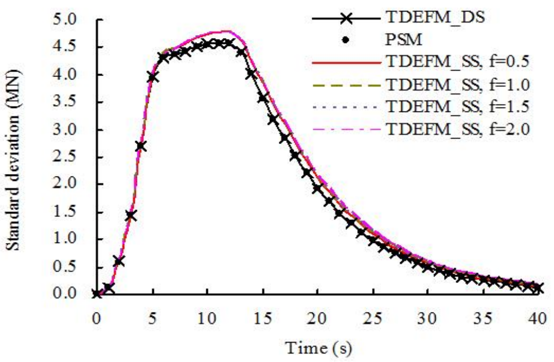


(a) Main-arch crown

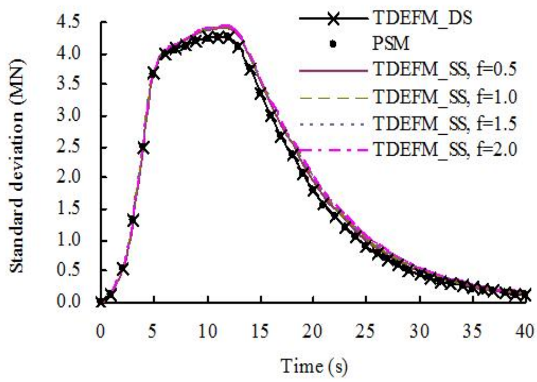


(b) Side-arch crown

Fig. 8 Standard deviations of longitudinal displacements of the main- and side-arch crown of the bridge

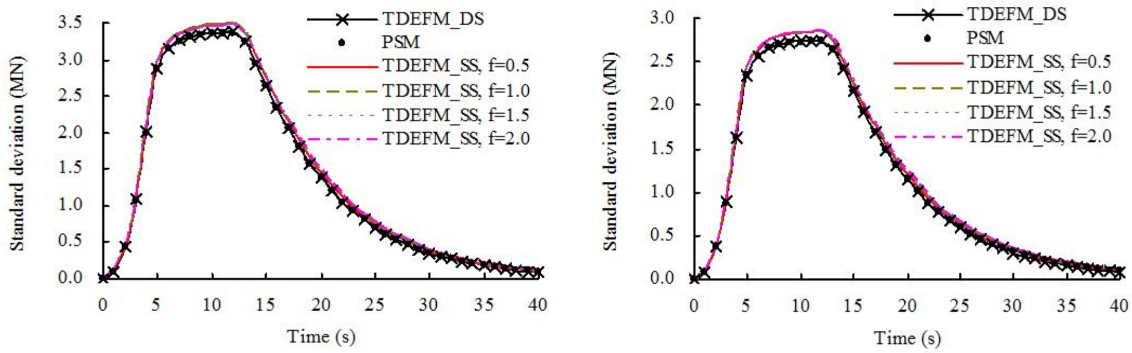


(a) Cross-section 1



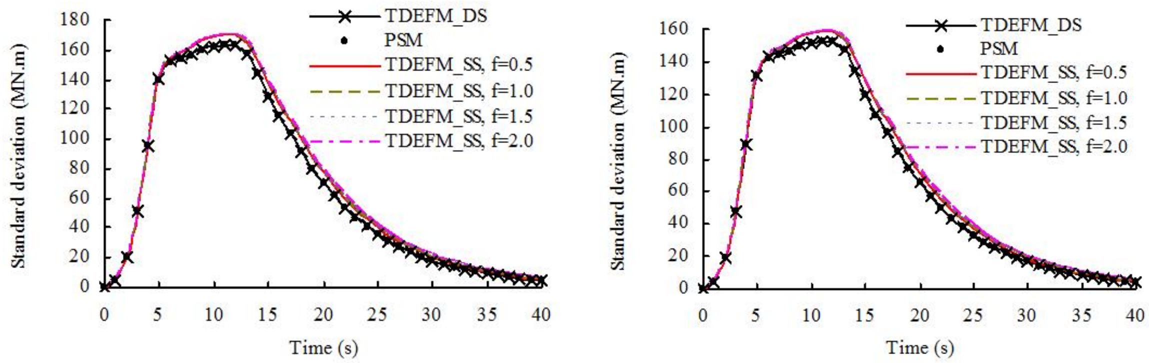
(b) Cross-section 2

Fig. 9 Standard deviations of in-plane shear forces of cross-sections 1 and 2 of the bridge



(a) Cross-section 1 (b) Cross-section 2

Fig. 10 Standard deviations of axial forces of cross-sections 1 and 2 of the bridge



(a) Cross-section 1 (b) Cross-section 2

Fig. 11 Standard deviations of in-plane bending moments of cross-sections 1 and 2 of the bridge

Similar conclusions as those found in the preceding example can be drawn from the above figures. It can be seen that the results obtained by TDEFM_DS and PSM are in good agreement, and few discrepancies can be observed from the results obtained by TDEFM_SS with different values of *f*. Again, one can see that the standard deviations of the responses are larger when structural uncertainties are also considered. However, for this engineering example, it can be seen that the variation of the structural responses is mainly due to the random seismic excitation, and the structural uncertainties only contribute not more than 5% to the standard deviation responses of the bridge. This might be true for most engineering structures subjected to random excitations. But if nonlinear analysis is required, the uncertainties due to the nonlinear behavior of the material, e.g. the nonlinear restoring force of the structure, may be much more noticeable, and may contribute more to the variation of the structural responses. This will be investigated in the subsequent papers of the authors. Note that, for the above example, even though the changes of the standard deviation responses are not as large as expected when considering structural uncertainties, such changes will lead to considerably large changes to the reliability results of the bridge if reliability analysis is further conducted. For example, for cross-sections 1 and 2 in Fig. 6, if the limit state is defined as that the cross-section is not allowed to enter the elastic-plastic state, the reliability indexes for these two cross-sections will decrease respectively from 3.89 to 3.48 and from 3.11 to 2.71 when

Table 7 CPU time elapsed by different methods for analysis of the Xinguang Bridge (s)

Method	Deterministic structure	Stochastic structure
TDEFM_DS	173	—
PSM	8958	—
TDEFM_SS	—	2046

structural uncertainties are considered, indicating an evident increase of the failure probability of the structure.

As for the computational effort, the CPU time elapsed by different methods for this engineering example is given in Table 7, It can be seen that, even for this large-scale structure modeled with a large number of beam and shell elements, the non-stationary random seismic response analysis can be carried out efficiently using the proposed method.

5. Conclusions

Due to the computational inefficiency of the existing methods, an explicit formulation approach is developed to analyze the non-stationary random vibration of stochastic structures. Based on the explicit expressions of structural dynamic responses, the mean and standard deviation responses of deterministic structures are easily obtained through the operation rules of moments. Combined with the total expectation theorem and the response surface techniques, the method for deterministic structures is extended to solve random responses of stochastic structures. Closed-form solutions to the mean and mean square responses considering both uncertain effects resulting from random excitations and structural uncertainties are obtained in the present study. Two numerical examples are given to demonstrate the performance of the proposed method. The results show that the predicted random responses obtained by the present approach are in good agreement with those obtained by MCS, and the proposed method is applicable to large-scale structures with high efficiency. Finally, it should be noted that the present approach is applicable to linear structures, and the authors are now extending the method for analysis of nonlinear systems based on an efficient time-domain explicit iteration scheme (Su *et al*. 2014).

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