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Vibration analysis of laminated plates with various boundary conditions using extended Kantorovich method

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Abstract. In this study, an extended Kantorovich method, employing multi-term displacement functions, is applied to analyze the vibration problem of symmetrically laminated plates with arbitrary boundary conditions. The vibration behaviors of laminated plates are determined based on the variational principle of total energy minimization and the iterative Kantorovich method. The out-of-plane displacement is represented in the form of a series of a sum of products of functions in x and y directions. With a known function in the x or y directions, the formulation for the variation of total potential energy is transformed to a set of governing equations and a set of boundary conditions. The equations and boundary conditions are then numerically solved for the natural frequency and vibration mode shape. The solutions are verified with available solutions from the literature and solutions from the Ritz and finite element analysis. In most cases, the natural frequencies compare very well with the reference solutions. The vibration mode shapes are also very well modeled using the multi-term assumed displacement function in the terms of a power series. With the method used in this study, it is possible to solve the angle-ply plate problem, where the Kantorovich method with single-term displacement function is ineffective.

Keywords: vibration; natural frequency; laminated plate; extended Kantorovich method; various boundary conditions

1. Introduction

Composite materials are being increasingly utilized in advanced engineering applications because of their high specific strength and stiffness. In addition to tensile or compressive failures, other modes of failure, such as dynamic and stability failures, are also important, especially in thin-walled structures. The vibration behavior of plate-like structures is one of the topics that have been extensively studied during the past decades. Since analytical solutions are available only for a certain type of problems, the studies in the past mainly included either numerical or experimental approaches. Both numerical and experimental approaches have particular advantages and disadvantages. Experimental methods are expensive and time consuming, but complications, imperfections, and unforeseen effects of the structures are accounted for. Numerical methods, which are usually based on energy criteria, are less expensive compared with experimental

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approaches; however, its accuracy depends on several factors. For stability and dynamic problems of plates, the well-known numerical methods include the Ritz method, Galerkin method, and finite element method. Inappropriate use of assumed functions or an inadequate number of elements can be the cause of numerical discrepancies. Recently, the extended Kantorovich method, which is considered a “semi-analytical numerical method”, was employed in several bending and buckling problems of plates. The technique, which was originally proposed by Kerr (1969), derives the governing equation of the problem from energy criteria, and then solves using a numerical technique with the derived boundary conditions. If the governing equation is solved accurately, the obtained solutions can be considered as exact solutions, since they are obtained by solving the governing equations directly. In this study, the extended Kantorovich method with a multi-term displacement function is proposed to solve the vibration problem of symmetric laminated plates with various boundary conditions. With multi-term assumed functions, the technique is applicable not only for specially orthotropic plates but also for angle-ply laminates.

The extended Kantorovich technique has been adopted to solve bending, buckling and vibration problems of plate-like structures. The technique was employed in stress analysis problems of Mindlin plates by Yuan *et al.* (1998), and bending problems of plates with various geometric configurations (Aghdam *et al.* 2003, 2007, Abouhamze *et al.* 2007). Yuan and Jin (1998) applied the extended Kantorovich method to determine the buckling load of rectangular isotropic plates. An iterative procedure and multi-term trial functions have been employed. Eisenberger and Alexandrov (2003) applied the Kantorovich method to the buckling problem of variable thickness thin isotropic plates. The governing equation for thin plates with variable flexural rigidity was derived and solved using the single-term extended Kantorovich method. The obtained solutions were verified with available results and found to be more accurate with less computation effort. The technique was applied to the buckling problem of laminated composite plates by Ungbhakorn and Singhatanadgid (2006). The partial differential equation, derived from minimum potential energy conditions, was reduced to an ordinary differential equation. The ODE, in the form of an eigenvalue problem, was then solved analytically to obtain the buckling load and mode. However, the out-of-plane displacement was assumed as a single-term function in that study. Thus, the technique was applicable only for specially orthotropic plates. Later, Shufrin *et al.* (2008a and 2008b) extended the technique by assuming the out-of-plane displacement in the form of multi-term functions. The governing equations in this case are not a single ODE, but are a system of ODEs that can be solved numerically. A system of ODEs was solved using the exact element method and successfully determined the buckling load and modes of angle-ply plates.

The extended Kantorovich method was also applied to study the vibration behavior of plates. Unlike the buckling problem, the number of studies on the vibration of composite plates using the Kantorovich method is limited. Application of the technique to the vibration of isotropic plates can be found in the studies by Shufrin and Eisenberger (2005 and 2006). The authors applied the single-term extended Kantorovich method to the vibration of shear-deformable plates with constant and variable thickness, respectively. Dalaei and Kerr (1996) and Bercin (1996) successfully employed the Kantorovich method to solve the vibration of clamped orthotropic plates using a single-term function. It is seen that the scope of most of the studies are limited to isotropic or orthotropic plates with particular boundary conditions. However, Lee (1997) attempted to solve the vibration problem of angle-ply plates using the extended Kantorovich method. In that study, the vibration of rectangular laminated plates with all edges elastically restrained against rotation was studied using the iterative Kantorovich method and the Ritz method, with three different sets of trial functions. It was found that the solution from the Kantorovich method was

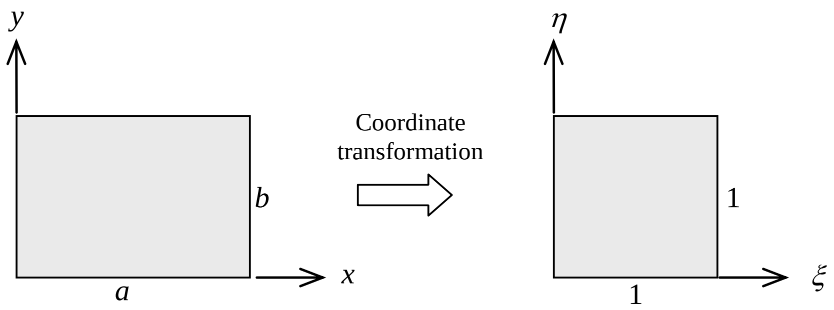


Fig. 1 Dimensionless coordinate of a rectangular plate

more accurate than those from the Ritz method for orthotropic cross-ply laminates. The technique was inapplicable to the vibration analysis of angle-ply laminates because this type of laminate requires inclined nodal lines of the vibration mode. This finding corresponds to a remark in a study by Ungbhakorn and Singhatanadgid (2006), which stated that a multi-term displacement function is required for the modeling of angle-ply plates. Therefore, an extended Kantorovich method is employed to solve the vibration problem of rectangular symmetric laminated plates, including angle-ply plates, in this study. Boundary conditions of the specimens are any combinations of simple support, clamped support, or free edge. Analytical closed-form solutions are not possible for these types of specimens. The out-of-plane displacement is assumed in the form of a multi-term function to accommodate the skew nodal line of the vibration mode shape. Conceptually, if the system of ODEs is accurately solved, the obtained solutions are considered as exact solutions. Thus, they might serve as benchmark solutions for other numerical methods.

2. Derivation of the governing equations

The governing equations for free vibration of a rectangular plate can be derived using the principle of minimum total potential energy. The laminated composite plates considered in this study are $a \times b$ rectangular plates. For convenience and generality, all of the following derivations are based on the dimensionless coordinates of $\xi = x/a$ and $\eta = y/b$, as shown in Fig. 1. After coordinate transformation, the specimen in the dimensionless coordinates is a unit square plate. The total potential energy of the symmetrically laminated composite plates can be determined by subtracting the kinetic energy from the strain energy of the vibrating plate, and is written in the form of (Whitney 1987)

$$\begin{aligned}\Pi = \frac{1}{2}\int_0^a\int_0^b & D_{11}w_{,xx}^2 + 2D_{12}w_{,xx}w_{,yy} + D_{22}w_{,yy}^2 + 4D_{16}w_{,xx}w_{,xy} + 4D_{26}w_{,yy}w_{,xy} \\ & + 4D_{66}w_{,xy}^2 - \rho h\omega^2 w^2 dxdy\end{aligned} \tag{1}$$

where w is the transverse displacement of the mid-surface of the plate, D_{ij} are the flexural rigidities of the plate, ρ is the density of the plate, h is the thickness of the plate and ω is the frequency of the vibration, which needs to be determined. This potential energy can be rewritten in term of dimensionless parameters as

$$\Pi = \frac{1}{2}\int_0^1\int_0^1\left[\frac{b}{a^3}D_{11}w_{,\xi\xi}^2 + \frac{2}{ab}D_{12}w_{,\xi\xi}w_{,\eta\eta} + \frac{a}{b^3}D_{22}w_{,\eta\eta}^2 + \frac{4}{ab}D_{66}w_{,\xi\eta}^2 \right.$$
$$\left. +4\left(\frac{D_{16}}{a^2}w_{,\xi\xi} + \frac{D_{26}}{b^2}w_{,\eta\eta}\right)w_{,\xi\eta} - ab\rho h\omega^2 w^2\right]d\xi d\eta \tag{2}$$

To determine the natural frequencies using the multi-term Kantorovich method, the out-of-plane displacement is assumed in the form of a series similar to the study by Shufrin *et al*. (2008a) as

$$w(\xi,\eta) = \sum_{i=1}^{N} X_i(\xi)Y_i(\eta) = \{X\}^T\{Y\} \tag{3}$$

where $X_i(\xi)$ are functions of ξ that satisfy the boundary conditions at ξ=0 and ξ=1, and $Y_i(\eta)$ are functions of η that satisfy the boundary conditions at η=0 and η=1. If $Y_i(\eta)$ is previously specified, the total potential energy in term of dimensionless parameters is written as

$$\Pi = \frac{1}{2}\int_0^1\Big[D_{11}\{X_{,\xi\xi}\}^T[S_1]\{X_{,\xi\xi}\} + 2D_{12}\{X_{,\xi\xi}\}^T[S_2]\{X\} + D_{22}\{X\}^T[S_4]\{X\}$$
$$+4D_{66}\{X_{,\xi}\}^T[S_6]\{X_{,\xi}\} + 4D_{16}\{X_{,\xi\xi}\}^T[S_3]\{X_{,\xi}\} + 4D_{26}\{X\}^T[S_5]\{X_{,\xi}\}$$
$$-\rho h\omega^2\{X\}^T[S_7]\{X\}\Big]d\xi \tag{4}$$

where

$[S_1] = \frac{b}{a^3}\int_0^1\{Y\}\{Y\}^T d\eta$, $[S_2] = \frac{1}{ab}\int_0^1\{Y\}\{Y_{,\eta\eta}\}^T d\eta$, $[S_3] = \frac{1}{a^2}\int_0^1\{Y\}\{Y_{,\eta}\}^T d\eta$

$[S_4] = \frac{a}{b^3}\int_0^1\{Y_{,\eta\eta}\}\{Y_{,\eta\eta}\}^T d\eta$, $[S_5] = \frac{1}{b^2}\int_0^1\{Y_{,\eta\eta}\}\{Y_{,\eta}\}^T d\eta$, $[S_6] = \frac{1}{ab}\int_0^1\{Y_{,\eta}\}\{Y_{,\eta}\}^T d\eta$, and

$[S_7] = ab\int_0^1\{Y\}\{Y\}^T d\eta$.

Since $Y_i(\eta)$ are functions of η only, [S_i] are $N\times N$ square matrices without any unknown parameters. The total potential energy in Eq. (4) is now written in form of only unknowns $\{X\}$. The variational principle (Reddy 2003) requires the stationary condition for the functional Eq. (4), i.e., $\delta\Pi$=0. The variation of potential energy Π can be written as

$$\delta\Pi = \int_0^1\Big[\Big(D_{11}\{X_{,\xi\xi}\}^T[S_1] + D_{12}\{X\}^T[S_2]^T + 2D_{16}\{X_{,\xi}\}^T[S_3]^T\Big)\{\delta X_{,\xi\xi}\}$$
$$+\Big(4D_{66}\{X_{,\xi}\}^T[S_6] + 2D_{16}\{X_{,\xi\xi}\}^T[S_3] + 2D_{26}\{X\}^T[S_5]\Big)\{\delta X_{,\xi}\}$$
$$+\Big(D_{12}\{X_{,\xi\xi}\}^T[S_2] + D_{22}\{X\}^T[S_4] + 2D_{26}\{X_{,\xi}\}^T[S_5]^T$$
$$-\rho h\omega^2\{X\}^T[S_7]\Big)\{\delta X\}\Big]d\xi = \{0\} \tag{5}$$

Eq. (5) is consisted of N equations where N is the number of terms in the assumed displacement function, Eq. (3). By performing integration by part to Eq. (5), the governing equations are obtained as

$$D_{11}[S_1]^T\{X_{,\xi\xi\xi\xi}\}+2D_{16}\left([S_3]-[S_3]^T\right)\{X_{,\xi\xi\xi}\}+\left(D_{12}\left([S_2]+[S_2]^T\right)-4D_{66}[S_6]^T\right)\{X_{,\xi\xi}\}$$
$$+\left(2D_{26}\left([S_5]-[S_5]^T\right)\right)\{X_{,\xi}\}+\left(D_{22}[S_4]^T-\rho h\omega^2[S_7]^T\right)\{X\}=\{0\}, \tag{6}$$

where the boundary conditions along the ξ=0 and ξ=1 edges are:
either

$$\left(D_{11}[S_1]^T\{X_{,\xi\xi}\}+2D_{16}[S_3]\{X_{,\xi}\}+D_{12}[S_2]\{X\}\right)\Big|_{\xi=0}^{\xi=1}=\{0\} \tag{7a}$$

or

$$\{X_{,\xi}\}\Big|_{\xi=0}^{\xi=1}=\{0\} \tag{7b}$$

and, either

$$\left(-D_{11}[S_1]^T\{X_{,\xi\xi\xi}\}-2D_{16}\left([S_3]-[S_3]^T\right)\{X_{,\xi\xi}\}\right.$$
$$\left.-\left(D_{12}[S_2]-4D_{66}[S_6]^T\right)\{X_{,\xi}\}+\left(2D_{26}[S_5]^T\right)\{X\}\right)\Big|_{\xi=0}^{\xi=1}=\{0\} \tag{7c}$$

or

$$\{X\}\Big|_{\xi=0}^{\xi=1}=\{0\} \tag{7d}$$

The governing equations, Eq. (6), can be simplified as

$$\{X_{,\xi\xi\xi\xi}\}+[A_1]\{X_{,\xi\xi\xi}\}+[A_2]\{X_{,\xi\xi}\}+[A_3]\{X_{,\xi}\}+\left([A_4]-\omega^2[A_5]\right)\{X\}=\{0\} \tag{8}$$

where:

$$[A_1]=\frac{2D_{16}}{D_{11}}[S_1]^{-1}\left([S_3]-[S_3]^T\right),\qquad [A_2]=\frac{[S_1]^{-1}}{D_{11}}\left(D_{12}\left([S_2]+[S_2]^T\right)-4D_{66}[S_6]^T\right),$$
$$[A_3]=\frac{2D_{26}}{D_{11}}[S_1]^{-1}\left([S_5]-[S_5]^T\right),\qquad [A_4]=\frac{D_{22}}{D_{11}}[S_1]^{-1}[S_4]^T\text{, and}$$
$$[A_5]=\frac{\rho h}{D_{11}}[S_1]^{-1}[S_7]^T$$

The governing equations shown in Eq. (8) are a system of ODEs consisting of N equations. For a single-term Kantorovich method, there is only one governing equation, which can be solved analytically as shown in a report by Ungbhakorn and Singhatanadgid (2006). With known boundary conditions in Eq. (7), the system of governing equations can be solved for the natural frequency ω and the vibration mode shape $\{X\}$. These boundary conditions correspond to the edge supports as follows:

simply supported edge: Eq. (7a) and Eq. (7d)

clamped edge: Eq. (7b) and Eq. (7d)

free edge: Eq. (7a) and Eq. (7c)

With specified $Y_i(\eta)$, the natural frequencies and associated mode shapes can be determined by solving the governing equations, Eq. (8), along with appropriate boundary conditions. With a similar procedure, the other set of governing equations and boundary conditions can be derived for the previously assumed functions $X_i(\xi)$. The governing equations are obtained as

$$\begin{aligned} &D_{22}[R_1]^T\{Y_{,\eta\eta\eta\eta}\}+2D_{26}\left([R_3]-[R_3]^T\right)\{Y_{,\eta\eta\eta}\}+\left(D_{12}\left([R_2]+[R_2]^T\right)-4D_{66}[R_6]^T\right)\{Y_{,\eta\eta}\} \\ &+\left(2D_{16}\left([R_5]-[R_5]^T\right)\right)\{Y_{,\eta}\}+\left(D_{11}[R_4]^T-\rho h\omega^2[R_7]^T\right)\{Y\}=\{0\} \end{aligned} \tag{9}$$

where

$$[R_1]=\frac{a}{b^3}\int_0^1\{X\}\{X\}^T d\xi, \qquad [R_2]=\frac{1}{ab}\int_0^1\{X\}\{X_{,\xi\xi}\}^T d\xi,$$

$$[R_3]=\frac{1}{b^2}\int_0^1\{X\}\{X_{,\xi}\}^T d\xi, \qquad [R_4]=\frac{b}{a^3}\int_0^1\{X_{,\xi\xi}\}\{X_{,\xi\xi}\}^T d\xi,$$

$$[R_5]=\frac{1}{a^2}\int_0^1\{X_{,\xi\xi}\}\{X_{,\xi}\}^T d\xi, \quad [R_6]=\frac{1}{ab}\int_0^1\{X_{,\xi}\}\{X_{,\xi}\}^T d\xi, \text{ and}$$

$$[R_7]=ab\int_0^1\{X\}\{X\}^T d\xi$$

The governing equations, Eq. (9) are simplified as

$$\{Y_{,\eta\eta\eta\eta}\}+[B_1]\{Y_{,\eta\eta\eta}\}+[B_2]\{Y_{,\eta\eta}\}+[B_3]\{Y_{,\eta}\}+\left([B_4]-\omega^2[B_5]\right)\{Y\}=\{0\} \tag{10}$$

where

$$[B_1]=\frac{2D_{26}}{D_{22}}[R_1]^{-1}\left([R_3]-[R_3]^T\right), \quad [B_2]=\frac{[R_1]^{-1}}{D_{22}}\left(D_{12}\left([R_2]+[R_2]^T\right)-4D_{66}[R_6]^T\right),$$

$$[B_3]=\frac{2D_{16}}{D_{22}}[R_1]^{-1}\left([R_5]-[R_5]^T\right), \quad [B_4]=\frac{D_{11}}{D_{22}}[R_1]^{-1}[R_4]^T, \text{ and}$$

$$[B_5]=\frac{\rho h}{D_{22}}[R_1]^{-1}[R_7]^T$$

Similar to the previous case, the derivation also returns the boundary conditions of functions $Y_i(\eta)$, which are summarized as follows:
either

$$\left(D_{22}[R_1]^T\{Y_{,\eta\eta}\}+2D_{26}[R_3]\{Y_{,\eta}\}+D_{12}[R_2]\{Y\}\right)\Big|_{\eta=0}^{\eta=1}=\{0\} \tag{11a}$$

or

$$\{Y_{,\eta}\}\Big|_{\eta=0}^{\eta=1}=\{0\} \tag{11b}$$

and, either

$$\left(-D_{22}\left[R_1\right]^T\left\{Y_{,\eta\eta\eta}\right\}-2D_{26}\left(\left[R_3\right]-\left[R_3\right]^T\right)\left\{Y_{,\eta\eta}\right\}-\left(D_{12}\left[R_2\right]-4D_{66}\left[R_6\right]^T\right)\left\{Y_{,\eta}\right\}\right.$$
$$\left.+2D_{16}\left[R_5\right]^T\left\{Y\right\}\right)\Big|_{\eta=0}^{\eta=1}=\left\{0\right\} \tag{11c}$$

or

$$\left\{Y\right\}\Big|_{\eta=0}^{\eta=1}=\left\{0\right\} \tag{11d}$$

Similar to the previous case, the boundary conditions for each edge support are:

simply supported edge: Eq. (11a) and Eq. (11d)

clamped edge: Eq. (11b) and Eq. (11d)

free edge: Eq. (11a) and Eq. (11c)

At this moment, two sets of ODEs and corresponding boundary conditions are derived and are complete for iterative calculations. The first set of equations, i.e., Eqs. (7)-(8), are the boundary conditions and the governing equations in the case of functions $Y_i(\eta)$ being specified. Similarly, Eqs. (10)-(11) are the other set of governing equations and boundary conditions, if functions $X_i(\xi)$ are specified. In the iterative procedure, either $Y_i(\eta)$ or $X_i(\xi)$ are assumed as initial solutions which may or may not satisfy the boundary conditions of the problem. If functions $Y_i(\eta)$ are assumed in the first iteration, the first set of equations are solved for the natural frequencies and corresponding functions $X_i(\xi)$, which satisfy the boundary conditions at ξ=0 and ξ=1. Solutions from the first iteration are almost certainly incorrect, because the assumed functions $Y_i(\eta)$ are randomly chosen. The second iteration is then performed using the solutions from the first iteration, i.e., functions $X_i(\xi)$, as the specified functions. In this calculation, the obtained displacement functions $Y_i(\eta)$ will satisfy the boundary conditions at η=0 and η=1. The third iteration can be performed using $Y_i(\eta)$ obtained from the second iteration. These iterative calculations can be performed until the natural frequency has converged to a particular value, which is the solution of the problem.

3. Solution procedures

The systems of N ODEs, Eq. (8) and Eq. (10), are solved numerically in this study. The displacement functions $\{X\}$ or $\{Y\}$ are assumed as infinite power series, similar to the study by Shufrin *et al*. (2008a). For the first set of the governing equations, Eq. (8), the displacement functions $\{X\}$ are assumed in the form of an infinite power series as

$$\left\{X\right\}=\begin{Bmatrix}X_1(\xi)\\ \vdots\\ X_N(\xi)\end{Bmatrix}=\begin{Bmatrix}\sum_{i=0}^{\infty}aa_{1,i}\xi^i\\ \vdots\\ \sum_{i=0}^{\infty}aa_{N,i}\xi^i\end{Bmatrix}=\sum_{i=0}^{\infty}\begin{Bmatrix}aa_1\\ \vdots\\ aa_N\end{Bmatrix}_i\xi^i=\sum_{i=0}^{\infty}\left\{AA\right\}_i\xi^i \tag{12}$$

where $\{AA\}_i$ are the unknown coefficients to be determined. By substituting the assumed function from Eq. (12) in the governing equations, the unknown coefficients $\{AA\}_{i+4}$ can be written as

$$\{AA\}_{i+4} = -\frac{\left\{\begin{matrix}[A_1]\{AA\}_{i+3}(i+1)(i+2)(i+3)+[A_2]\{AA\}_{i+2}(i+1)(i+2)\\ +[A_3]\{AA\}_{i+1}(i+1)+\left([A_4]-\omega^2[A_5]\right)\{AA\}_i\end{matrix}\right\}}{(i+1)(i+2)(i+3)(i+4)} \tag{13}$$

With this formulation, the number of unknown coefficients is reduced to $4N$. The displacement functions $\{X\}$ are written in terms of the unknown coefficients $\{AA\}_3$, $\{AA\}_2$, $\{AA\}_1$ and $\{AA\}_0$. These unknown coefficients can be solved using the boundary conditions shown in Eq. (7). There are four sets of boundary conditions, two on the ξ=0 edge and the other two on the ξ=1 edge. Each set of boundary conditions is composed of N equations, so the total number of equations are $4N$, which is equal to the number of unknown coefficients; $\{AA\}_3$, $\{AA\}_2$, $\{AA\}_1$ and $\{AA\}_0$. However, there is an additional unknown of ω in the equations. Therefore, there are $4N$ linear equations in terms of $4N$+1 unknowns. The system of equations can be simplified and written in a matrix form as

$$[\Omega]\{AA\} = 0 \tag{14}$$

where $[\Omega]$ is a $4N\times4N$ square matrix having only ω as an unknown, $\{AA\}$ is a $4N\times1$ column matrix of $4N$ unknown coefficients. The solution of Eq. (14) is not a trivial one only if all members of $\{AA\}$ are not zero, simultaneously. Thus, the equation is in a form similar to an eigenvalue problem. The value of ω or eigenvalue is determined from

$$\det[\Omega] = 0 \tag{15}$$

Theoretically, Eq. (15) has an infinite number of solutions. Each solution corresponds to the natural frequency of each vibration mode. Moreover, there is an eigenvector corresponding to each eigenvalue. The obtained eigenvector, together with the previously specified function of $Y_i(\eta)$, represents the vibration mode shape. At this point, the iteration using the initially assumed functions $Y_i(\eta)$ is concluded.

The calculation procedure for the case of specified $X_i(\xi)$ is similar to the other case. The undetermined function $\{Y\}$ in the governing equation, Eq. (10), is assumed in a power series form of

$$\{Y\} = \begin{Bmatrix} Y_1(\eta) \\ \vdots \\ Y_N(\eta) \end{Bmatrix} = \begin{Bmatrix} \sum_{i=0}^{\infty} bb_{1,i}\eta^i \\ \vdots \\ \sum_{i=0}^{\infty} bb_{N,i}\eta^i \end{Bmatrix} = \sum_{i=0}^{\infty} \begin{Bmatrix} bb_1 \\ \vdots \\ bb_N \end{Bmatrix}_i \eta^i = \sum_{i=0}^{\infty} \{BB\}_i \eta^i . \tag{16}$$

With a similar procedure to the previous case, the unknown coefficients $\{BB\}_{i+4}$ can be written as

$$\{BB\}_{i+4} = -\frac{\left\{\begin{matrix}[B_1]\{BB\}_{i+3}(i+1)(i+2)(i+3)+[B_2]\{BB\}_{i+2}(i+1)(i+2)\\ +[B_3]\{BB\}_{i+1}(i+1)+\left([B_4]-\omega^2[B_5]\right)\{BB\}_i\end{matrix}\right\}}{(i+1)(i+2)(i+3)(i+4)} . \tag{17}$$

The subsequent procedures are the same as those described previously. A matrix equation similar to Eq. (14) is arranged and solved by considering the determinant of the square matrix to be zero. The eigenvalues and corresponding eigenvectors are obtained for each vibration mode.

In each iterative calculation, a previously specified function of $X_i(\xi)$ or $Y_i(\eta)$ is required. This function can be arbitrarily chosen in the first iteration. If the function $X_i(\xi)$ is specified in the first iteration, the function $Y_i(\eta)$ in form of an eigenvector is obtained from the calculation along with the eigenvalue. This eigenvector will be used as the specified $Y_i(\eta)$ in the next iteration. The iterations are repeated until the obtained eigenvalues from each calculation have converged. The converged eigenvalue and eigenvector represent the natural frequency and vibration mode shape of the problem, respectively.

4. Example of the iteration procedures

A $[\pm 45]_{2S}$ composite plate with SCSF boundary conditions is chosen as a sample to show the iterative procedures. The symbols S, C and F represent boundary conditions with simple support, clamped support and free edges, respectively. The first and third letters represent the boundary condition on the x=0 and x=a edges, respectively. Similarly, the boundary conditions of the y=0 and y=b edges are indicated by the second and fourth letters. Thus, the SCSF specimen is the specimen with simple support on the x=0 and x=a edges, clamp support on the y=0 edge, and no support or free edge on the y=b edge. The material mainly used in this study is a graphite-epoxy laminated composite, whose mechanical and physical properties are shown in Table 1. The specimen is a square plate with a=30 cm and b=30 cm. The number of terms used in the displacement function are 3, that is, the displacement function shown in Eq. (3) is used with N=3. The iterative calculations are shown in Table 2. The first iteration begins with assumed functions $\{Y\}$ as $Y_1(y)=y^4$, $Y_2(y)=y^5$and $Y_3(y)=y^6$. These functions are chosen arbitrarily. However, using the initial functions that satisfy the boundary conditions of the problem can reduce the number of iterations. It is noticed that the chosen functions of $Y_1(y)=y^4$, $Y_2(y)=y^5$and $Y_3(y)=y^6$ satisfy the clamped boundary condition on the y = 0 edge. The natural frequency and displacement functions in terms of x are determined from governing equations, Eq. (6), and boundary conditions, Eq. (7), using the solution procedures explained in the previous section. The natural frequency of the 2nd mode is obtained as 289.503 rad/s for the first iteration. The displacement functions in terms of x are also obtained as:

$$\{X\} = \begin{Bmatrix} X_1^{(1)}(\xi) \\ X_2^{(1)}(\xi) \\ X_3^{(1)}(\xi) \end{Bmatrix}$$

The superscripted numbers in parentheses refer to the set of functions involved in the calculation. The vibration mode shape from the first iteration is plotted from the product of the assumed function $\{Y\}$ and the obtained functions $\{X\}$ as defined in Eq. (3). The last column of Table 2 presents the vibration mode shape obtained from each calculation. The vibration mode shape is presented as a 3D surface plot in the x-y coordinate domain. It is noticed that the vibration mode obtained in the first iteration is incorrect, because the assumed functions $Y_N^{(1)}(y)$ do not satisfy the boundary conditions, although the obtained functions $X_N^{(1)}(\xi)$ satisfy the respective boundary conditions. The second iteration begins with the assumed functions $X_N^{(1)}(\xi)$, which are the functions obtained from the first iteration. This iteration yields new natural frequencies and functions $Y_N^{(2)}(y)$, which can be used to plot the vibration mode shape. The obtained functions $Y_N^{(2)}(y)$ are also employed as the assumed function in the next iteration. These iterative calculations continue until the obtained natural frequencies converge. It is seen that the difference between the natural frequencies obtained from the third and fourth iterations is approximately 0.25%. A more accurate solution can be achieved with a higher number of iterations. From Table 2, the natural frequency is constant to the third digit after the 11th calculation. Similarly, the vibration mode converges at the third or fourth calculation. In this example, the specimen is a $[\pm 45]_{2S}$ composite plate, which has a higher degree of anisotropy compared to other specially orthotropic specimens. Thus, a higher number of iterations is required for this specimen. Unidirectional or cross-ply specimens need only a few iterations before converged solutions are obtained.

Table 1 Ply properties of graphite-epoxy composite used in this study

E_{11} (GPa)	E_{22} (GPa)	G_{12} (GPa)	v_{12}	ρ (kg/m^3)	Ply thickness (mm)
132	10.8	5.65	0.24	1540	0.127

Table 2 Iterative calculations for the 2nd vibration mode of the $[\pm 45]_{2S}$ composite plate

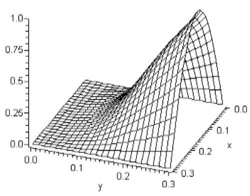
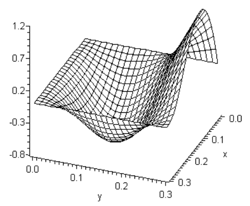
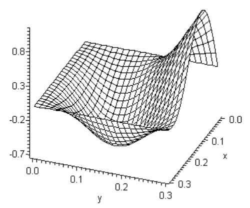
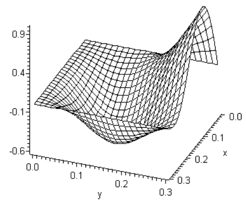
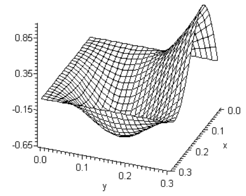
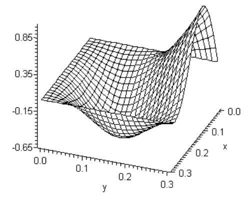
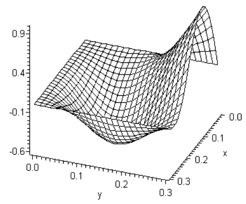
Iteration No.	Assumed functions (Input)	Obtained functions (Output)	ω (rad/s)	Mode shape
1	$Y_1^{(1)}(y)=y^4$ $Y_2^{(1)}(y)=y^5$ $Y_3^{(1)}(y)=y^6$	$X_1^{(1)}(\xi)$ $X_2^{(1)}(\xi)$ $X_3^{(1)}(\xi)$	289.503	
2	$X_1^{(1)}(\xi)$ $X_2^{(1)}(\xi)$ $X_3^{(1)}(\xi)$	$Y_1^{(2)}(\eta)$ $Y_2^{(2)}(\eta)$ $Y_3^{(2)}(\eta)$	679.073	
3	$Y_1^{(2)}(\eta)$ $Y_2^{(2)}(\eta)$ $Y_3^{(2)}(\eta)$	$X_1^{(2)}(\xi)$ $X_2^{(2)}(\xi)$ $X_3^{(2)}(\xi)$	666.702	

Table 2 Continued

4	$X_1^{(2)}(\xi)$ $X_2^{(2)}(\xi)$ $X_3^{(2)}(\xi)$	$Y_1^{(3)}(\eta)$ $Y_2^{(3)}(\eta)$ $Y_3^{(3)}(\eta)$	665.038	
5	$Y_1^{(3)}(\eta)$ $Y_2^{(3)}(\eta)$ $Y_3^{(3)}(\eta)$	$X_1^{(3)}(\xi)$ $X_2^{(3)}(\xi)$ $X_3^{(3)}(\xi)$	664.990	
6	$X_1^{(3)}(\xi)$ $X_2^{(3)}(\xi)$ $X_3^{(3)}(\xi)$	$Y_1^{(4)}(\eta)$ $Y_2^{(4)}(\eta)$ $Y_3^{(4)}(\eta)$	664.967	
7	$Y_1^{(4)}(\eta)$ $Y_2^{(4)}(\eta)$ $Y_3^{(4)}(\eta)$	$X_1^{(4)}(\xi)$ $X_2^{(4)}(\xi)$ $X_3^{(4)}(\xi)$	664.953	

It should be noted that, this study extends the solution procedures of the buckling problem by Shufrin *et al*. (2008a) to the vibration problem of symmetric laminated plate. For particular input functions of X_N or Y_N in the vibration study, there are an infinite number of eigenvalues obtained from the governing equations and boundary conditions. The obtained solutions from an iterative calculation are only solutions which are corresponding to the specific input functions X_N or Y_N. Solutions of other vibration modes will be obtained if the other input functions are employed. The solutions of interest in vibration problem are every eigenvalues and their corresponding eigenvectors, while only the lowest eigenvalue is required for the buckling problem. As a result, not only the lowest eigenvalue but also higher eigenvalues must be examined in the vibration problem. Therefore, the solution procedures of vibration problem is slightly complicated than that of buckling problem in which only lowest eigenvalue is of concern.

5. Convergence study

In addition to convergence of the solution in the aspect of number of iteration, the convergence study was also conducted to determine the appropriate number of terms used in the displacement function, w. As shown in Eq. (3), the displacement function is a linear combination of the product of function $X_i(\xi)$ and function $Y_i(\eta)$, while N is the number of terms used to approximate the function. If a higher number of terms are used in the calculation, the obtained solution is closer to the exact solution; however, more computational resources are required. So, it is necessary to determine an appropriate number of terms used in the approximate function so that reasonably accurate solutions are obtained with reasonable resources. The result of the convergence studies are shown in Table 3. The fundamental natural frequencies of specimens with various boundary conditions and stacking sequences are determined using displacement functions with a number of terms ranging from N=1 to N=5. The specimens are either isotropic, cross-ply, or angle-ply plates with dimensions of 30×30 cm^2. Mechanical properties of the laminated specimens are shown in Table 1, while E=70 GPa and v=0.3 are used for isotropic plates. From the table, isotropic and cross-ply plates require displacement functions with only one or two terms so that the natural frequencies are constant or converged. For example, the fundamental natural frequency of CCCC isotropic plate is constant at 103.60 rad/s when N=2, 3, 4, or 5 are used in the displacement functions. Besides isotropic and cross-ply specimens, solutions of unidirectional specimen converge using only two terms in the displacement functions. These types of specimens require a minimum number of terms in the displacement functions, because the coupling bending stiffness D_{16} and D_{26} of the specimens are absent. As a result, the vibration modes are symmetric in both x and y directions, so that a one-term displacement function is sufficient to describe the vibration mode shape.

For angle-ply specimens, the natural frequencies from calculation with N=1 are much higher than those of using a higher number of terms. The associated vibration modes are also incorrect because displacement functions with N=1 can not accommodate the non-symmetric mode shape. Mode shapes are getting closer to the correct solutions when the displacement functions with N=2 are used. The computations with N=2, 3, 4, or 5 result in a comparable mode shape and nearly unchanged natural frequencies. Unlike the preceding types of specimens, the obtained natural frequencies are not exactly constant with an increasing number of terms used in the displacement function. However, the variation of the solutions is minimal, and can be considered as a converged

Table 3 Convergence study in terms of number of terms used in the displacement function

B.C.	Stacking sequence	Natural frequency (rad/s)				
		N=1	N=2	N=3	N=4	N=5
CCCC	Isotropic	103.64	103.60			
	$[0/90]_{2S}$	723.95	723.93			
	$[\pm 45]_{2S}$	703.36	693.78	693.60	693.59	693.58
	[45/-45/45]	263.26	238.30	236.87	236.75	236.71
CCSS	Isotropic	77.90	77.89			
	$[0/90]_{2S}$	510.31	510.30			
	$[\pm 45]_{2S}$	560.05	551.83	551.65	551.64	551.62
	[45/-45/45]	210.02	186.98	184.65	184.03	183.84
SCSF	Isotropic	36.53				
	$[0/90]_{2S}$	272.18				
	$[\pm 45]_{2S}$	248.74	246.03	245.95	245.93	245.92
	[45/-45/45]	85.50	77.98	77.25	77.03	76.95

solution. To balance the computational time and the accuracy of the solutions, displacement functions with N=2 and N=3 are used for plates with symmetric and non-symmetric vibration mode shapes, respectively.

6. Numerical verification

From the convergence studies, the appropriate numbers of terms used in the displacement functions in order to obtain a converged solution are determined. However, the accuracy of the proposed technique must be verified with other available solutions in the literature. In this section, natural frequencies of laminated plates determined from the multi-term Kantorovich method are compared with the solutions of Chen *et al.* (2003), Leissa and Narita (1989), and Chow *et al.* (1992), as shown in Table 4 and 5. The natural frequencies of the specimens with stacking sequences of $[0]_3$, [15/-15/15], [30/-30/30], [45/-45/45], and [0/90/0] are compared in the tables. The dimensions of the specimens used in this verification are 10×10 m^2 with 0.06 m thickness. The material properties are: E_1/E_2=2.45, G_{12}/E_2=0.48, v_{12}=0.23, and mass density ρ=8000 kg/m^3. In order to compare the solutions with those in the literature, natural frequency is presented in term of a dimensionless frequency parameter which is defined according to

$$\beta = \sqrt{\frac{\rho h \omega^2 a^4}{D_0}} \tag{18}$$

where $D_0 = \dfrac{E_1 h^3}{12(1 - v_{12} v_{21})}$.

Tables 4 and 5 present the dimensionless frequency parameters of the first five vibration modes of the SSSS and CCCC specimens, respectively. It is observed that majority of the solutions from the extended Kantorovich method, denoted as “EKM,” agree very well with those of the other three studies. Most of the solutions differ from the previous studies by less than 1 %. Moreover, a majority of them are lower than those of other studies. Thus, it is confirmed that the obtained solutions using assumed power series are converged and very well compared to other available solutions.

The other set of studies are presented in Tables 6-9, which compare the natural frequencies of typical graphite/epoxy laminated plates determined from different methods. Besides the solutions from the present extended Kantorovich method, the natural frequency is also determined using the Ritz method and finite element method, which are represented by “Ritz” and “FEM,” respectively, in the table. The Ritz method is based on the principle of minimum total potential energy. In this method, the out-of-plane displacement w is assumed in terms of a summation of a product of two functions in the x and y directions with unknown coefficients. The functions in the x and y directions are chosen from the Bernoulli-Euler beam functions that satisfy the boundary conditions of the specimen. The out-of-plane displacement function used in this study is a 12×12 or 144-term displacement function. By minimizing the plate’s total potential energy, the unknown coefficients in the displacement function and natural frequencies of the plate are determined. The summary of the application of the Ritz method to the vibration of laminated plates is concisely presented by Maheri and Adams (2003) and Shen *et al.* (2003). The beam functions for various combinations of boundary conditions are also systematically tabulated in the latter reference. Besides the Ritz method, the finite element method is also performed, using commercially available software. The problems are modeled using 8-node multi-layered quadratic shell elements with a total number of elements of 900, 1800, and 2700 for specimens with aspect ratio of 1, 2 and 3, respectively. Both the natural frequency and its corresponding vibration mode shape are determined.

Natural frequencies of all-edge-clamp (CCCC) composite laminated plates with a dimension of $a \times b$ are presented in Table 6. In this simulation, the plate width is selected as b=300 mm, and the plate length a is varied from 300 mm, 600 mm, and 900 mm, i.e. the plate aspect ratios are 1, 2 and 3, respectively. Specimens used in this part of the study include both cross-ply and angle-ply plates. Their stacking sequences are $[0]_8$, $[0/90]_{2S}$, $[\pm 45]_{2S}$ and $[0/30/60/90]_S$. The material and ply properties of the graphite-epoxy lamina used in this investigation are presented in Table 1. Natural frequencies of the first eight modes determined from the extended Kantorovich method, the Ritz method, and finite element method are shown in the table. It is observed that most of the solutions from the extended Kantorovich method are closely consistent with the solutions from the Ritz method and FEM. The percentage of discrepancies of the solutions from Kantorovich method, compared to those of the other two solutions, are negligibly small. For laminates with $[0]_8$, $[0/90]_{2S}$, or $[\pm 45]_{2S}$ stacking sequences, a majority of the solutions from Kantorovich method are lower than those of from the Ritz method, and higher than those from FEM. On the other hand, for $[0/30/60/90]_S$ laminates, the Kantorovich method returns higher natural frequencies compared to those of both methods. However, there are two cases that the solutions from Kantorovich method deviate from the reference solutions by more than one percent. For the $[\pm 45]_{2S}$ plate with aspect ratio of 1, the natural frequency of vibration modes 5 and 8 deviate from the reference solutions by as much as 2.4 % and 1.3 %, respectively. Since it is not obvious that the solutions from the extended Kantorovich method are always higher or lower than those of the other two solution methods in particular cases, the errors in numerical calculation are probably the cause of discrepancy. Similar comparisons of the natural frequencies determined by the three methods are presented in Tables 7-9 for specimens with CCCF, SCSC and SCSF boundary conditions. Similar to CCCC specimens, the solutions from the Kantorovich method correspond very well to the reference solutions. In a few cases of angle ply specimens, the natural frequencies from the Kantorovich method slightly diverge from the solutions of the other two methods. The highest degree of discrepancy is 1.6 %.

It is also noticed that all of those cases in Tables 6-9 with a difference percentage of more than 1 percent are cases of $[\pm 45]_{2S}$ or $[0/30/60/90]_S$ laminates. For unidirectional and cross-ply laminates, the solutions from Kantorovich method are different from the other two methods with a discrepancy percentage of less than 0.15 %. This observation agrees with the number of iterations required to obtain the converged solution. For specially orthotropic plates with zero laminate bending stiffness of D_{16} and D_{26}, only 4 to 5 iterative calculations are required to get a converged solution. On the contrary, some cases of the angle-ply laminates need more than 10 iterations to achieve a converged natural frequency. The present of terms D_{16} and D_{26} results in a more complicated energy expression, thus, more iterative calculations are need. From a physical view point, the vibration mode shapes of the angle-ply specimens are not symmetric; therefore, a more complicated displacement function is needed. Therefore, it is reasonable to obtain a higher degree of discrepancy in some cases of the simulation. This discrepancy can be decreased with a higher number of terms used in the displacement function.

The accuracy of the Kantorovich method is not only considered from the natural frequency, but also determined from the vibration mode shape. Thus, the vibration mode shapes of a $[\pm 45]_{2S}$ laminated plate with SCSC and SCSF boundary conditions and plate aspect ratio of 2 are

Table 4 Frequency parameters β of SSSS laminated composite square plates

Stacking Sequence	Solution	Frequency parameters (β)				
		Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
$[0]_3$	Chen *et al*. (2003)	15.18	33.34	44.51	60.79	64.80
	Leissa and Narita (1989)	15.19	33.3	44.42	60.77	64.53
	Chow *et al*. (1992)	15.19	33.31	44.52	60.78	64.55
	Present study, EKM	15.17	33.25	44.39	60.68	64.46
[15/-15/15]	Chen *et al*. (2003)	15.41	34.15	43.93	60.91	66.94
	Leissa and Narita (1989)	15.43	34.09	43.87	60.85	66.67
	Chow *et al*. (1992)	15.37	34.03	43.80	60.8	66.56
	Present study, EKM	15.40	34.03	43.83	60.75	66.62
[30/-30/30]	Chen *et al*. (2003)	15.88	35.95	42.63	61.54	72.12
	Leissa and Narita (1989)	15.90	35.86	42.62	61.45	71.71
	Chow *et al*. (1992)	15.86	35.77	42.48	61.27	71.41
	Present study, EKM	15.86	35.77	42.58	61.31	71.88
[45/-45/45]	Chen *et al*. (2003)	16.11	37.04	41.8	61.94	78.03
	Leissa and Narita (1989)	16.14	36.93	41.81	61.85	77.04
	Chow *et al*. (1992)	16.08	36.83	41.67	61.65	76.76
	Present study, EKM	16.10	36.87	41.82	61.68	78.57
[0/90/0]	Chen *et al*. (2003)	15.18	33.82	44.14	60.79	66.12
	Present study, EKM	15.17	33.73	44.02	60.68	65.77

Table 5 Frequency parameters β of CCCC laminated composite square plates

Stacking Sequence	Solution	Frequency parameters (β)				
		Mode 1	Mode 2	Mode 3	Mode 4	Mode 5
$[0]_3$	Chen *et al*. (2003)	29.27	51.21	67.94	86.25	87.97
	Chow *et al*. (1992)	29.13	50.82	67.29	85.67	87.14
	Present study, EKM	29.08	50.78	67.26	85.59	87.06
[15/-15/15]	Chen *et al*. (2003)	29.07	51.82	66.54	85.17	90.56
	Chow *et al*. (1992)	28.92	51.43	65.92	84.55	89.76
	Present study, EKM	28.89	51.39	65.89	84.49	89.68
[30/-30/30]	Chen *et al*. (2003)	28.69	53.57	63.24	84.43	96.13
	Chow *et al*. (1992)	28.55	53.15	62.71	83.83	95.21
	Present study, EKM	28.51	53.11	62.67	83.79	95.42
[45/-45/45]	Chen *et al*. (2003)	28.50	55.11	60.91	84.25	103.2
	Chow *et al*. (1992)	28.38	54.65	60.45	83.65	102.0
	Present study, EKM	28.33	54.64	60.47	83.62	103.9
[0/90/0]	Chen *et al*. (2003)	29.27	51.93	67.40	86.25	89.76
	Present study, EKM	29.08	51.48	66.72	85.59	88.82

determined from the Kantorovich method and finite element method. Tables 10 and 11 demonstrate the vibration mode shapes determined from both studies. The specimen is intentionally selected as an angle-ply laminate with a combination of various boundary conditions. From the tables, vibration mode shapes from the extended Kantorovich method match the finite element solutions exceptionally well. For the SCSC specimen, the inclination of the mode shape contour due to the inclined fiber angle can be clearly observed in vibration modes 2, 3, 4, 7, and 8. An unusual mode shape of mode 5 determined from both methods compare very well. Therefore, the out-of-plane displacement of the vibrating specimens is effectively simulated using the assumed displacement function in terms of power series shown in Eq. (12) and Eq. (16).

In conclusion, a computer routine was generated to handle the computational algorithms derived in the previous sections. The extended Kantorovich method using power series as the assumed displacement functions has been verified with solutions from the previous studies. It is found that most of the solutions deviate from the solutions of the previous studies with a discrepancy percentage of less than 1 %. The solutions from the present method are also compared to the solutions of the well-known Ritz method and finite element method. The comparisons showed that the proposed Kantorovich method is an effective semi-analytical numerical method for solving the vibration problem of laminated plates.

Table 6 Natural frequency in Hz of CCCC laminated composite plates

Aspect ratio	Mode	$[0]_8$			$[0/90]_{2S}$			$[\pm 45]_{2S}$			$[0/30/60/90]_S$		
		EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM
1	1	115.18	115.18	115.11	115.22	115.22	115.22	110.39	110.39	110.40	112.71	112.71	112.71
	2	146.71	146.71	146.66	208.54	208.54	208.55	212.40	212.15	212.15	172.91	172.90	172.92
	3	212.45	212.45	212.47	264.23	264.23	264.24	231.89	231.67	231.70	273.83	273.66	273.72
	4	302.80	302.80	302.37	324.87	324.87	324.95	338.09	337.76	337.80	277.58	277.57	277.58
	5	312.03	312.03	312.15	372.77	372.77	372.80	392.37	383.10	383.12	333.95	333.77	333.91
	6	324.65	324.66	324.30	457.90	457.90	458.14	399.74	399.03	399.18	392.03	391.90	392.09
	7	371.10	371.11	370.98	501.50	501.50	501.52	496.98	493.55	493.66	454.84	453.89	454.21
	8	442.60	442.60	442.90	543.91	543.91	544.11	550.38	543.19	543.38	520.29	519.61	519.96
2	1	43.07	43.071	43.07	70.53	70.53	70.53	71.55	71.56	71.55	54.53	54.52	54.524
	2	84.45	84.45	84.42	94.27	94.27	94.27	99.22	99.23	99.21	87.40	87.40	87.398
	3	92.77	92.77	92.77	144.34	144.34	144.34	140.38	140.39	140.36	131.98	131.96	131.95
	4	123.52	123.52	123.50	183.52	183.52	183.52	180.80	180.81	180.78	140.81	140.73	140.73
	5	153.45	153.45	153.35	197.39	197.39	197.40	194.40	194.39	194.34	163.36	163.16	163.16
	6	172.79	172.79	172.79	219.19	219.19	219.19	215.03	215.10	214.99	198.83	198.73	198.74
	7	183.93	183.93	183.85	230.40	230.40	230.42	260.53	260.44	260.39	233.91	233.20	233.21
	8	196.61	196.61	196.60	288.56	288.57	288.61	265.54	265.63	265.46	249.31	249.31	249.31
3	1	34.24	34.24	34.24	66.77	66.77	66.77	65.44	65.44	65.44	47.91	47.91	47.911
	2	48.36	48.36	48.35	73.18	73.18	73.18	77.50	77.53	77.50	59.13	59.13	59.125
	3	76.00	76.00	75.98	88.94	88.94	88.94	96.19	96.23	96.18	80.84	80.84	80.833
	4	87.45	87.45	87.44	116.42	116.42	116.42	120.71	120.77	120.70	112.84	112.83	112.82
	5	97.06	97.06	97.06	155.53	155.53	155.53	150.74	150.82	150.73	126.73	126.73	126.72
	6	116.12	116.12	116.07	181.35	181.35	181.34	173.92	173.94	173.91	136.17	136.15	136.13
	7	117.57	117.58	117.56	185.44	185.44	185.44	186.28	186.37	186.24	154.53	154.39	154.37
	8	151.16	151.17	151.12	194.71	194.72	194.71	189.23	189.34	189.21	155.00	154.94	154.92

Table 7 Natural frequency in Hz of CCCF laminated composite plates

Aspect ratio	Mode	$[0]_8$			$[0/90]_{2S}$			$[\pm 45]_{2S}$			$[0/30/60/90]_S$		
		EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM
1	1	108.59	108.61	108.52	92.59	92.59	92.59	67.38	68.20	67.31	96.09	96.20	96.06
	2	118.89	118.89	118.80	118.76	118.76	118.76	129.18	129.36	128.93	119.75	119.98	119.72
	3	151.68	151.68	151.59	212.12	212.12	212.12	174.01	176.15	173.48	179.95	180.32	179.90
	4	217.25	217.25	217.20	252.20	252.21	252.19	232.58	232.21	231.77	261.14	261.38	261.01
	5	297.83	297.87	297.39	270.37	270.38	270.37	261.41	262.17	260.88	280.34	280.45	279.92
	6	307.67	307.70	307.23	333.41	333.41	333.45	332.48	337.04	331.74	288.10	288.39	287.93
	7	316.20	316.21	316.25	375.51	375.51	375.52	368.99	368.48	367.87	349.47	349.73	349.00
	8	332.72	332.73	332.30	466.39	466.40	466.55	407.27	406.94	406.01	400.43	400.84	400.21
2	1	28.24	28.24	28.23	25.98	25.98	25.98	23.85	24.02	23.81	26.28	26.31	26.27
	2	45.41	45.41	45.40	64.96	64.96	64.95	50.99	51.58	50.91	56.97	57.06	56.94
	3	75.46	75.47	75.43	71.31	71.31	71.31	79.03	79.15	78.96	68.36	68.47	68.32
	4	89.13	89.14	89.09	98.25	98.25	98.25	89.72	90.94	89.59	94.50	94.63	94.40
	5	94.73	94.72	94.70	124.81	124.81	124.80	119.77	120.05	119.56	130.09	129.70	129.49
	6	129.00	129.00	128.94	150.33	150.33	150.32	140.73	142.71	140.50	135.27	135.72	135.49
	7	146.68	146.70	146.58	184.57	184.56	184.56	171.18	171.83	170.91	148.42	147.54	147.28
	8	158.83	158.84	158.72	200.92	200.92	200.91	186.23	186.40	186.18	175.47	174.29	173.73
3	1	13.76	13.76	13.76	15.21	15.21	15.21	15.90	15.97	15.89	14.15	14.18	14.15
	2	34.50	34.50	34.49	31.10	31.10	31.10	28.85	29.10	28.81	31.63	31.68	31.62
	3	35.33	35.33	35.32	57.14	57.14	57.14	46.55	47.11	46.50	49.41	49.63	49.54
	4	51.94	51.95	51.92	66.59	66.59	66.59	68.32	68.40	68.16	57.24	57.23	57.15
	5	66.05	66.05	66.02	75.00	75.00	75.00	70.04	70.74	69.80	66.63	66.72	66.46
	6	80.96	80.96	80.92	92.47	92.48	92.47	89.03	89.23	88.92	86.30	86.35	86.18
	7	88.38	88.37	88.36	92.96	92.96	92.96	98.20	99.59	97.98	98.82	98.68	98.39
	8	100.17	100.16	100.13	122.08	122.07	122.06	116.22	116.54	116.05	120.07	120.05	119.82

Table 8 Natural frequency in Hz of SCSC laminated composite plates

Aspect ratio	Mode	$[0]_8$			$[0/90]_{2S}$			$[\pm 45]_{2S}$			$[0/30/60/90]_S$		
		EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM
1	1	61.54	61.54	61.53	80.46	80.46	80.45	91.55	91.62	91.51	71.71	71.75	71.681
	2	107.99	107.99	107.98	179.87	179.87	179.87	172.01	172.19	171.97	145.89	145.96	145.82
	3	186.28	186.28	186.26	190.54	190.54	190.54	211.58	211.63	211.42	190.41	190.38	190.27
	4	198.29	198.29	198.18	258.90	258.90	258.91	293.14	293.10	292.77	237.79	237.96	237.72
	5	228.02	228.02	227.91	362.25	362.25	362.24	324.57	324.39	323.96	278.45	277.69	277.47
	6	287.55	287.56	287.43	373.89	373.89	373.89	379.77	380.08	379.80	342.01	342.16	341.8
	7	293.58	293.59	293.58	411.79	411.79	411.82	440.83	440.23	439.67	395.34	395.48	395.39
	8	380.34	380.35	380.24	427.21	427.22	427.27	497.50	492.55	492.05	429.15	426.85	426.61
2	1	35.17	35.17	35.16	67.26	67.26	67.24	69.10	69.13	69.09	49.55	49.57	49.521
	2	61.54	61.54	61.52	80.46	80.46	80.43	91.23	91.31	91.21	70.89	70.92	70.807
	3	88.89	88.89	88.87	117.22	117.22	117.17	125.77	125.90	125.74	114.72	114.72	114.56
	4	107.99	107.99	107.95	179.87	179.87	179.78	172.01	172.19	171.98	129.12	129.11	129.00
	5	116.89	116.90	116.85	182.06	182.06	181.93	179.96	179.94	179.88	149.99	149.91	149.64
	6	153.45	153.45	153.37	190.54	190.54	190.40	209.13	209.21	209.03	172.94	173.02	172.72
	7	170.39	170.39	170.34	213.53	213.53	213.36	232.18	232.16	231.91	198.59	198.19	197.81
	8	186.28	186.28	186.21	258.90	258.90	258.70	254.39	254.53	254.20	235.34	235.33	234.83

Table 8 Continued

3	1	32.28	32.28	32.28	66.04	66.04	66.04	64.73	64.74	64.73	46.68	46.69	46.68
	2	40.68	40.68	40.68	69.68	69.69	69.68	74.96	75.00	74.95	54.15	54.18	54.14
	3	61.54	61.54	61.53	80.46	80.46	80.46	91.14	91.22	91.12	70.59	70.64	70.57
	4	86.50	86.50	86.50	102.00	102.00	102.00	112.82	112.95	112.80	97.38	97.44	97.34
	5	92.92	92.92	92.92	135.34	135.34	135.34	139.85	140.01	139.83	125.94	125.94	125.92
	6	95.43	95.43	95.40	179.87	179.87	179.87	171.92	172.08	171.87	131.50	131.50	131.40
	7	107.99	107.99	107.98	180.99	180.99	180.99	174.06	173.92	173.85	136.40	136.26	136.18
	8	135.03	135.03	135.00	183.82	183.82	183.82	187.41	187.46	187.37	148.26	148.30	148.13

Table 9 Natural frequency in Hz of SCSF laminated composite plates

Aspect ratio	Mode	$[0]_8$			$[0/90]_{2S}$			$[\pm 45]_{2S}$			$[0/30/60/90]_S$		
		EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM	EKM	Ritz	FEM
1	1	49.23	49.24	49.22	43.32	43.32	43.32	39.14	39.68	39.11	45.62	45.77	45.60
	2	67.04	67.04	67.00	84.40	84.41	84.39	105.83	106.40	105.72	80.89	81.18	80.81
	3	113.49	113.49	113.42	162.58	162.59	162.58	122.12	124.00	121.92	151.23	151.40	151.03
	4	190.81	190.82	190.74	187.58	187.59	187.57	201.16	201.85	200.91	170.51	170.91	170.43
	5	191.38	191.41	191.27	193.78	193.78	193.76	230.66	231.31	230.07	205.76	206.28	205.51
	6	204.71	204.73	204.58	268.15	268.15	268.13	252.94	256.71	251.94	246.01	246.42	245.78
	7	237.92	237.94	237.75	362.84	362.86	362.82	329.64	326.81	325.56	291.96	292.17	291.06
	8	297.31	297.31	297.24	364.60	364.61	364.58	369.58	367.33	364.87	350.58	350.79	349.87
2	1	14.31	14.31	14.30	15.77	15.77	15.77	18.70	18.81	18.69	15.43	15.49	15.42
	2	37.45	37.45	37.44	43.32	43.32	43.32	38.95	39.44	38.93	43.73	43.80	43.72
	3	49.23	49.24	49.22	67.77	67.77	67.77	70.50	71.63	70.45	54.48	54.62	54.37
	4	67.04	67.04	67.01	84.40	84.40	84.40	76.07	76.21	76.03	77.57	77.78	77.50
	5	90.62	90.62	90.60	92.71	92.72	92.71	111.53	111.91	111.38	99.03	98.99	98.65
	6	108.37	108.39	108.34	123.80	123.80	123.79	114.60	116.50	114.44	123.09	123.13	122.79
	7	113.49	113.48	113.43	162.58	162.59	162.58	157.10	157.81	156.85	131.95	132.14	131.89
	8	123.24	123.25	123.18	182.96	182.96	182.95	171.26	173.92	170.88	160.94	159.25	158.61
3	1	8.36	8.36	8.36	12.02	12.02	12.02	14.14	14.19	14.14	10.35	10.39	10.35
	2	23.19	23.19	23.19	22.31	22.31	22.31	24.31	24.52	24.30	22.70	22.77	22.69
	3	33.18	33.18	33.17	43.32	43.32	43.32	38.96	39.45	38.94	44.00	44.06	43.96
	4	44.35	44.35	44.33	65.73	65.73	65.73	58.59	59.48	58.57	48.06	48.13	48.03
	5	49.23	49.24	49.22	71.25	71.25	71.25	67.77	67.86	67.75	60.74	60.93	60.61
	6	67.04	67.04	67.01	73.95	73.95	73.95	83.27	83.95	82.91	72.07	72.13	71.99
	7	86.00	86.02	85.98	84.40	84.41	84.40	87.37	87.90	87.18	84.47	84.63	84.17
	8	87.31	87.31	87.30	107.90	107.90	107.89	109.98	110.45	109.77	103.83	103.98	103.69

Table 10 Vibration mode shapes of SCSC $[\pm 45]_{2S}$ laminated composite plates

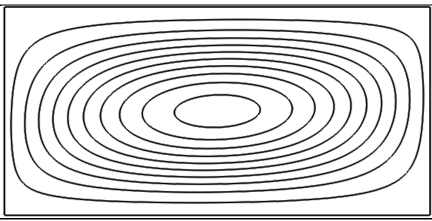
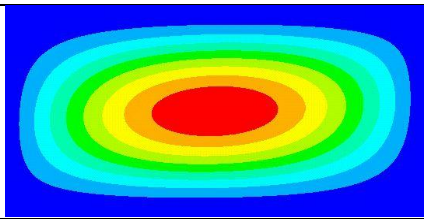
Mode	EKM	FEM
1		

Table 10 Continued

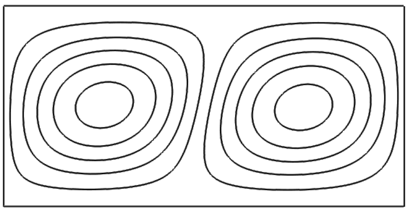
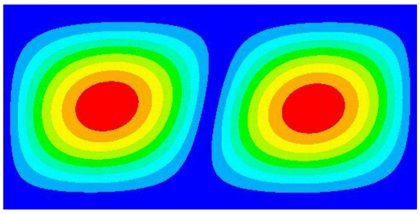
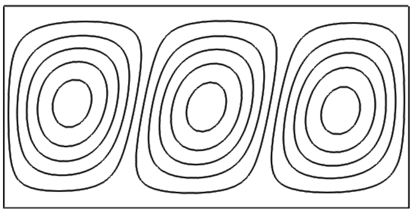
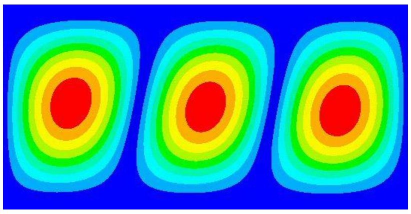
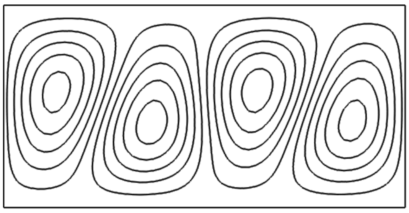
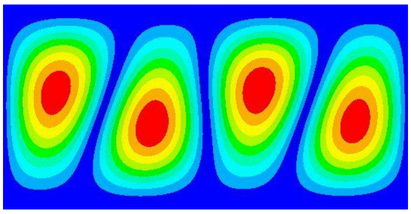
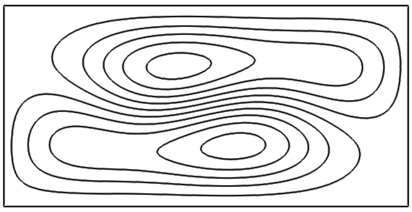
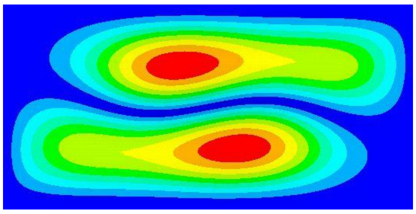
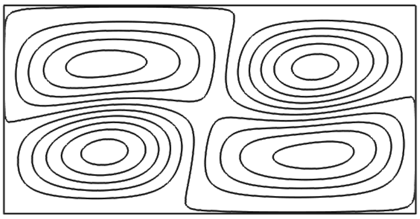
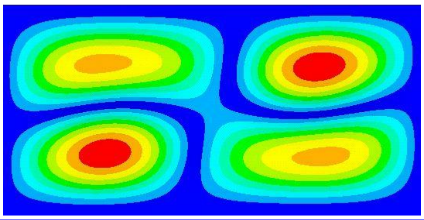
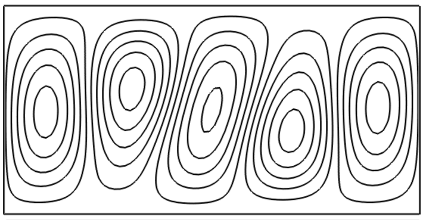
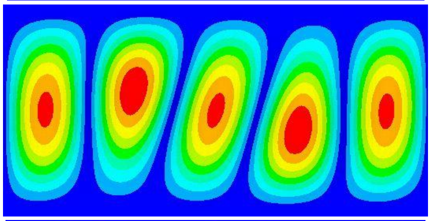
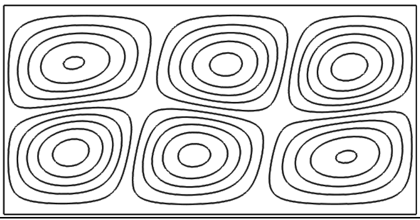
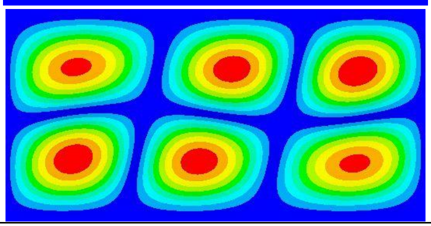
2		
3		
4		
5		
6		
7		
8		

Table 11 Vibration mode shapes of SCSF $[\pm 45]_{2S}$ laminated composite plates

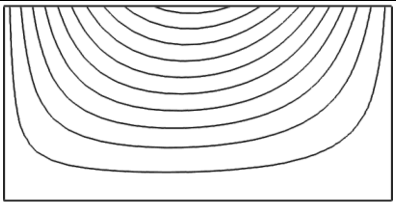
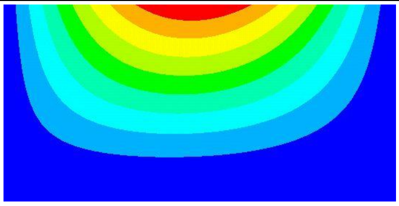
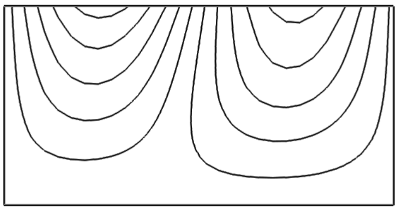
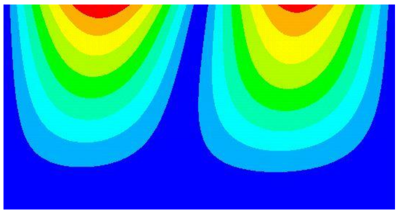
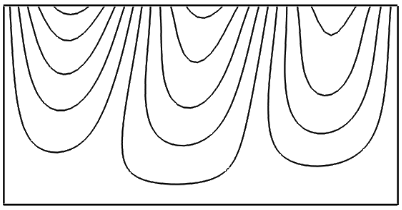
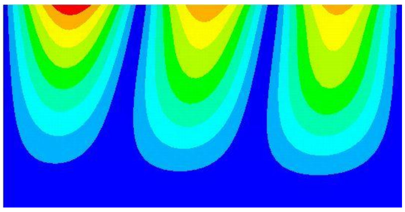
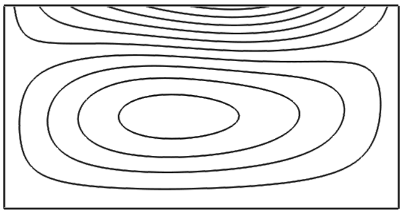
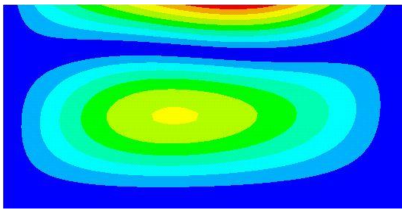
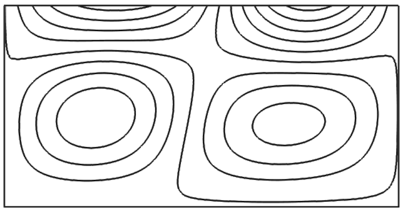
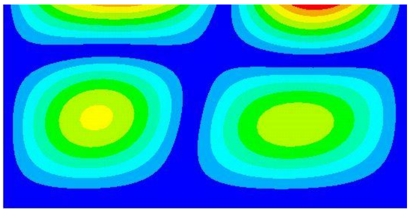
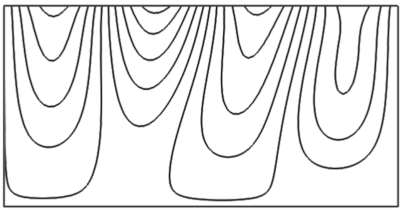
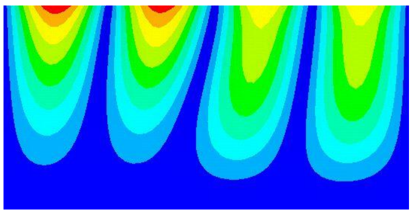
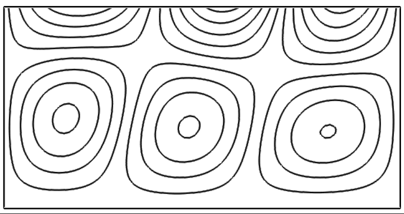
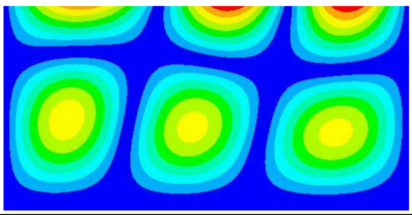
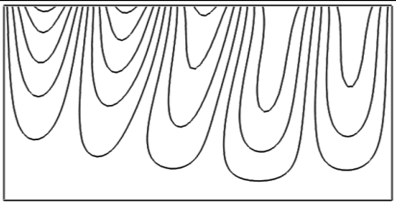
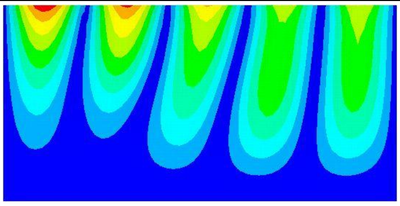
Mode	EKM	FEM
1		
2		
3		
4		
5		
6		
7		

Table 11 Continued

8		

7. Conclusions

Vibration behaviors of symmetrically laminated plates are investigated using the Kantorovich method, employing multi-term displacement functions in the form of a power series. The solution procedure is based on the variational principle of total potential energy minimization. With a known displacement function in terms of functions of *x* or functions of *y*, a system of governing equations in the form of ordinary differential equations is obtained with a set of boundary conditions. The governing equations are solved by assuming the undetermined displacement functions in terms of a power series. These calculation procedures are repeated until the natural frequency of the problem converges to specific values. The obtained natural frequencies have been verified with the solutions from the previous studies, the Ritz method, and the finite element method and found to be accurate. The vibration mode shapes obtained from the Kantorovich method compare very well to those determined from the finite element analysis. Even for an uncommon mode shape of an angle-ply specimen, the solution of the proposed technique compares very well with the finite element solution. Thus, the out-of-plane displacement of an angle-ply plate is perfectly modeled using the multi-term assumed displacement function in terms of a power series. This type of problem is impossible to handle using the conventional single-term Kantorovich method.

In conclusion, although higher computational resources may be required for the extended Kantorovich method compared to other numerical methods, the multi-term extended Kantorovich method applied in this study derived and solved the governing equations directly. Since the equations are solved numerically, the solution method can be considered as a semi-analytical numerical method. If the numerical method used to solve the governing equations returns an accurate numerical solution, the solution can be considered as an exact solution and can be used as a benchmark solution for other numerical methods. It is interesting to apply the technique to other types of structural problems, including vibration problems of complicated plate structures.

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