Structural Engineering and Mechanics, *Vol. 51, No. 4 (2014) 651-662*
DOI: http://dx.doi.org/10.12989/sem.2014.51.4.651

Relaxed Saint-Venant principle for thermoelastic micropolar diffusion

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(Received March 7, 2014, Revised June 11, 2014, Accepted June 19, 2014)

Abstract. The main goal of this study is to extend the domain of influence result to cover the micropolar thermoelastic diffusion. So, we prove that for a finite time $t>0$ the displacement field u_i, the microrotation vector φ_i, the temperature θ and the chemical potential P generate no disturbance outside a bounded domain B_t.

Keywords: thermoelastic; micropolar; diffusion; domain of influence

1. Introduction

The problems connected with the diffusion of matter in thermoelastic bodies and the interaction of mechano-diffusion processes have become the subject of research by many authors. At elevated and low temperatures, the processes of heat and mass transfer play the decisive role in many problems of satellites, returning space vehicles, and landing on water or land. These days, oil companies are interested in the process of thermodiffusion for more efficient extraction of oil from oil deposits. Diffusion can be defined as the random walk, of an ensemble of particles, from the regions of higher concentration to the regions of lower concentration. Thermodiffusion in an elastic solid is due to coupling of the fields of temperature, mass diffusion and that of strain. Nowacki (1974), Nowacki (1976) developed the theory of thermoelastic diffusion. In this theory, the coupled thermoelastic model is used.

Uniqueness and reciprocity theorems for the equations of generalized thermoelastic diffusion problem, in isotropic media, was proved by Sherief *et al.* (2004) on the basis of the variational principle equations, under restrictive assumptions on the elastic coefficients. Aouadi (2009) proved this theorem in the Laplace transform domain. Aouadi (2010) derived the uniqueness and reciprocity theorems for the generalized thermoelastic diffusion problem in anisotropic media.

In the paper Chirita and Ciarletta (2007) derived necessary and sufficient conditions for strong

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http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

ellipticity in several classes of anisotropic linearly elastic materials. Other important results regarding generalized thermoelastic bodies are present in papers Marin *et al*. (2013), Marin (1994), Marin (1998), Marin (2010a), Marin (2010b).

Abbas and his co-workers applied finite element method for different problems with different theories of thermoelasticity in the papers Abbas and Othman (2012 a), Abbas and Othman (2012 b), Abbas (2012), Kumar *et al* (2013), Abbas and Kumar (2013).

An intelligent supersize finite element method, was employed in the paper Kim *et al*. (2013) for the ultimate longitudinal strength analysis.

In the paper Takabatake (2012), the existence and effect of dead loads are proven by numerical calculations based on the Galerkin method.

In the present paper we first consider the basic equations and conditions of the mixed initial-boundary value problem in the context of micropolar thermoelastic diffusion. Next we define the domain of influence B_t of the data at time t associated with the problem. We adopt the method used in Carbonaro and Russo (1984) to establish a domain of influence theorem. The main result asserts that in the context of theory considered, the solutions of the mixed initial-boundary value problem vanishes outside B_t, for a finite time t>0.

2. Basic equations

An anisotropic elastic material is considered. Assume a such body that occupies a properly regular region B of three-dimensional Euclidian space R^3 bounded by a piecewise smooth surface ∂B and we denote the closure of B by $\bar{B}$. We use a fixed system of rectangular Cartesian axes Ox_i, (i=1,2,3) and adopt Cartesian tensor notation. A superposed dot stands for the material time derivate while a comma followed by a subscript denotes partial derivatives with respect to the spatial coordinates. Einstein summation on repeated indices is also used. Also, the spatial argument and the time argument of a function will be omitted when there is no likelihood of confusion.

The basic equations for micropolar thermoelastic diffusion are

- equations of motion

$$\begin{aligned} t_{ij,\,j} + \rho F_i &= \rho \ddot{u}_i, \\ m_{ij,\,j} + \varepsilon_{ijk} t_{jk} + \rho M_i &= I_{ij} \ddot{\varphi}_j; \end{aligned} \tag{1}$$

- equation of energy

$$\rho T_0 \dot{\eta} = q_{i,\,i} + \rho\, h; \tag{2}$$

- equation of conservation of mass

$$\eta_{i,\,i} + r = \dot{\gamma}. \tag{3}$$

We complete the above equations with

- the constitutive equations

$$\begin{aligned} t_{ij} &= a_{ijmn}\varepsilon_{mn} + b_{ijmn}\mu_{mn} + b_{ij}\gamma - f_{ij}\ \theta, \\ m_{ij} &= b_{mnij}\varepsilon_{mn} + c_{ijmn}\mu_{mn} + c_{ij}\gamma - g_{ij}\ \theta, \end{aligned}$$

$$
\begin{aligned}
&\rho\eta = f_{ij}\varepsilon_{ij} + g_{ij}\mu_{ij} + m\gamma + c\,\theta,\\
&q_i = K_{ij}\,\theta_{,j}\\
&\eta_i = d_{ij}\,P_{,j}\\
&P = b_{ij}\varepsilon_{ij} + c_{ij}\mu_{ij} + \rho\gamma - m\,\theta,
\end{aligned}
\tag{4}
$$

- the kinetic relations

$$\varepsilon_{ij} = u_{j,i} + \varepsilon_{jik}\varphi_k,\ \mu_{ij} = \varphi_{j,i}. \tag{5}$$

In the above equations we have used the following notations: ρ-the constant mass density in the reference state; η-the specific entropy; T-the absolute temperature of the medium; T_0-the constant absolute temperature of the body in its reference state; θ-the temperature variation measured from the reference temperature T_0; I_{ij}-coefficients of microinertia; u_i-the components of displacement vector; φ_i-the components of microrotation vector; ε_{ij}, μ_{ij}-kinematic characteristics of the strain; t_{ij}-the components of the stress tensor; m_{ij}-the components of the couple stress tensor; K_{ij}-the components of the thermal conductivity tensor; q_i-the components of the heat conduction vector; η_{ij}-the components of the diffusion; F_i-the components of the body forces; M_i-the components of the body couple; d_{ij} is the diffusion tensor; h-the heat supply per unit mass and unit time; r is the diffusion supply per unit of initial volume; C is the concentration of the diffusive material in the elastic body; P is the chemical potential per unit mass; a_{ijmn}, b_{ijmn}, …, a are the characteristic functions of the material.

By using the chemical potential as a state variable instead of the concentration of the diffusive material, we can obtain an alternative form of the above equations. So, the basic equations become

$$
\begin{aligned}
&\rho\ddot{u}_i = t_{ij,\,j} + \rho F_i,\\
&I_{ij}\ddot{\varphi}_j = m_{ij,\,j} + \varepsilon_{ijk}t_{jk} + \rho M_i;\\
&\rho T_0\dot{\eta} = q_{i,\,i} + \rho\,h;\\
&\dot{\gamma} = \eta_{i,\,i} + r.
\end{aligned}
\tag{6}
$$

The constitutive equations received the form

$$
\begin{aligned}
&t_{ij} = A_{ijmn}\varepsilon_{mn} + B_{ijmn}\mu_{mn} + B_{ij}\gamma - \beta_{ij}\,\theta,\\
&m_{ij} = B_{mnij}\varepsilon_{mn} + C_{ijmn}\mu_{mn} + C_{ij}\gamma - \alpha_{ij}\,\theta,\\
&\rho\eta = \beta_{ij}\varepsilon_{ij} + \alpha_{ij}\mu_{ij} + m\,P + a\,\theta,\\
&q_i = K_{ij}\,\theta_{,j}\\
&\eta_i = d_{ij}\,P_{,j}\\
&\gamma = -B_{ij}\varepsilon_{ij} - C_{ij}\mu_{ij} + l\,P + d\,\theta,
\end{aligned}
\tag{7}
$$

where new sizes have the following meanings

$$A_{ijmn} = a_{ijmn} - \frac{1}{\rho}b_{ij}\,b_{mn},\ B_{ijmn} = b_{ijmn} - \frac{1}{\rho}b_{ij}\,c_{mn},$$

$$B_{ij}=\frac{1}{\rho}b_{ij},\,C_{ij}=\frac{1}{\rho}c_{ij},\,\beta_{ij}=f_{ij}-\frac{1}{\rho}m\,b_{ij},\,\alpha_{ij}=g_{ij}-\frac{1}{\rho}m\,c_{ij},$$
$$d=\frac{m}{\rho},\,a=\frac{m^2}{\rho}+c,\,l=\frac{1}{\rho}. \tag{8}$$

We assume that the constitutive coefficients satisfy the following symmetry relations

$$A_{ijmn}=A_{mnij},\,C_{ijmn}=C_{mnij},\,K_{ij}=K_{ji},$$
$$B_{ij}=B_{ji},\,C_{ij}=C_{ji},\,\beta_{ij}=\beta_{ji},\,\alpha_{ij}=\alpha_{ji}. \tag{9}$$

The entropy inequality (the second law of thermodynamics) implies that

$$K_{ij}\,\xi_{,i}\,\xi_{,j}\geq 0, \tag{10}$$

for all $\xi_{,i}$.

The components of the surface traction, the heat flux and the diffusion flux at regular points of ∂B are given by

$$t_i=t_{ij}\,n_j,\,m_i=m_{ij}\,n_j,\,q=q_i\,n_i,\,S=\eta_i\,n_i,$$

respectively.

By n_i we denoted the components of the outward unit normal of surface ∂B.

To the system of field Eq. (7) we adjoin the following initial conditions

$$u_i(x,0)=u_i^0(x),\;\;\dot{u}_i(x,0)=u_i^1(x),\;\;\varphi_i(x,0)=\varphi_i^0(x),$$
$$\dot{\varphi}_i(x,0)=\varphi_i^1(x),\;\;\theta(x,0)=\theta^0(x),\;\;\gamma(x,0)=\gamma^0(x),\;\;x\in\overline{B}, \tag{11}$$

and the following prescribed boundary conditions

$$u_i=\overline{u}_i\ on\ \partial B_1\times[0,t_0),\ t_i\equiv t_{ij}n_j=\bar{t}_i\ on\ \partial B_1^c\times[0,t_0),$$
$$\varphi_i=\overline{\varphi}_i\ on\ \partial B_2\times[0,t_0),\ m_i\equiv m_{ij}n_j=\overline{m}_i\ on\ \partial B_2^c\times[0,t_0),$$
$$\theta=\overline{\theta}\ on\ \partial B_3\times[0,t_0),\ q\equiv q_in_i=\overline{q}\ on\ \partial B_3^c\times[0,t_0),$$
$$P=\overline{P}\ on\ \partial B_4\times[0,t_0),\ S\equiv\eta_in_i=\overline{S}\ on\ \partial B_4^c\times[0,t_0), \tag{12}$$

where ∂B_1, ∂B_2, ∂B_3 and ∂B_4 with respective complements $\partial B_1^c,\partial B_2^c,\partial B_3^c$ and ∂B_4^c are subsets of ∂B, so that

$$\partial\overline{B}_1\cup\partial B_1^c=\partial\overline{B}_2\cup\partial B_2^c=\partial\overline{B}_3\cup\partial B_3^c=\partial\overline{B}_4\cup\partial B_4^c=\partial B,$$
$$\partial B_1\cap\partial B_1^c=\partial B_2\cap\partial B_2^c=\partial B_3\cap\partial B_3^c=\partial B_4\cap\partial B_4^c=\varnothing,$$

n_i are the components of the unit outward normal to ∂B, t_0 is some instant that may be infinite, $u_i^0,u_i^1,\varphi_i^0,\varphi_i^1,\theta^0,\sigma^0,\sigma^1,\overline{u}_i,\bar{t}_i,\overline{\varphi}_i,\overline{m}_i,\overline{\sigma},\overline{\theta},\overline{q}$ and $\overline{h}$ are prescribed functions in their domains.

Introducing the constitutive Eq. (7) into Eq. (6) we obtain the following system of equations

$$\rho\ddot{u}_i=\left(A_{ijmn}\varepsilon_{mn}+B_{ijmn}\mu_{mn}+B_{ij}\;P-\beta_{ij}\theta\right)_{,j}+\rho F_i,$$
$$I_{ij}\ddot{\varphi}_j=(B_{mnij}\varepsilon_{mn}+C_{ijmn}\mu_{mn}+C_{ij}\;P-\alpha_{ij}\theta)_{,j}+$$

$$+\varepsilon_{ijk}\left(A_{jkmn}\varepsilon_{mn}+B_{jkmn}\mu_{mn}+B_{jk}\ P-\beta_{jk}\theta\right)+\rho M_i,$$

$$a\ \dot{\theta}=\frac{1}{T_0}\left(K_{ij}\theta_{,j}\right)_{,i}+\frac{\rho}{T_0}h-\beta_{ij}\dot{\varepsilon}_{ij}-\alpha_{ij}\dot{\mu}_{ij}-d\ \dot{P}. \tag{13}$$

$$l\ \dot{P}=d_{ij}\ P_{,ij}+B_{ij}\dot{\varepsilon}_{ij}+C_{ij}\dot{\mu}_{ij}-d\ \dot{\theta}+r.$$

By a solution of the mixed initial boundary value problem of the micropolar thermoelastic diffusion in the cylinder $\Omega_0=B\times[0,t_0]$ we mean an ordered array (u_i, φ_i, θ, P) which satisfies the system of Eq. (13) for all $(x,t)\in\Omega_0$, the boundary conditions (12) and the initial conditions (11).

3. Main result

In the beginning of this section we define the notion of the domain of influence. Next, we will establish a domain of influence inequality, which is a counterpart of the inequality established in Hetnarski and Ignaczak (1999) and which is the basis of demonstration of main result of this study: the domain influence theorem in the context of micropolar thermoelastic diffusion.

We shall use the following assumptions on the material properties

i) $\rho>0, I_{ij}>0, T_0>0, a>0;$

ii) $A_{ijmn}x_{ij}x_{mn}+2B_{ijmn}x_{ij}y_{mn}+C_{ijmn}y_{ij}y_{mn}+2B_{ij}x_{ij}\omega+$

$+2C_{ij}y_{ij}\omega+l\ \omega^2\geq\alpha\left(x_{ij}x_{ij}+y_{ij}y_{ij}+\omega^2\right),$ for all $x_{ij}, y_{ij}, \omega;$

iii) $K_{ij}\eta_i\eta_j\geq c_0\eta_i\eta_i,$ for all η_i with $c_0>0$.

These assumptions are in agreement with the usual restrictions imposed in the mechanics of continua. The assumption *iii*) represent a considerable strenghtening of the consequence (10) of the entropy production inequality.

For a sufficiently small $\varepsilon>0$, let $W_\varepsilon(z)$ be a smooth nondecreasing function, defined as follows

$$W_\varepsilon(z)=\begin{cases}0 & ,z\in(-\infty,\varepsilon]\\ 1 & ,z\in[\varepsilon,\infty).\end{cases}$$

For $0\leq s\leq t$ we define the function $G(x, s)$ by

$$G(x,s)=W_\varepsilon\left(\frac{R-r}{c}+t-s\right) \tag{14}$$

for some fixed positive R and t, where $r=|x-x_0|$, x_0 is an arbitrary fixed point, c is a positive constant to be determined later.

$G(x, s)$ is a smooth function defined on $B\times[0, t]$, vanishing outside the set Σ, where Σ is defined as

$$\Sigma=\bigcup_{s\in[0,t]}S[x_0,R+c(t-s)].$$

Here $S(x_0, R)$ is a sphere of the form

$$S(x_0, R) = \{x \in R^3 : | x - x_0 | < R\}. \tag{15}$$

Let $U(x, s)$ be the function defined by

$$\begin{aligned} U(x,s) = \frac{1}{2}\Big[&\rho\dot{u}_i\dot{u}_i + I_{ij}\dot{\varphi}_{ij}\dot{\varphi}_{ij} + a\,\theta^2 + l\,P^2 + A_{ijmn}\varepsilon_{ij}\varepsilon_{mn} + \\ &+ 2B_{ijmn}\varepsilon_{ij}\mu_{mn} + C_{ijmn}\mu_{ij}\mu_{mn} + 2B_{ij}\,P\varepsilon_{ij} + 2C_{ij}\,P\mu_{ij}\Big](x,s). \end{aligned} \tag{16}$$

We also define the function $K(x, s)$ by

$$K(x,s) = \frac{1}{2}\Big[\rho\dot{u}_i\dot{u}_i + I_{ij}\dot{\varphi}_i\dot{\varphi}_i + a\,\theta^2 + l\,P^2 + \varepsilon_{ij}\varepsilon_{ij} + \mu_{ij}\mu_{ij}\Big](x,s). \tag{17}$$

Taking into account the assumptions *i*), *ii*) and *iii*) from the form (16) and (17) of the functions $U(x, s)$ and $K(x, s)$, respectively, we deduce

$$K(x,s) \le U(x,s). \tag{18}$$

In the next theorem we prove the domain of influence inequality which is a necessary step to prove the main result.

Theorem 1. Let (u_i, φ_i, θ, P) be a solution to the system of Eq. (13) with the initial conditions (11) and the boundary conditions (12). Then for any R>0, t>0 and $x_0 \in B$, we have that

$$\begin{aligned} \int_{D[x_0,\,R]} U(x,t)\,dV + \frac{1}{T_0}\int_0^t\int_{D[x_0,\,R+c(t-s)]} K_{ij}\,\theta_{,i}\,\theta_{,j}\,dV \le \int_{D[x_0,\,R+ct]} U(x,0)\,dV + \\ + \int_0^t\int_{D[x_0,\,R+c(t-s)]}\rho\left(F_i\,\dot{u}_i + M_i\,\dot{\varphi}_i + \frac{1}{T_0}h\,\theta + r\,P\right)dVds + \\ + \int_0^t\int_{\partial D[x_0,\,R+c(t-s)]}\left[\bar{t}_i\dot{u}_i + \bar{m}_i\dot{\varphi}_i + \frac{1}{T_0}\bar{q}\,\theta + \bar{S}\,P\right]dSds, \end{aligned} \tag{19}$$

where

$$\begin{aligned} D(x_0,R) &= \{x \in B : | x - x_0 | < R\}, \\ \partial D(x_0,R) &= \{x \in \partial B : | x - x_0 | < R\}. \end{aligned}$$

Proof. Multiplying the Eq. $(13)_1$ by $G\dot{u}_i$ and making use of the constitutive Eq. $(7)_1$, we obtain

$$\begin{aligned} \frac{1}{2}G\frac{d}{dt}\left(\rho\dot{u}_i\dot{u}_i\right) = \rho G F_i\dot{u}_i + \left(G t_{ij}\dot{u}_i\right)_{,j} - G_{,j}t_{ij}\dot{u}_i - \\ - G\left(A_{ijmn}\varepsilon_{mn} + B_{ijmn}\mu_{mn} + B_{ij}\,P - \beta_{ij}\theta\right)\dot{\varepsilon}_{ij}. \end{aligned} \tag{20}$$

Multiplying the Eq. $(13)_2$ by $G\dot{\varphi}_i$ and making use of the constitutive Eqs. $(7)_1$ and $(7)_2$, we obtain

$$\begin{aligned} \frac{1}{2}G\frac{d}{dt}\left(I_{ij}\dot{\varphi}_i\dot{\varphi}_j\right) = \rho G M_i\dot{\varphi}_i + (G m_{ij}\dot{\varphi}_i)_{,j} - G_{,j}m_{ij}\dot{\varphi}_i - \\ - G\left(B_{mnij}\varepsilon_{mn} + C_{ijmn}\mu_{mn} + C_{ij}\,P - \alpha_{ij}\theta\right)\dot{\mu}_{ij} + \\ + \varepsilon_{ijk}\left(A_{jkmn}\varepsilon_{mn} + B_{jkmn}\mu_{mn} + B_{jk}\,P - \beta_{jk}\,\theta\right)\dot{\varphi}_i. \end{aligned} \tag{21}$$

Multiplying the Eq. $(13)_3$ by $G\theta$ and making use of the constitutive Eqs. $(7)_3$ and $(7)_4$, we are lead to

$$\frac{1}{2}G\frac{d}{dt}\left(a\theta^2\right)=\frac{1}{T_0}G\,h\,\theta+\frac{1}{\rho T_0}\left[\left(G\,\theta\,q_i\right)_{,i}-G_{,i}\,\theta\,q_i\right]-$$

$$-\frac{1}{\rho T_0}G\,K_{ij}\,\theta_{,i}\,\theta_{,j}-G\left(\beta_{ij}\,\theta\,\dot{\varepsilon}_{ij}+\alpha_{ij}\,\theta\,\dot{\mu}_{ij}+d\,\theta\,\dot{P}\right). \tag{22}$$

At last, multiplying the Eq. $(13)_4$ by GP and making use of the constitutive Eqs. $(7)_5$ and $(7)_6$, we obtain

$$\begin{aligned}\frac{1}{2}G\frac{d}{dt}\left(l\,P^2\right)&=G\,r\,P+G\left(\eta_i\,P\right)_{,i}-G_{,i}\,\eta_i\,P-\\&-G\left(B_{ij}\varepsilon_{ij}\,\dot{P}+C_{ij}\mu_{ij}\,\dot{P}-d\,\theta\,\dot{P}\right).\end{aligned} \tag{23}$$

Adding Eqs. (20), (21), (22) and (23) together, we are lead to

$$\begin{aligned}&\frac{1}{2}G\frac{d}{dt}\left(\rho\dot{u}_i\dot{u}_i+I_{ij}\dot{\varphi}_i\dot{\varphi}_j+a\,\theta^2+l\,p^2\right)=\rho GF_i\dot{u}_i+\rho GM_i\dot{\varphi}_i+\\&+G\,r\,P+\frac{\rho}{T_0}G\,h\,\theta+G\left(t_{ij}\dot{u}_i+m_{ij}\dot{\varphi}_i+\frac{1}{\rho T_0}\,\theta\,q_j+\eta_j\,P\right)_{,j}-\\&-G\left[A_{ijmn}\varepsilon_{mn}\dot{\varepsilon}_{ij}+B_{ijmn}\left(\varepsilon_{mn}\dot{\mu}_{ij}+\dot{\varepsilon}_{mn}\mu_{ij}\right)+C_{ijmn}\mu_{mn}\dot{\mu}_{ij}+\right.\\&\left.+B_{ij}\left(\dot{\varepsilon}_{ij}\,P+\varepsilon_{ij}\,\dot{P}\right)+C_{ij}\left(\dot{\mu}_{ij}\,P+\mu_{ij}\,\dot{P}\right)\right]-G_{,j}t_{ij}\dot{u}_i-\\&-G_{,j}m_{ij}\dot{\varphi}_i-G_{,i}\eta_i\,P-\frac{1}{\rho T_0}G_{,i}\,q_i\,\theta-\frac{1}{\rho T_0}GK_{ij}\,\theta_{,i}\,\theta_{,j}.\end{aligned} \tag{24}$$

The relation (24) may be restated as follows

$$\begin{aligned}&\frac{1}{2}G\frac{d}{dt}\left(\rho\dot{u}_i\dot{u}_i+I_{ij}\dot{\varphi}_i\dot{\varphi}_j+a\,\theta^2+l\,p^2+A_{ijmn}\varepsilon_{mn}\varepsilon_{ij}+\right.\\&\left.+2B_{ijmn}\mu_{mn}\varepsilon_{ij}+C_{ijmn}\mu_{mn}\mu_{ij}+2B_{ij}\varepsilon_{ij}\,P+2C_{ij}\mu_{ij}\,P\right)=\\&=\rho G\left(F_i\dot{u}_i+M_i\dot{\varphi}_i+\frac{1}{T_0}\,h\,\theta+r\,P\right)+\\&+G\left(t_{ij}\dot{u}_i+m_{ij}\dot{\varphi}_i+\frac{1}{\rho T_0}\,\theta\,q_j+\eta_j\,P\right)_{,j}-\\&-G_{,j}t_{ij}\dot{u}_i-G_{,j}m_{ij}\dot{\varphi}_i-G_{,i}\frac{1}{\rho T_0}\,\theta\,q_i-G_{,i}\eta_i\,P-\frac{1}{\rho T_0}\,K_{ij}\,\theta_{,i}\,\theta_{,j}.\end{aligned} \tag{25}$$

Taking into account the definition (16) of function $H(x,\ s)$ we can rewrite relation (25) in the form

$$\frac{1}{2}G\dot{U}+\frac{1}{\rho T_0}\,K_{ij}\,\theta_{,i}\,\theta_{,j}=\rho G\left(F_i\dot{u}_i+M_i\dot{\varphi}_i+\frac{1}{T_0}\,h\,\theta+r\,P\right)+$$

$$
\begin{aligned}
&+G\left(t_{ij}\dot{u}_i + m_{ij}\dot{\varphi}_i + \frac{1}{\rho T_0}\,\theta\, q_j + \eta_j\ P\right)_{,\,j} - \\
&-G_{,\,j}\left(t_{ij}\dot{u}_i + m_{ij}\dot{\varphi}_i + \frac{1}{\rho T_0}\,\theta\, q_i + \eta_j\ P\right) - \frac{1}{\rho T_0} K_{ij}\ \theta_{,\,i}\ \theta_{,\,j}.
\end{aligned}
\tag{26}
$$

Integrating both sides of Eq. (26) over $B\times[0,\ t]$ and making use of the divergence theorem and the boundary conditions (12), we deduce that

$$
\begin{aligned}
&\int_B GU(x,t)dV + \frac{1}{\rho T_0}\int_0^t\int_B G\ K_{ij}\theta_{,\,i}\theta_{,\,j}dVds = \int_B G\ U(x,0)\,dV + \\
&+\int_0^t\int_{\partial B} G\left(\bar{t}_i\dot{u}_i + \overline{m}_i\dot{\varphi}_i + \frac{1}{\rho T_0}\,\overline{q}\ \theta + \bar{S}\ P\right)dVds + \\
&+\int_0^t\int_B \rho G\left(F_i\dot{u}_i + M_i\dot{\varphi}_i + \frac{1}{T_0}\,h\ \theta + r\ P\right)dVds + \\
&+\int_0^t\int_B \dot{G}\ U(x,s)dVds - \int_0^t\int_B G_{,\,j}\left(t_{ij}\dot{u}_i + m_{ij}\dot{\varphi}_i + \frac{1}{\rho T_0}\,q_j\ \theta + \eta_j\ P\right)dVds.
\end{aligned}
\tag{27}
$$

Taking into account the definition (16) of the function G, we find that

$$
\begin{aligned}
&\left|-G_{,\,j}t_{ij}\dot{u}_i - G_{,\,j}m_{ij}\dot{\varphi}_i - \frac{1}{\rho T_0}G_{,\,i}q_i\theta - G_{,\,i}\eta_i\ P\right| = \\
&= \left|\frac{1}{c}W_\varepsilon^{'}\frac{x_j}{r}\,t_{ij}\dot{u}_i + \frac{1}{c}W_\varepsilon^{'}\frac{x_j}{r}\,m_{ij}\dot{\varphi}_i + \frac{1}{c}\frac{1}{\rho T_0}W_\varepsilon^{'}\frac{x_i}{r}\,q_i\theta + \frac{1}{c}W_\varepsilon^{'}\frac{x_i}{r}\,\eta_i\ P\right| = \\
&= \left|\frac{1}{c}W_\varepsilon^{'}\frac{1}{r}\Big[\big(A_{ijmn}\varepsilon_{mn}x_j + B_{ijmn}\mu_{mn}x_j + B_{ij}\ P\ x_j - \beta_{ij}\ \theta\ x_j\big)\dot{u}_i + \right. \\
&+\big(\text{-}B_{mnij}\varepsilon_{mn}\ x_j + C_{ijmn}\mu_{mn}\ x_j + C_{ij}\ P\ x_j - \beta_{ij}\ \theta\ x_j\big)\dot{\varphi}_i + \\
&\left. + d_{ij}\ \dot{x}_{,\,j}\ P_{,\,i} + \frac{1}{\rho T_0}K_{ij}\ \theta_{,\,j}\ \theta\ x_j\Big]\right|,
\end{aligned}
\tag{28}
$$

where

$$
W_\varepsilon^{'} = \frac{dW_\varepsilon}{dr}.
$$

We now make use of arithmetic-geometric mean inequality

$$
ab \le \frac{1}{2}\left(\frac{a^2}{p^2} + b^2 p^2\right),
\tag{29}
$$

where p is an arbitrarly positive parameter.

If we use this inequality to the last terms of relation (28) and by choosing suitable parameters *p* we can find *c* such that

$$\left| -G_{,j} t_{ij} \dot{u}_i - G_{,j} m_{ij} \dot{\varphi}_i - \frac{1}{T_0} G_{,i}\ q_i\ \theta - G_{,i}\ \eta_i\ P \right| \le W_\varepsilon^{'}\ K(x,s), \tag{30}$$

and that

$$\int_0^t \int_B \dot{G} U(x,s) dV ds - \int_0^t \int_B \left(G_{,j} t_{ij} \dot{u}_i + G_{,j} m_{ij} \dot{\varphi}_i + \frac{1}{T_0} G_{,i}\ q_i \theta + G_{,i}\ \eta_i\ P \right) dV ds \le$$

$$\le \int_0^t \int_B W_\varepsilon^{'}(x,s) \left[K(x,s) - U(x,s) \right] dV ds \le 0. \tag{31}$$

From Eq. (27), taking into account the inequality (31), we deduce that

$$\begin{aligned} &\int_B GU(x,t) dV + \frac{1}{T_0} \int_0^t \int_B G\ K_{ij} \theta_{,i} \theta_{,j} dV ds \le \int_B GU(x,0) dV + \\ &+ \int_0^t \int_B \rho G \left(F_i \dot{u}_i + M_i \dot{\varphi}_i + \frac{1}{\rho^2 T_0}\ h\ \theta + r\ P \right) dV ds + \\ &+ \int_0^t \int_{\partial B} G \left(\bar{t}_i \dot{u}_i + \overline{m}_i \dot{\varphi}_i + \frac{1}{\rho T_0}\ \overline{q}\ \theta + \overline{S}\ P \right) dS ds. \end{aligned} \tag{32}$$

Letting $\varepsilon \to 0$ into relation (32), *G* tends boundedly to the characteristic function of Σ and we get the inequality (19) and the proof of Theorem 1 is complete.

Based on the above estimations, we can now prove the main result of our study: the domain of influence theorem.

Let *B*(*t*) be the set of points $x \in \overline{B}$ such that:

(1) $x \in B \Rightarrow u_i^0 \ne 0\ or\ u_i^1 \ne 0\ or\ \varphi_i^0 \ne 0\ or\ \varphi_i^1 \ne 0\ or\ \theta^0 \ne 0\ or\ \gamma^0 \ne 0\ or$
$\exists \tau \in [0,t]\ such\ that\ F_i(x,\tau) \ne 0\ or\ M_i(x,\tau) \ne 0\ or\ h(x,\tau) \ne 0\ or\ r(x,\tau) \ne 0;$

(2) $x \in \partial B_1 \Rightarrow \exists \tau \in [0,t]\ such\ that\ \bar{u}_i(x,\tau) \ne 0;$

(3) $x \in \partial B_1^c \Rightarrow \exists \tau \in [0,t]\ such\ that\ \bar{t}_i(x,\tau) \ne 0;$

(4) $x \in \partial B_2 \Rightarrow \exists \tau \in [0,t]\ such\ that\ \overline{\varphi}_i(x,\tau) \ne 0;$

(5) $x \in \partial B_2^c \Rightarrow \exists \tau \in [0,t]\ such\ that\ \overline{m}_i(x,\tau) \ne 0;$

(6) $x \in \partial B_3 \Rightarrow \exists \tau \in [0,t]\ such\ that\ \overline{\theta}(x,\tau) \ne 0;$

(7) $x \in \partial B_3^c \Rightarrow \exists \tau \in [0,t]\ such\ that\ \overline{q}(x,\tau) \ne 0;$

(8) $x \in \partial B_2 \Rightarrow \exists \tau \in [0,t]\ such\ that\ \overline{P}(x,\tau) \ne 0;$

(9) $x \in \partial B_2^c \Rightarrow \exists \tau \in [0,t]\ such\ that\ \overline{S}(x,\tau) \ne 0.$

The domain of influence of the data at instant *t* is defined as

$$B_t = \{ x_0 \in \overline{B} : B(t) \cap \overline{S}(x_0, ct) \ne \Phi \}, \tag{33}$$

where Φ is the empty set.

In the next theorem ve prove the existence of the domain of influence in the context of micropolar thermoelastic diffusion.

Theorem 2. Let (u_i, φ_i, θ, P) be a solution to the system of Eq. (13) with the initial conditions (11) and the boundary conditions (12). Then we have

$$u_i = 0,\ \varphi_i = 0,\ \theta = 0,\ P = 0,\ on\ \{\overline{B} \setminus B_t\} \times [0,t].$$

Proof. For any $x_0 \in \overline{B} \setminus B_t$ and $\tau \in [0,t]$, by using the inequality (19) with $t=\tau$ and $R=c(t-\tau)$, we obtain

$$\begin{aligned} &\int_{D\left[x_0,\, c(t-\tau)\right]} U(x,\tau)dV + \frac{1}{T_0}\int_0^\tau \int_{D[x_0,\, c(t-s)]} K_{ij}\,\theta_{,i}\,\theta_{,j} dVds \le \\ &\le \int_{D\left[x_0,\, ct\right)]} U(x,0)dV + \int_0^\tau \int_{D\left[x_0,\, c(t-s)\right]} \rho\left(F_i\dot{u}_i + M_i\dot{\varphi}_i + \frac{1}{T_0}\, h\,\theta + r\,P \right) dVds + \\ &+ \int_0^\tau \int_{\partial D\left[x_0,\, c(t-s)\right]} \rho\left(\bar{t}_i\dot{u}_i + \overline{m}_i\dot{\varphi}_i + \frac{1}{T_0}\,\overline{q}\,\theta + \overline{S}\,P \right) dSds. \end{aligned} \tag{34}$$

Since $x_0 \in \overline{B} \setminus B_t$, we have $x \in D(x_0, ct) \Rightarrow x \notin B(t)$ and hence

$$\int_{D\left[x_0,\, ct\right]} U(x,0)dV = 0. \tag{35}$$

Moreover, since $D[x_0, c(t-s)] \subseteq D(x_0, ct)$, we have

$$\int_0^\tau \int_{D\left[x_0,\, c(t-s)\right]} \rho\left(F_i\dot{u}_i + M_i\dot{\varphi}_i + \frac{1}{T_0}\, h\,\theta + r\,P \right) dVds = 0, \tag{36}$$

and

$$\int_0^\tau \int_{D\left[x_0,\, c(t-s)\right]} \left(\bar{t}_i\dot{u}_i + \overline{m}_i\dot{\varphi}_i + \frac{1}{T_0}\,\overline{q}\,\theta + \overline{S}\,P \right) dVds = 0. \tag{37}$$

Taking into account the assumption *iii*) and the relations (34) - (38) we obtain

$$\int_{D\left[x_0,\, c(t-\tau)\right]} U(x,\tau)dV \le 0. \tag{39}$$

Based on inequality (18), we deduce that

$$\int_{D\left[x_0,\, c(t-\tau)\right]} K(x,\tau)dV \le 0, \tag{40}$$

By using the definition (17) of the function $K(x, t)$ we are lead to

$$\dot{u}_i(x_0,\tau) = 0,\ \dot{u}_i(x_0,\tau) = 0,\ \theta(x_0,\tau) = 0,\ P(x_0,\tau) = 0,$$

for any $(x_0,\tau) \in \{\overline{B} \setminus B_t\} \times [0,t]$.

Finally, since $u_i(x_0,0)=0, \varphi_i(x_0,0)=0$ for any $x_0 \in \overline{B} \setminus B_t$, we deduce

$$u_i(x_0,\tau)=0, \varphi_i(x_0,\tau)=0, \theta(x_0,\tau)=0, \sigma(x_0,\tau)=0,$$

for any $(x_0,\tau) \in \{\overline{B} \setminus B_t\} \times [0,t]$ and the proof of Theorem 2 is complete.

4. Conclusions

In fact, the main result of the paper is an extension of known Saint-Venant's principle from classical Elasticity. We have shown in the paper that this principle remains valid even if we exceeded the framework of classical mechanics.

The essence of the principle remains the same even if we have taken into consideration and the effect of thermal treatment, the effect of micropolar structure and the effect of diffusion.

Acknowledgments

The authors are very grateful to the reviewers for useful observations that led to improvements to this paper.

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