Structural Engineering and Mechanics, *Vol. 51, No. 3 (2014) 377-402*
DOI: http://dx.doi.org/10.12989/sem.2014.51.3.377

Minimum dynamic response of cantilever beams supported by optimal elastic springs

Ersin Aydin*

Faculty of Engineering, Department of Civil Engineering, Nigde University, Nigde, Turkey

(Received April 17, 2012, Revised April 1, 2014, Accepted April 4, 2014)

Abstract. In this study, optimal distribution of springs which supports a cantilever beam is investigated to minimize two objective functions defined. The optimal size and location of the springs are ascertained to minimize the tip deflection of the cantilever beam. Afterwards, the optimization problem of springs is set up to minimize the tip absolute acceleration of the beam. The Fourier Transform is applied on the equation of motion and the response of the structure is defined in terms of transfer functions. By using any structural mode, the proposed method is applied to find optimal stiffness and location of springs which supports a cantilever beam. The stiffness coefficients of springs are chosen as the design variables. There is an active constraint on the sum of the stiffness coefficients and there are passive constraints on the upper and lower bounds of the stiffness coefficients. Optimality criteria are derived by using the Lagrange Multipliers. Gradient information required for solution of the optimization problem is analytically derived. Optimal designs obtained are compared with the uniform design in terms of frequency responses and time response. Numerical results show that the proposed method is considerably effective to determine optimal stiffness coefficients and locations of the springs.

Keywords: optimal stiffness; beam vibrations; transfer functions; optimal support location; support stiffness

1. Introduction

Beams are widely used in different engineering applications. They are basic structural components. Dynamic beam problems have been investigated by researchers. In practice, the design of structural supports is equally of great importance. This may arise in most structural engineering designs, especially in building constructions, work piece machining fixture, welding or rivet joints of marine and aircraft structures. It is well known that support conditions play a crucial role in structural analysis. A small amount of adjustments in support positions can influence the structural performance significantly and should be designed carefully in favor of the structural performance. Supports are not only expected to hold a structure firmly, but can also be redesigned to improve the structural performance.

In vibration optimization problems, Eigenfrequencies are usually maximized in optimization since resonance phenomena in a mechanical structure must be avoided, and maximizing

*Corresponding author, Assistant Professor, E-mail: eaydin@nigde.edu.tr

http://www.techno-press.org/?journal=sem&subpage=8 ISSN: 1225-4568 (Print), 1598-6217 (Online)

Eigenfrequencies can provide a high probability of dynamic stability. However, vibrating mechanical structures can provide additional useful dynamic functions or performance, if the desired Eigen frequencies and Eigen mode shapes in the structures can be implemented. Support position optimization may arise in almost all structural design projects, especially in building constructions, workpiece machining, printed circuit boards, marine and aircraft structures. However, support position optimization for decreasing structural deflections has not been fully investigated yet. It still remains the most challenging task for researchers, because the maximum deflection of a structure, as the objective function of the problem in this paper, is highly nonlinear and non-glossy with respect to support positions. Design variables are generally chosen such as structural topology, geometry and size of structural elements in optimization of dynamic structural systems. The support location and stiffness can also be taken as design variables to minimize the proposed structural dynamic response. The variation of support location and stiffness can cause an important influence (positive or negative) on the dynamic response of the structure. Moreover, the support location and the stiffness should be taken as factors to be considered for structural design.

There are some investigations about optimal design for beam structures including position and stiffness of supports (Mroz and Rozvany 1975, Rozvany 1975, Prager and Rozvany 1975, Szelag and Mroz 1978, Mroz and Lekszycki 1982, Garstecki and Mroz 1987, Dems and Turant 1997, Bojczuk and Mroz 1998, Mroz and Haftka’s 1994, Akessson and Olhof 1988, Wang 1993, Wang and Chen 1996, Wang 2003). Bojczuk and Mroz (1998) had presented a method for trusses. Chuang and Hou (1992) developed various sensitivity equations for eigenvalue sensitivity analysis of planar frames with variable joint and support locations. Liu *et al.* (1996) presented a method to derive the equations of eigenvalue rate with respect to the support location using the generalized variational principles of the Rayleigh quotient. Son and Kwak (1993) developed a sensitivity formula of eigenvalues with respect to the change of boundary conditions by using material derivative concept based on variational formulation. Won and Park (1998) presented a procedure to find the loci of optimal support positions for a structure to maximize its fundamental eigenvalue by increasing the support stiffness. A theoretical formulation was presented by Sinha and Friswell (2001) for estimating support location. A different approach was proposed by Imam and Shihri (1996) to determine the optimum topological locations of supports or columns in a structure. Marcelin (2001) used genetic algorithms to find support positions in machining of mechanical parts. Olhof and Akesson (1991) studied support optimization of a column to maximize the buckling load. Buhl (2001) demonstrated a method for support distribution using a continuum type topology optimization. Olhof and Taylor (1998) studied optimal design of non-uniform, elastic, continuous columns with unspecified number of available interior supports. Jihong and Weihong (2006) studied to maximize the natural frequency of structures and presented the support layout design that corresponds to optimization of boundary conditions. The position optimization of simple support was studied by Wang *et al*. (2004) to maximize the fundamental frequency of a beam or plate structure. Albaracin *et al*. (2004) investigated the problem of a uniform beam with intermediate constraints and ends elastically restrained against rotation and translation. Friswell and Wang (2007) developed a procedure to calculate the minimum stiffness and the optimal position of one or two elastic supports lying along the free edge opposite to the restrained boundary edge of the plate. In order to improve the structural performance, an optimization scheme of minimization of maximal absolute bending moment in a planar frame was presented by Wang (2006) to find optimal design of the support position. In another study, Wang (2004) investigated the design sensitivity analysis for the deflection of a beam or plate structure with respect to the position of a simple support using the discrete method. The minimum stiffness of a simple support that increase a natural frequency of a beam to its upper limit was developed by Wang *et al*. (2006) for various boundary conditions. Kong (2009) analysed the vibration of plates with various boundary and internal support conditions and proposed a computational technique to determine the optimal location and stiffness of discrete elastic supports in maximizing the fundamental frequency of both isotropic plates and composite plates. Wang *et al*. (2010) applied the Rayleigh-Ritz method to analyse the optimal configuration of additional supports in plates. The magnitude of mass and stiffness of a linear spring that supports a beam element are well known key parameters affecting the free vibration characteristics of a beam in the existing literature. In addition to this respect, the offset of each linear spring, which supports a beam, is also the predominant parameter (Lin 2010). Zhu and Zhang (2010) proposed an integrated layout optimization method to deal with the simultaneous design of structure and support layout. Fayyah and Razak (2012) studied the effect of deterioration in the elastic bearing support stiffness on the dynamic properties of structural elements, in order to determine the sensitivity of dynamic properties as a tool for monitoring the condition of supports.

In this study, a new method is proposed to find the optimal location and size of the stiffness coefficient of the supporting springs. The transfer function amplitudes of the tip displacement evaluated at one of the undamped natural frequency of a cantilever beam is chosen as the first objective function; and afterwards, the second objective function is taken as the transfer function amplitude of the absolute acceleration of the tip of cantilever beam evaluated at one of the undamped natural frequency. The objective functions defined are minimized being subjected to a constraint on the sum of the stiffness coefficients of the springs that support the cantilever beam. The dynamic response of the beam is defined by the transfer functions based on one of the undamped natural frequencies of the cantilever beam. Optimality conditions are derived by using the Lagrange Multipliers. In order to find the optimal designs for two objectives under constraints, the first and second order sensitivity formulations are derived, and the steepest direction search algorithm, which was proposed by Takewaki (1998) to determine the optimal damper position in a cantilever beam, is used to find optimal spring stiffness and locations in case of tip deflection minimization in the first mode. Moreover, two simple algorithms, which are used as the first order sensitivities, are proposed to determine optimal spring distribution. A simple algorithm is presented in Section 3.5 to find the optimal spring coefficient in case of tip deflection evaluated at the second and third natural circular frequencies of the beam. A new simple algorithm for finding optimal spring allocation is also proposed to minimize the tip absolute acceleration. Numerical results show that the proposed algorithms can be effective to find optimal placement of elastic springs supporting a cantilever beam.

2. Formulation of problem

Consider a cantilever Timoshenko beam of length L, solid square cross section A and bending stiffness EI; and elastic springs supporting the beam. The beam is subjected to base acceleration. There is a lumped mass at tip of the beam shown in Fig. 1 as well. In order to model the cantilever beam and the supporting springs, the cantilever beam is divided into n FE elements of equal length, and the potential locations of the springs are defined at each node. The initial node is defined from the left except for the fixed end. The spring supports act only in the vertical direction. In the specified locations of the beam, stiffness coefficients $\mathbf{k}=\{k_j\}$ of supporting springs indicate the design variables; and n presents the number of design variables.

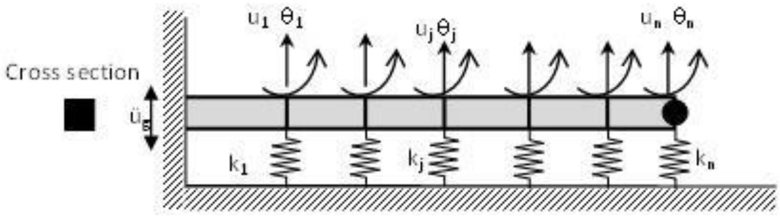


Fig. 1 Cantilever beam supported by elastic springs

Let u_j and θ_j be assigned as the dynamic transverse displacement and the angle of rotation at the j^{th} node of the beam. As shown in Fig. 1, while the dynamic displacement vector can be presented as $\mathbf{u}=\{u_1, \theta_1 \dots u_j, \theta_j \dots u_n, \theta_n\}^T$, **M** denotes the mass matrix and **C** denotes the structural damping matrix that are determined as either mass proportional damping or stiffness proportional damping.

The element stiffness matrix for Timoshenko beams is given as (Prezemieniecki 1968)

$$\boldsymbol{k} = \frac{EI}{L^3(1+\phi)}\begin{bmatrix} 12 & & & Sym \\ 6L & L^2(4+\phi) & & \\ -12 & -6L & 12 & \\ 6L & L^2(2-\phi) & -6L & L^2(4+\phi) \end{bmatrix} \tag{1}$$

where I denote the second moment of area and $\phi = \frac{12EI}{G\kappa AL^2}$, the element mass matrix is written as follows

$$\boldsymbol{m} = \frac{\rho AL}{(1+\phi)^2}\begin{bmatrix} m_1 & & & Sym \\ m_2 & m_5 & & \\ m_3 & -m_4 & m_1 & \\ m_4 & m_6 & -m_2 & m_5 \end{bmatrix} + \frac{\rho AL}{(1+\phi)^2}\left(\frac{r}{L}\right)^2\begin{bmatrix} m_7 & & & Sym \\ m_8 & m_9 & & \\ -m_7 & -m_8 & m_7 & \\ m_8 & m_{10} & -m_8 & m_9 \end{bmatrix} \tag{2}$$

Let $r = \sqrt{\frac{I}{A}}$ denote the radius of gyration of cross-section and the elements of element mass matrix be given as

$$m_1 = \frac{13}{35} + \frac{7\phi}{10} + \frac{\phi^2}{3} \qquad m_2 = \left(\frac{11}{210} + \frac{11\phi}{120} + \frac{\phi^2}{24}\right)L \qquad m_3 = \frac{9}{70} + \frac{3\phi}{10} + \frac{\phi^2}{6}$$

$$m_4 = -\left(\frac{13}{420} + \frac{3\phi}{40} + \frac{\phi^2}{24}\right)L \quad m_5 = \left(\frac{1}{105} + \frac{\phi}{60} + \frac{\phi^2}{120}\right)L^2 \quad m_6 = -\left(\frac{1}{140} + \frac{\phi}{60} + \frac{\phi^2}{120}\right)L^2$$

$$m_7 = \frac{6}{5} \quad m_8 = \left(\frac{1}{10} - \frac{\phi}{2}\right)L \quad m_9 = \left(\frac{2}{15} + \frac{\phi}{6} + \frac{\phi^2}{3}\right)L^2 \quad m_{10} = \left(-\frac{1}{30} - \frac{\phi}{6} + \frac{\phi^2}{6}\right)L^2 \tag{3}$$

In case of without elastic springs, the equation of motion is given as

$$\boldsymbol{M}\ddot{\boldsymbol{u}}(t) + \boldsymbol{C}\dot{\boldsymbol{u}}(t) + \boldsymbol{K}\boldsymbol{u}(t) = -\boldsymbol{M}\boldsymbol{r}\ddot{u}_g(t) \tag{4}$$

where $\boldsymbol{u}(t)$, $\dot{\boldsymbol{u}}(t)$ and $\ddot{\boldsymbol{u}}(t)$ denote displacement, velocity and acceleration vector of the beam model, $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ denote mass matrix, damping matrix and stiffness matrix of structural model. Let $r = \{1 \quad 0 \quad \dots \quad 1 \quad 0\}^T$ denote the influence coefficient vector and $\ddot{u}_g(t)$ be a fixed base acceleration. Let ω denote the circular frequency of the base excitation.

The Fourier transform of Eq. (4) can be presented as

$$(\boldsymbol{K} + i\omega\boldsymbol{C} - \omega^2\boldsymbol{M})\boldsymbol{U}(\omega) = -\boldsymbol{M}\boldsymbol{r}\ddot{U}_g(\omega) \tag{5}$$

where $\boldsymbol{U}(\omega)$ and $\ddot{U}_g(\omega)$ denote the Fourier transform of $\boldsymbol{u}(t)$ and $\ddot{u}_g(t)$; and $i = \sqrt{-1}$ is the imaginary unit.

If the beam is supported by the elastic springs as shown in Fig. 1, Eq. (5) can be rearranged as

$$\left((\boldsymbol{K} + \boldsymbol{K_{sp}}) + i\omega\boldsymbol{C} - \omega^2\boldsymbol{M}\right)\boldsymbol{U_{sp}}(\omega) = -\boldsymbol{M}\boldsymbol{r}\ddot{U}_g(\omega) \tag{6}$$

where $\boldsymbol{K_{sp}}$ is the stiffness matrix that belongs to the supporting springs and covers the design parameters $\mathbf{k}=\{k_j\}$. It is given as

$$\boldsymbol{K_{sp}} = \begin{bmatrix} k_1 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & k_j & 0 & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 0 & \dots & k_n & 0 \\ 0 & 0 & \dots & 0 & 0 & \dots & 0 & 0 \end{bmatrix}_{2nx2n} \tag{7}$$

where $\boldsymbol{U_{sp}}(\omega)$ represents the Fourier Transform of the displacement vector with added springs. A new quantity defined by Takewaki (1998) is given as

$$\widehat{\boldsymbol{U}}(\omega) = \frac{\boldsymbol{U_{sp}}(\omega)}{\ddot{U}_g(\omega)} \tag{8}$$

where ω is equal to $\omega_{s,}$ the support motion will have a harmonic excitation frequency that equals to s^{th} natural frequency of the beam. Eq. (6) can be rewritten by using Eq. (8)

$$\boldsymbol{A}\widehat{\boldsymbol{U}} = -\boldsymbol{M}\boldsymbol{r} \tag{9}$$

where

$$\boldsymbol{A} = (\boldsymbol{K}+\boldsymbol{K_{sp}}) + i\omega_s\boldsymbol{C} - \omega_s^2\boldsymbol{M} \tag{10}$$

Let $\widehat{\boldsymbol{U}}(\omega_s)$ denote the transfer function of displacement which is independent from excitation. Eq. (9) may be rearranged as follows

$$\widehat{\boldsymbol{U}} = -\boldsymbol{A}^{-1}\boldsymbol{M}\boldsymbol{r} \tag{11}$$

It can be seen that $\boldsymbol{M}$, $\boldsymbol{C}$ and $\boldsymbol{K}$ are prescribed, and ω_s may be calculated when a total stiffness matrix ($\boldsymbol{K}$+$\boldsymbol{K_{sp}}$) is given. At the optimization stages presented in this paper, while $\boldsymbol{K_{sp}}$ changes in each step the new value of ω_s is recalculated accordingly.

The formulations so far were derived without $\boldsymbol{K_{sp}}$ by applying Takewaki (1998) to find the optimal damping coefficients of dampers supporting a cantilever beam. Recently, these formulations were also used in some optimization studies (Aydin *et al.* 2007, Aydin and Boduroglu 2008).

The transfer function vector of the absolute acceleration at the s^{th} natural frequency of a shear building structure was derived by Cimellaro (2007) to find the placement of the optimal visco-elastic dampers. The transfer function vector of absolute acceleration can be written for a cantilever beam supported by elastic springs as follows

$$\hat{\ddot{\boldsymbol{U}}} = -\boldsymbol{M}^{-1}\left(\boldsymbol{K} + \boldsymbol{K}_{sp} + \boldsymbol{i}\omega_s\boldsymbol{C}\right)\hat{\boldsymbol{U}} \tag{12}$$

3. Problem of optimal spring location for a cantilever beam

3.1 Definition of the optimal support location problem

A general structural optimization problem is defined to minimize/maximize one or multi objective functions that show the structural performance and cost under constraints on design variables and structural response. There are many possible objective functions for structural optimization, such as weight, stiffness, displacement, stress, vibration frequency, buckling load, cost, acceleration, force and specified damage index.

In this section, the objective functions required for the supporting design of vibrating beam are formulated. The key idea is that these objective functions, when implemented with the proposed method, minimize the transfer function of both tip deflection and tip absolute acceleration in a cantilever beam.

The optimization problem can be stated mathematically as

$$\text{Minimum} \qquad f(\boldsymbol{k}) = f(k_1, \ldots, k_N) \tag{13}$$

subjected to the inequality constraints both on the upper and lower bounds of stiffness coefficients of each spring as follows

$$0 \le k_j \le \bar{k}_j \qquad \text{(j=1,2,…,n)} \tag{14}$$

where $\bar{k}_j$ is the upper bound for stiffness coefficient of the spring in j^{th} node, and an equality constraint on the sum of stiffness coefficients is written as

$$\sum_{j=1}^{n} k_j = \bar{K} \tag{15}$$

where $\bar{K}$ is the total stiffness coefficients of supporting springs.

To find the optimal support position by minimizing the dynamic response in a cantilever beam as mentioned before, the objective functions are chosen as tip displacement (Takewaki 1998) and tip absolute acceleration (Cimellaro 2007) in this study.

The first optimization problem based on the minimization of the transfer function amplitude of the tip displacement for cantilever beam is expressed as

$$\min \qquad f_1 = \left|\hat{U}_t(k_j)\right| \qquad \text{(j=1,2,..., n)} \tag{16}$$

where $\left|\hat{U}_t(k_j)\right|$ corresponds to the transfer function amplitude of the tip displacement evaluated at any s^{th} natural frequency which is shown in displacement vector $\hat{\boldsymbol{U}}$.

The second optimization problem based on the minimization of the transfer function amplitude of the tip absolute acceleration at any s^{th} natural frequency of the cantilever beam is described as follows

$$\min \qquad f_2 = \left|\hat{\ddot{U}}_t(k_j)\right| \qquad \text{(j=1,2,..., n)} \tag{17}$$

where $\left|\hat{\ddot{U}}_t(k_j)\right|$ corresponds to the transfer function amplitude of the tip absolute acceleration evaluated at the s^{th} undamped natural frequency of the cantilever beam which is shown in vector $\hat{\ddot{\boldsymbol{U}}}$. This objective function was proposed to calculate the optimal distribution of the visco-elastic dampers in shear building structures (Cimellaro 2007). The first objective function was used to find the optimal damper positioning for a cantilever beam (Takewaki 1998). These performance functions are adapted to find optimal supporting spring positions in the current study.

3.2 Optimality criteria

The optimality criteria for the optimal supporting spring problem can be derived using the Lagrange Multipliers Method. The generalized Lagrangian, L for the problem of optimal spring placement for each of the objective functions can be written in terms of the Lagrange Multipliers λ, $\boldsymbol{\mu}$ and $\boldsymbol{\nu}$ as follows

$$L(k_j, \lambda, \mu_j, \upsilon_j) = f_h(\boldsymbol{k}) + \lambda\left(\sum_{j=1}^{n}(k_j - \bar{K})\right) + \sum_{j=1}^{n}\mu_j(0 - k_j) + \sum_{i=1}^{n}\upsilon_j(k_i - \bar{k}_j) \tag{18}$$

where h can be equal to either 1 or 2 which corresponds to each one of the objective functions. The optimality criteria without upper and lower bound constraints on the stiffness coefficients can be derived from the stationary conditions of the Lagrangian $L(\boldsymbol{\mu}=\boldsymbol{0},\ \boldsymbol{\nu}=\boldsymbol{0})$ with respect to λ and k_i

$$\frac{\partial f_h}{\partial k_j} + \lambda = 0 \quad for\ \ 0 < k_j < \bar{k}_j \quad (j = 1,2,\ldots,n)\ \ and\ \ (h = 1\ ,2) \tag{19}$$

$$\sum_{j=1}^{n} k_j - \bar{K} = 0 \tag{20}$$

where $\frac{\partial f_h}{\partial k_j}$ represents partial differentiation of the h^{th} objective function with respect to the design variable k_j. When the upper and lower bound constraints are available, Eq. (19) can be rewritten as

$$\frac{\partial f_h}{\partial k_j} + \lambda \geq 0 \qquad for\ \ k_j = 0 \tag{21}$$

$$\frac{\partial f_h}{\partial k_j} + \lambda \leq 0 \qquad for\ \ k_j = \bar{k}_j \tag{22}$$

These nonlinear equations can be solved by the steepest direction search algorithm (SDSA) (Takewaki 1998). Sensitivities should be derived to be used in the optimization algorithm.

3.3 Derivation of the sensitivity formulations

To determine the optimal position of spring supports, both the sensitivity of objective functions and the natural frequency corresponding to the position of the support must be estimated. This sensitivity information allows both the search direction and the optimal position of the spring support to be determined.

If Eq. (9) is differentiated with respect to k_i as given below

$$\frac{\partial \boldsymbol{A}}{\partial k_j}\widehat{\boldsymbol{U}} + \boldsymbol{A}\frac{\partial \widehat{\boldsymbol{U}}}{\partial k_j} = \boldsymbol{0} \quad \text{(j=1…,n)} \tag{23}$$

The first order sensitivity of $\widehat{\boldsymbol{U}}$ is written as:

$$\frac{\partial \widehat{\boldsymbol{U}}}{\partial k_j} = -\boldsymbol{A}^{-1}\frac{\partial \boldsymbol{A}}{\partial k_j}\widehat{\boldsymbol{U}} \tag{24}$$

Eq. (23) and Eq. (24) were derived by Takewaki (1998). The first order derivatives of the absolute accelerations $\widehat{\ddot{\boldsymbol{U}}}$ at the s^{th} undamped natural circular frequency was derived by a partial differential of Eq. (12) (Cimellaro 2007) as follows

$$\frac{\partial\hat{\ddot{\boldsymbol{U}}}}{\partial k_j} = -\boldsymbol{A}^{-1}\frac{\partial \boldsymbol{A}}{\partial k_j}\omega_s{}^2\boldsymbol{U} \tag{25}$$

The quantities of $\hat{U}_i$ and $\hat{\ddot{U}}_i$, which are shown in Eqs. (9) and (12), can be written as

$$\hat{U}_i = Re[\hat{U}_i] + Im[\hat{U}_i] \tag{26}$$

$$\hat{\ddot{U}}_i = Re\left[\hat{\ddot{U}}_i\right] + Im\left[\hat{\ddot{U}}_i\right] \tag{27}$$

where $\hat{U}_i$, and $\hat{\ddot{U}}_i$ are the transfer function values of the i^{th} node displacement and i^{th} node absolute acceleration, respectively in complex form. The first order sensitivities of the quantities $\hat{U}_i$, and $\hat{\ddot{U}}_i$, in Eqs. (24)-(25), can be expressed as

$$\frac{\partial\hat{U}_i}{\partial k_j} = Re\left[\frac{\partial\hat{U}_i}{\partial k_j}\right] + Im\left[\frac{\partial\hat{U}_i}{\partial k_j}\right] \tag{28}$$

$$\frac{\partial\hat{\ddot{U}}_i}{\partial k_j} = Re\left[\frac{\partial\hat{\ddot{U}}_i}{\partial k_j}\right] + Im\left[\frac{\partial\hat{\ddot{U}}_i}{\partial k_j}\right] \tag{29}$$

The absolute values of $\hat{U}_i$ and $\hat{\ddot{U}}_i$ can be written as:

$$\left|\hat{U}_i\right| = \sqrt{(Re[\hat{U}_i])^2 + (Im[\hat{U}_i])^2} \tag{30}$$

$$\left|\hat{\ddot{U}}_i\right| = \sqrt{(Re[\hat{\ddot{U}}_i])^2 + (Im[\hat{\ddot{U}}_i])^2} \tag{31}$$

If $\left|\hat{U}_i\right|$ and $\left|\hat{\ddot{U}}_i\right|$ are differentiated with respect to the j^{th} stiffness coefficient k_j, the first order sensitivities of the absolute values of transfer function amplitude of the i^{th} node displacement (Takewaki 1998) and the i^{th} absolute acceleration (Cimellaro 2007) are found to be

$$\frac{\partial|\hat{U}_i|}{\partial k_j} = \frac{1}{|\hat{U}_i|}\left\{Re[\hat{U}_i]\left(Re\left[\frac{\partial\hat{U}_i}{\partial k_j}\right]\right) + Im[\hat{U}_i]\left(Im\left[\frac{\partial\hat{U}_i}{\partial k_j}\right]\right)\right\} \tag{32}$$

$$\frac{\partial\left|\hat{\ddot{U}}_i\right|}{\partial k_j} = \frac{1}{\left|\hat{\ddot{U}}_i\right|}\left\{Re\left[\hat{\ddot{U}}_i\right]\left(Re\left[\frac{\partial\hat{\ddot{U}}_i}{\partial k_j}\right]\right) + Im\left[\hat{\ddot{U}}_i\right]\left(Im\left[\frac{\partial\hat{\ddot{U}}_i}{\partial k_j}\right]\right)\right\} \tag{33}$$

Partial differentiation of Eqs. (32)-(33) with respect to the other design parameter k_l leads to

$$\frac{\partial^2|\hat{U}_i|}{\partial k_j\partial k_l} = \frac{1}{|\hat{U}_i|^2}\Big(\left|\hat{U}_i\right|\left\{Re\left[\frac{\partial\hat{U}_i}{\partial k_l}\right]Re\left[\frac{\partial\hat{U}_i}{\partial k_j}\right] + Re[\hat{U}_i]Re\left[\frac{\partial^2\hat{U}_i}{\partial k_j\partial k_l}\right] + Im\left[\frac{\partial\hat{U}_i}{\partial k_l}\right]Im\left[\frac{\partial\hat{U}_i}{\partial k_j}\right] + Im[\hat{U}_i]Im\left[\frac{\partial^2\hat{U}_i}{\partial k_j\partial k_l}\right]\right\} - \frac{\partial|\hat{U}_i|}{\partial k_l}\left\{Re[\hat{U}_i]Re\left[\frac{\partial\hat{U}_i}{\partial k_j}\right] + Im[\hat{U}_i]Im\left[\frac{\partial\hat{U}_i}{\partial k_j}\right]\right\} \tag{34}$$

In Eq. (34), $Re\left[\frac{\partial^2\hat{U}_i}{\partial k_j\partial k_l}\right]$ and $Im\left[\frac{\partial^2\hat{U}_i}{\partial k_j\partial k_l}\right]$ are calculated from Eq. (35) which are the second derivatives of $\hat{\boldsymbol{U}}$ and $\hat{\ddot{\boldsymbol{U}}}$, respectively. Eqs. (23)-(37) were derived by Takewaki (1998). The absolute acceleration was incorporated into these equations for absolute accelerations by Cimellaro (2007). The differentiations of Eqs. (24)-(25) with respect to k_l are expressed in the following form

$$\frac{\partial^2 \widehat{\boldsymbol{U}}}{\partial k_j \partial k_l} = \boldsymbol{A}^{-1} \frac{\partial \boldsymbol{A}}{\partial k_l} \boldsymbol{A}^{-1} \frac{\partial \boldsymbol{A}}{\partial k_j} \widehat{\boldsymbol{U}} - \boldsymbol{A}^{-1} \frac{\partial \boldsymbol{A}}{\partial k_j} \frac{\partial \widehat{\boldsymbol{U}}}{\partial k_l} \tag{35}$$

The components of matrix A consist of $\boldsymbol{K}$, $\boldsymbol{K_{sp}}$, $\boldsymbol{M}$, $\boldsymbol{C}$ and ω_s. The added spring matrix $\boldsymbol{K_{sp}}$, the damping matrix $\boldsymbol{C}$ and the sth natural circular frequency of the beam ω_s are functions of design variables. In order to derive the first order sensitivity of matrix A, the partial derivative with respect to design variables k_j should be obtained.

For a cantilever beam model, both eigenvector and eigenvalues are calculated from the following equation

$$\left(\boldsymbol{K} + \boldsymbol{K_{sp}}\right)\boldsymbol{\Phi_s} = \Omega_s \boldsymbol{M} \boldsymbol{\Phi_s} \tag{36}$$

where $\boldsymbol{\Phi_s}$ and Ω_s are the sth eigenvector and the sth eigenvalue of the beam structure, respectively. If both sides of Eq. (36) are multiplied by $\boldsymbol{\Phi_s^T}$, Eq. (36) can be rearranged as follows

$$\boldsymbol{\Phi}_s^{\mathrm{T}} (\boldsymbol{K} + \boldsymbol{K_{sp}}) \boldsymbol{\Phi}_s = \Omega_s \boldsymbol{\Phi_s^T} \mathbf{M} \boldsymbol{\Phi_s} \tag{37}$$

Let $\overline{m}_s = \boldsymbol{\Phi_s^T} \mathbf{M} \boldsymbol{\Phi_s}$ and $\bar{k}_s = \boldsymbol{\Phi_s^T} (\mathbf{K} + \mathbf{K_{sp}}) \boldsymbol{\Phi_s}$ denote modal mass and modal stiffness evaluated at the s^{th} mode of the beam. Eq. (37) can be rewritten in the following form

$$\Omega_s = \frac{\bar{k}_s}{\overline{m}_s} \tag{38}$$

The first order sensitivity of the eigenvalue Ω_s with respect to design variable k_j is given as

$$\frac{\partial \Omega_s}{\partial k_j} = \frac{1}{\overline{m}_s} \frac{\partial \bar{k}_s}{\partial k_j} \tag{39}$$

where sensitivity of the modal stiffness with respect to design parameter k_j can be given as

$$\frac{\partial \bar{k}_s}{\partial k_j} = \boldsymbol{\Phi_s^T} \frac{\partial (\boldsymbol{K} + \boldsymbol{K_{sp}})}{\partial k_j} \boldsymbol{\Phi_s} \tag{40}$$

When $\Omega_s = \omega_s^2$ is substituted in Eq. (38), the first order sensitivity of the s^{th} natural circular frequency of the cantilever beam can be obtained in the following form:

$$\frac{\partial \omega_s}{\partial k_j} = \frac{1}{2 \overline{m}_s \omega_s} \frac{\partial \bar{k}_s}{\partial k_j} \tag{41}$$

To derive the first order sensitivity of the matrix A, in addition to the sensitivity of the eigenvalue and the natural circular frequency, the sensitivity of the damping matrix C should be obtained. Structural damping can be taken as either mass proportional or stiffness proportional damping. For both of the cases, structural damping matrix can be written as

$$\boldsymbol{C} = 2 \zeta_s \omega_s \boldsymbol{M} \tag{42}$$

$$\boldsymbol{C} = \frac{2 \zeta_s}{\omega_s} (\boldsymbol{K} + \boldsymbol{K_{sp}}) \tag{43}$$

where ζ_s denotes the damping ratio in the s^{th} mode. The partial derivatives of Eqs. (42)-(43) with respect to design variable k_j can be obtained as follows

$$\frac{\partial \boldsymbol{C}}{\partial k_j} = 2 \zeta_s \frac{\partial \omega_s}{\partial k_j} \mathbf{M} \tag{44}$$

$$\frac{\partial \boldsymbol{C}}{\partial k_j} = \frac{2 \zeta_s}{\omega_s} \frac{\partial \boldsymbol{K_{sp}}}{\partial k_j} - \frac{2 \zeta_s}{(\omega_s)^2} \frac{\partial \omega_s}{\partial k_j} \boldsymbol{K_{sp}} \tag{45}$$

where the elements of matrix $\boldsymbol{K_{sp}}$ are linear functions of design parameters k_j and the partial derivative of each term of $\boldsymbol{K_{sp}}$ matrix with respect to both k_1 and k_j can be obtained for the cantilever beam as

$$\frac{\partial \boldsymbol{K_{sp}}}{\partial k_1}=\begin{bmatrix}1&0&\dots&0&0&\dots&0&0\\0&0&\dots&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\dots&0&0&\dots&0&0\\0&0&\dots&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\dots&0&0&\dots&0&0\\0&0&\dots&0&0&\dots&0&0\end{bmatrix}_{2nx2n}\quad \frac{\partial \boldsymbol{K_{sp}}}{\partial k_j}=\begin{bmatrix}0&0&\dots&0&0&\dots&0&0\\0&0&\dots&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\dots&1&0&\dots&0&0\\0&0&\dots&0&0&\dots&0&0\\\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots&\vdots\\0&0&\dots&0&0&\dots&0&0\\0&0&\dots&0&0&\dots&0&0\end{bmatrix}_{2nx2n} \tag{46}$$

The first order sensitivity of the matrix A can be obtained by using the sensitivity formulations $\frac{\partial \boldsymbol{K_{sp}}}{\partial k_j}$, $\frac{\partial \boldsymbol{C}}{\partial k_j}$, $\frac{\partial \omega_s}{\partial k_j}$ and $\frac{\partial \Omega_s}{\partial k_j}$ as follows

$$\frac{\partial \boldsymbol{A}}{\partial k_j}=\frac{\partial \boldsymbol{K_{sp}}}{\partial k_j}+i\frac{\partial \omega_s}{\partial k_j}\boldsymbol{C}+i\omega_s\frac{\partial \boldsymbol{C}}{\partial k_j}-\frac{\partial \Omega_s}{\partial k_j}\boldsymbol{M} \tag{47}$$

In case of mass proportional structural damping, Eq. (47) can be written as

$$\frac{\partial \boldsymbol{A}}{\partial k_j}=\frac{\partial \boldsymbol{K_{sp}}}{\partial k_j}+i\frac{1}{2\overline{m}_s\omega_s}\frac{\partial \bar{k}_s}{\partial k_j}2\zeta_s\omega_s\boldsymbol{M}+i\omega_s 2\zeta_s\frac{\partial \omega_s}{\partial k_j}\mathbf{M}-\frac{1}{\overline{m}_s}\frac{\partial \bar{k}_s}{\partial k_j}\boldsymbol{M} \tag{48}$$

If the structural damping is chosen to be proportional to stiffness, Eq. (47) is as follows

$$\frac{\partial \boldsymbol{A}}{\partial k_j}=\frac{\partial \boldsymbol{K_{sp}}}{\partial k_j}+i\frac{1}{2\overline{m}_s\omega_s}\frac{\partial \bar{k}_s}{\partial k_j}\frac{2\zeta_s}{\omega_s}(\boldsymbol{K}+\boldsymbol{K_{sp}})+i\omega_s(\frac{2\zeta_s}{\omega_s}\frac{\partial \boldsymbol{K_{sp}}}{\partial k_j}-\frac{2\zeta_s}{(\omega_s)^2}\frac{\partial \omega_s}{\partial k_j}\boldsymbol{K_{sp}})-\frac{1}{\overline{m}_s}\frac{\partial \bar{k}_s}{\partial k_j}\boldsymbol{M} \tag{49}$$

3.4 Solution algorithm based on second order approximation for tip deflection (for $k_j<\bar{k}_j$)

Step 1. Assume the stiffness coefficients of all springs to be $k_j=0$ where $j=1,...,n$.
Assume $\Delta K=\frac{\bar{K}}{m}$ where m is the design step number.

Step 2. Compute $\frac{\partial f_1}{\partial k_j}$ using Eq (32).

Step 3. Find the index z satisfying $-\frac{\partial f_1}{\partial k_z}=Max\left(-\frac{\partial f_1}{\partial k_j}\right)$

Step 4. Update f by $f_1+\frac{\partial f_1}{\partial k_z}\Delta k_z$ where $\Delta k_z=\Delta K$

Step 5. Update $\frac{\partial f_1}{\partial k_j}$ by $\frac{\partial f_1}{\partial k_j}+\frac{\partial^2 f_1}{\partial k_j \partial k_z}\Delta k_z$ using Eq. (34).

Step 6. In Step 5, if there is a spring of an index i such that the following condition is satisfied: $-\frac{\partial f_1}{\partial k_z}=\max_{i,i\neq z}\left(-\frac{\partial f_1}{\partial k_i}\right)$, then stop tentatively. Compute $\overline{\Delta k_z}$ and update $\frac{\partial f_1}{\partial k_j}$ by $\frac{\partial^2 f_1}{\partial k_j \partial k_z}\overline{\Delta k_z}$ using Eq (34).

Step 7. Continue Steps 2 through 6 until the constraint $\sum_{i=1}^{n}k_i=\bar{K}$ is satisfied.

In Steps 3 and 4, under the constraint $\sum_{i=1}^{n}\Delta k_i=\Delta K$, the objective function is reduced and updates the direction according to index z. This algorithm was presented by Takewaki (1998). If there are multiple indices $z_1...z_m$, the objective function and its first derivative must be updated as

$$f_1 \rightarrow f_1 + \sum_{i=z_1}^{z_m} \frac{\partial f_1}{\partial k_i} \Delta k_i \tag{50}$$

$$\frac{\partial f_1}{\partial k_j} \rightarrow \frac{\partial f_1}{\partial k_j} + \sum_{i=z_1}^{z_m} \frac{\partial^2 f_1}{\partial k_j \partial k_l} \Delta k_l \tag{51}$$

In this study, this method is used to control the first mode in case of the tip deflection minimization. To control optimally second and third modes of the cantilever beam supported by the springs in case of the tip deflection minimization, a new search method is proposed such that it uses a first order approximation. In addition to minimizing the tip displacement of the beam, the tip absolute acceleration of the beam is minimized by adding the springs optimally for the control of the first three modes. For this purpose, a solution algorithm is also proposed in this study.

3.5 Solution algorithm based on first order approximation for tip deflection (for $k_j < \bar{k}_j$)

Step 1. Assume the stiffness coefficients of all springs to be $k_j=0$ where $j=1,...,n$.
Assume $\Delta K = \frac{\bar{K}}{m}$ where *m* is the design step number.
Step 2. Compute $\frac{\partial f_1}{\partial k_j}$ using Eqs. (32).
Step 3. Find the index *z* satisfying $-\frac{\partial f_1}{\partial k_z} = Max\left(-\frac{\partial f_1}{\partial k_j}\right)$.
Step 4. Update f by $f_1 + \frac{\partial f_1}{\partial k_z}\Delta k_z$ where $\Delta k_z = \Delta K$
Step 5. Continue Steps 2 through 4 until the constraint $\sum_{i=1}^{n} k_i = \bar{K}$ is satisfied.

A simplified feasible direction search algorithm can be used to calculate the optimal placement of springs using only the first order approximation. The classical Steepest Direction Search Algorithm is invalid, when it is applied to the optimal spring distribution problem based on the tip deflection for second and third mode control. To find the values of the stiffness coefficients of each spring in the first algorithm, in the case of adding the stiffness coefficient more than one in any step of the algorithm, the optimality conditions based on the second order sensitivity of the objective function are used. This algorithm does not use the second order sensitivity. Therefore, the increment of the stiffness coefficient added in each step is fixed by ΔK. The objective function is reduced and updated according to direction obtained until the constraint $\sum_{i=1}^{n} k_i = \bar{K}$ is satisfied.

3.6 Solution Algorithm based on first order approximation for tip absolute acceleration (for $k_j < \bar{k}_j$)

Step 1. Assume the stiffness coefficients of all springs to be $k_j=0$ where $j=1,...,n$.
Assume firstly $\Delta K = \frac{\bar{K}}{m}$ where *m* is the design step number.
Step 2. Compute $\frac{\partial f_2}{\partial k_j}$ using Eqs. (33).
Step 3. Find the index *z* satisfying $\frac{\partial f_2}{\partial k_z} = Max\left(\frac{\partial f_2}{\partial k_j}\right)$.
Step 4. Update f by $f_2 + \frac{\partial f_2}{\partial k_z}\Delta k_z$ where $\Delta k_z = \Delta K$

Step 5. If the new value of f_2 is higher than the previous value of f_2, stop the process and calculate the new $\sum_{i=1}^{n} k_i = \overline{K}$ as a summation of k_i values at that step.

Step 6. If the condition in Step 5 is not satisfied; continue Steps 2 through 4 until the constraint $\sum_{i=1}^{n} k_i = \overline{K}$ is satisfied.

This algorithm does not also use the second order sensitivity. Therefore, the increment of the stiffness coefficient added in each step is fixed by ΔK. The objective function is reduced and updated according to direction obtained until the constraint $\sum_{i=1}^{n} k_i = \overline{K}$ is satisfied. Moreover, if the condition in Step 5 is satisfied, the process should be stopped. When the process is continued after this step, the objective function will increase and it moves away from the minimum point. If the process is stopped in Step 5, the total stiffness capacity $\overline{K}$ chosen in initial stage of the algorithm will change. Accordingly, the minimum response of the beam depends on the total stiffness capacity.

In this study, a gradient based method is proposed. There are many optimization methods to solve engineering problem. The gradient based methods need sensitivity information. Sensitivities give us sensitive design parameters and sensitive locations above the structural response. These sensitivities may be discontinuous for various objective functions and constraints. The sensitivities in this paper are continuous for the proposed objective function and constraints. If the opposite situation is appeared, this difficulty can be overcome by using direct search approaches for optimization because direct search algorithms do not have many mathematical requirements (no derivatives needed, etc.) for the optimization problems. Also there are other optimization algorithms such as genetic algorithms, ant colony algorithms, particle swarm optimization that are capable of finding optimal design.

4. Numerical example

A fixed support Timoshenko cantilever beam which has a length of 6 m is shown in Fig. 1. It is selected in order to show the application of proposed methods on a numerical example (Takewaki 1998). The beam is modelled using finite elements of 1 m length for each. Considering a vertical and an angular displacement at each node, a total of 12 degrees of freedom are defined in the system. Material density, ρ=7.8 10^3 kg/m^3, elasticity modulus, E=2.06 10^{11} N/m^2, shear modulus, G=7.94 10^{10} N/m^2, shear correction factor, κ=5/6, cross sectional area, A=0.05m^2, second moment of area, I=2.08 10^{-4} m^4, and damping ratio for the first three modes, ζ=0.02 are selected for beam identification. In addition; a lumped mass of 100 kg's is added to the beam end. Structural damping matrix is calculated such that it is proportional to the mass matrix. Potential nodes to which springs will be placed or supported are defined as nodal points. The six spring stiffness coefficients, $\mathbf{k}=\{k_1, k_2, k_3, k_4, k_5, k_6\}$ are defined as design variables. First three natural circular frequencies of the beam are calculated as ω_1= 29.87 rad/s, ω_2=187 rad/s, and ω_3 = 526.31 rad/s.

Objective functions given by Eqs. (16) and (17) are minimized at the optimization stage. Firstly, the sum of spring stiffness coefficients is taken as $\overline{K} = 2282400\ N/m$ in the first mode control. The amplitude of transfer functions for the tip deflection evaluated at the first natural circular frequency is minimized by calculating optimum spring stiffness coefficients using the SDSA method.

In the case of optimal control of tip deflection at the first mode, total spring stiffness is placed to the tip of beam using the optimization algorithm given in Section 3.4, that is to say $k_6 = \overline{K}$ as

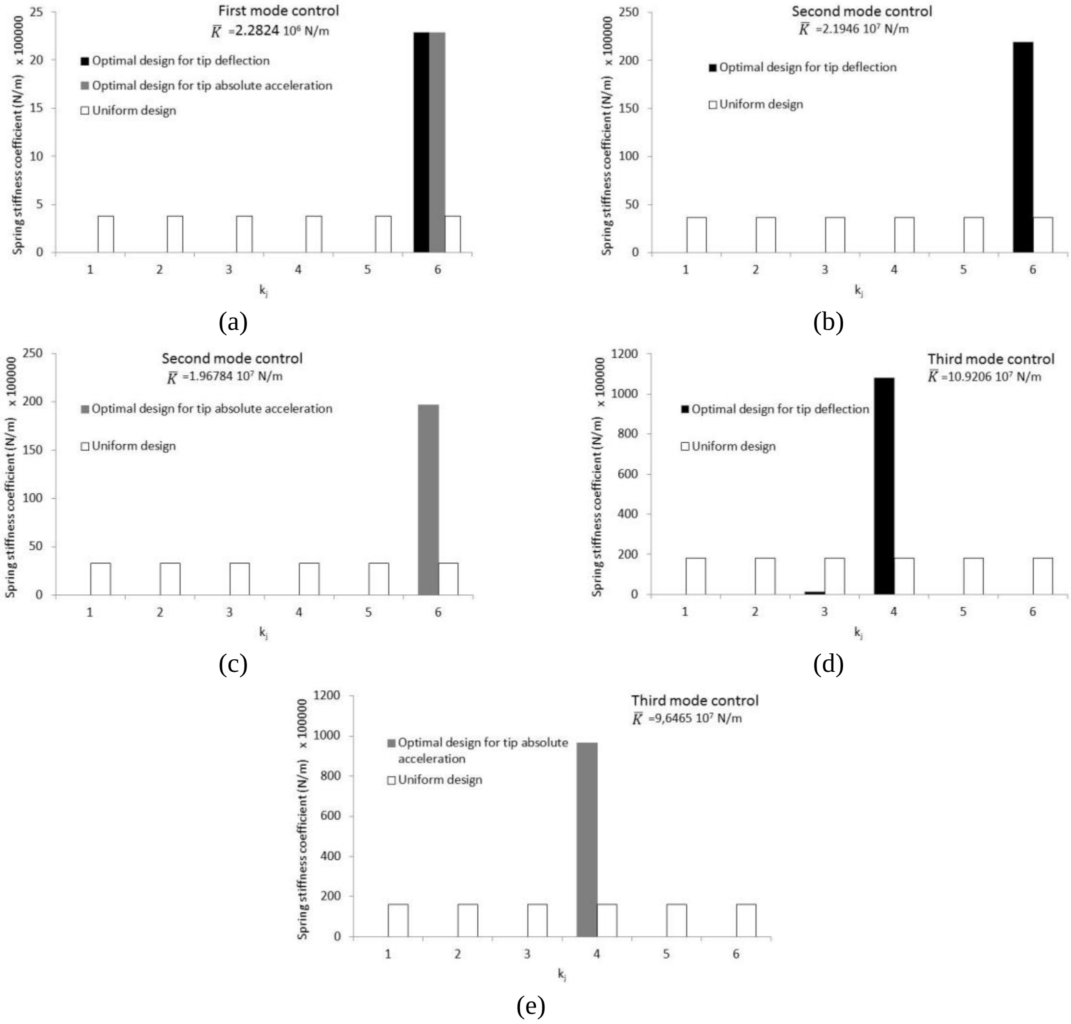
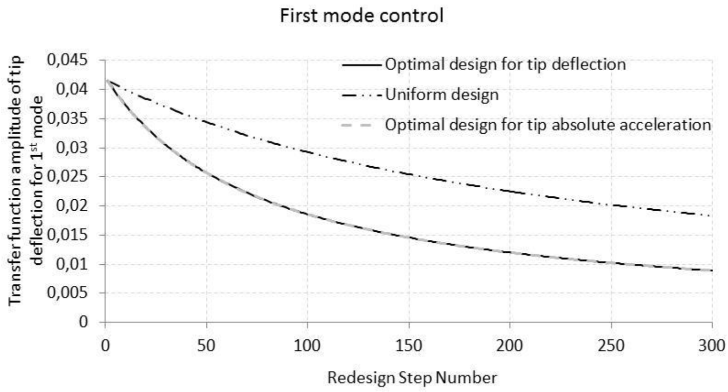


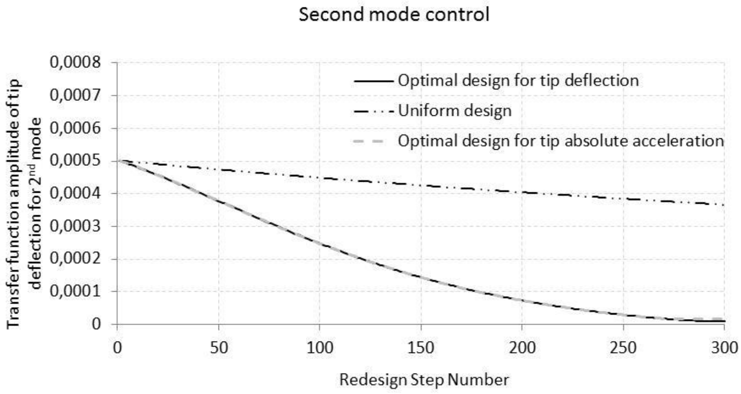
Fig. 2 Optimal spring designs for first-three modes

shown in Fig 2(a). Meanwhile; the other spring stiffness coefficients are equal to zero. This emphasizes that total stiffness should be placed to the end of cantilever beam in order to minimize the tip deflection response at the first mode. Afterwards, the optimal location of springs is determined by minimizing the transfer function amplitude of tip absolute acceleration provided that total stiffness coefficient is equal to $\overline{K} = 2282400\ N/m$. As a result of application of the optimization algorithm given in Section 3.6, optimal spring stiffness coefficients are found as $k_1 = k_2 = k_3 = k_4 = k_5 = 0$ and $k_6 = \overline{K}$ which is same as the spring design for tip deflection optimization (Fig. 2(a)).

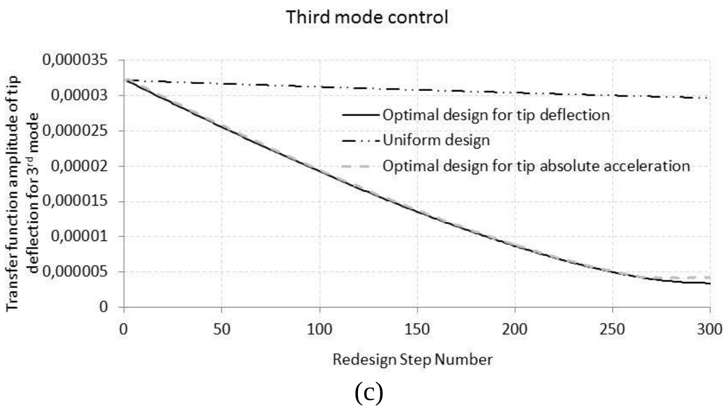
$\Delta K = \overline{K}/m$ value at each design step is distributed equally to each one of six springs; and a uniform design is defined, in order to make another comparison. At the end of uniform design, $k_1 = k_2 = k_3 = k_4 = k_5 = k_6 = \frac{2.2824\ 10^6}{6}$N/m; and these values are shown in Fig. 2(a). The amplitude of tip deflection at the first mode according to the redesign step number for each one of



(a)



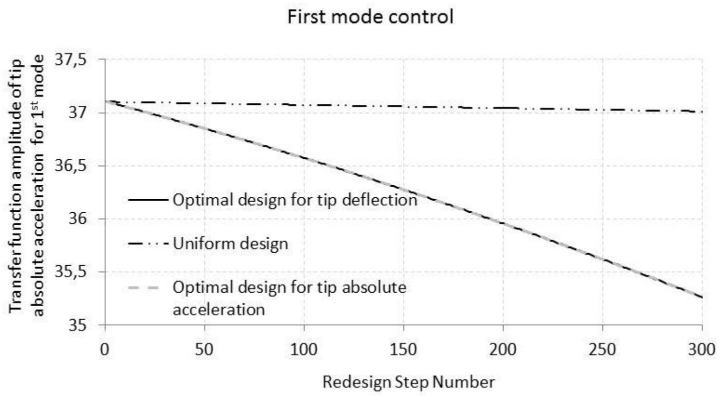
(b)



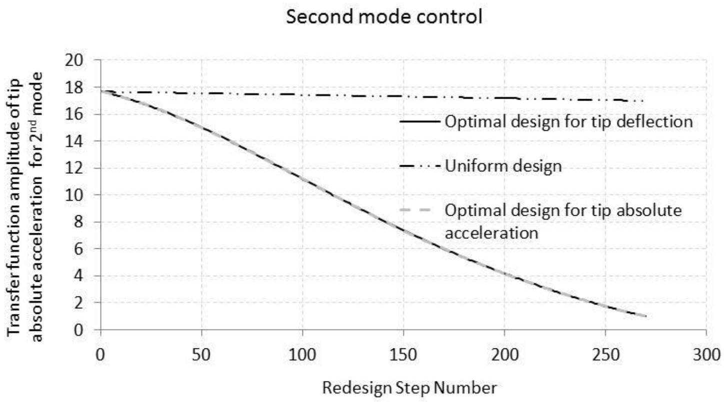
(c)

Fig. 3 Variation of objective function for tip deflection with respect to redesign step number in the first-three modes

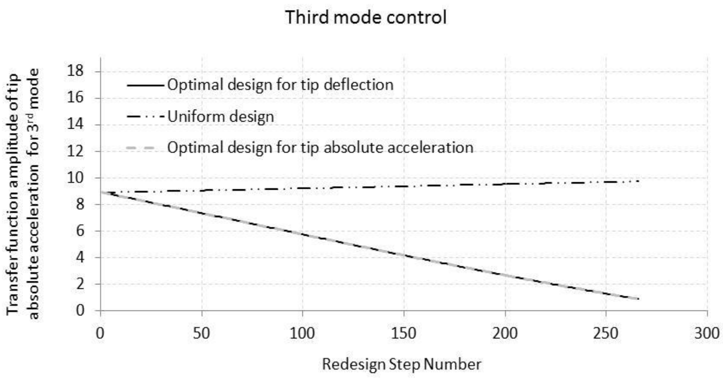
three designs is shown in Fig. 3(a). The same design results are observed in Fig. 3(a) showing the change of objective function f_1 for the minimization of tip deflection and tip absolute acceleration. Optimum designs show a better performance compared to uniform design. Fig. 4(a) shows the transfer function amplitude of tip absolute acceleration for three designs composed of two optimal



(a)



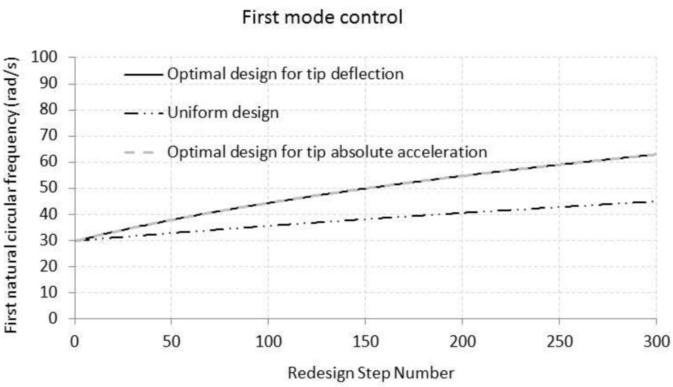
(b)



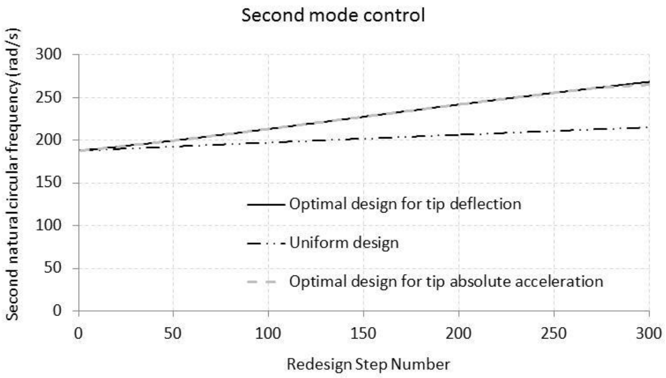
(c)

Fig. 4 Variation of objective function for tip absolute acceleration with respect to redesign step number in the first-three modes

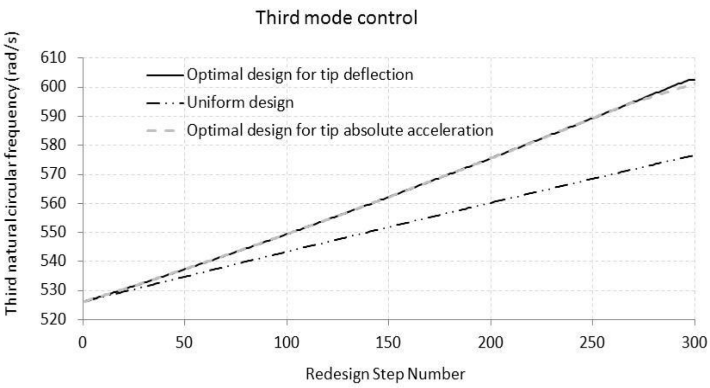
and one uniform designs. When compared with the uniform design, it is observed that both of the two optimal designs show a better performance with respect to the transfer function amplitude of the tip absolute acceleration at the first mode. It is shown in Fig. 5(a) that the first natural circular



(a)



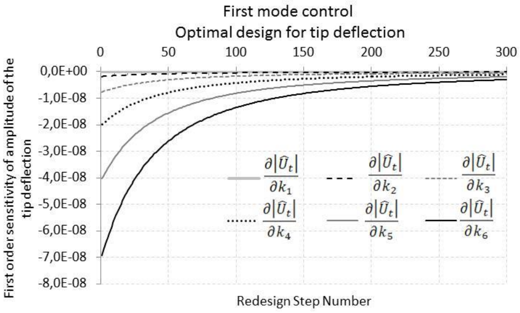
(b)



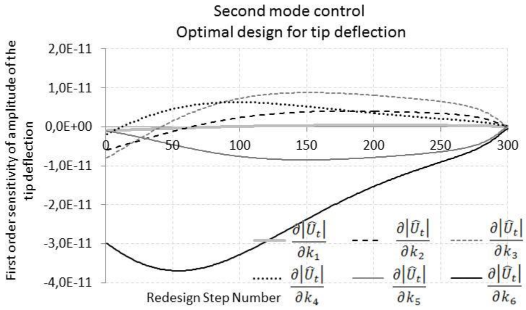
(c)

Fig. 5 Variation of natural circular frequency with respect to redesign step number in the first-three modes

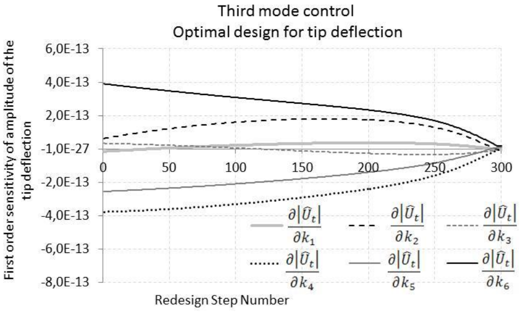
frequency is maximized during the optimal design at the first mode. The changes in partial differentials of f_1 at the first mode are plotted in Fig. 6(a) from which the convergence of optimization can be proven. A similar situation is shown in Fig. 7(a) in which the first order sensitivities of f_2 converge to zero at the end of design step. After the optimal and uniform designs at the first mode are evaluated, the change in transfer function amplitude of tip deflections



(a)

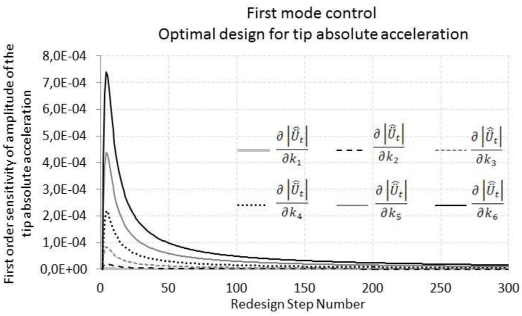


(b)

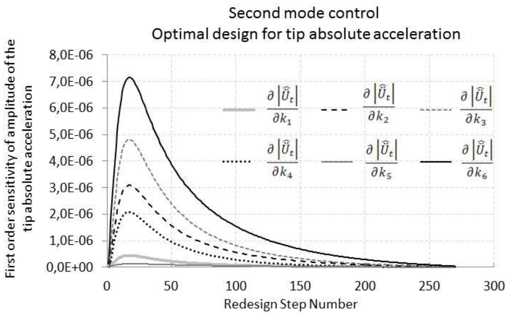


(c)

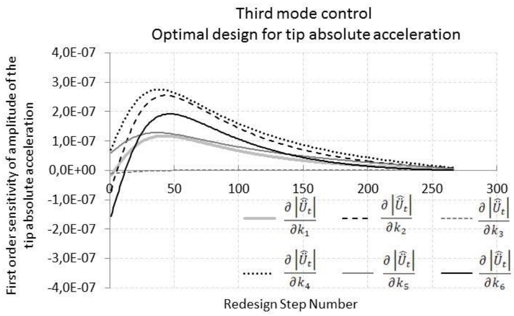
Fig. 6 Variation of the first order sensitivity of the amplitude of tip deflection with respect to redesign step number in the first-three modes



(a)

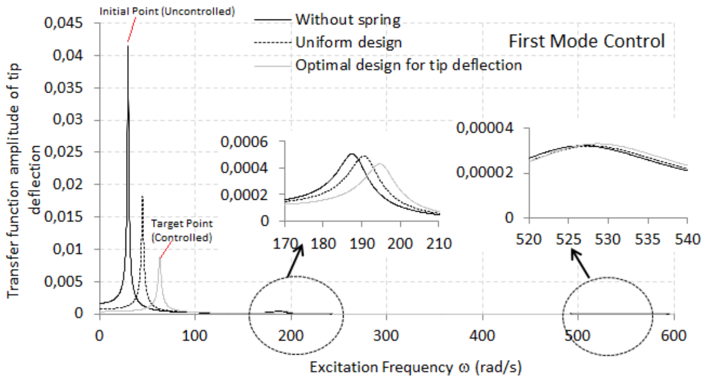


(b)

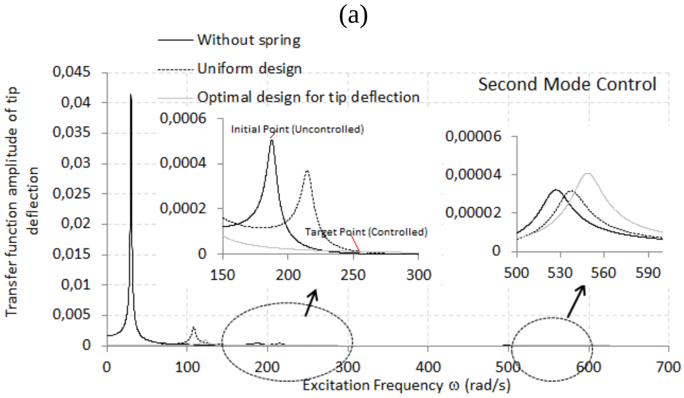


(c)

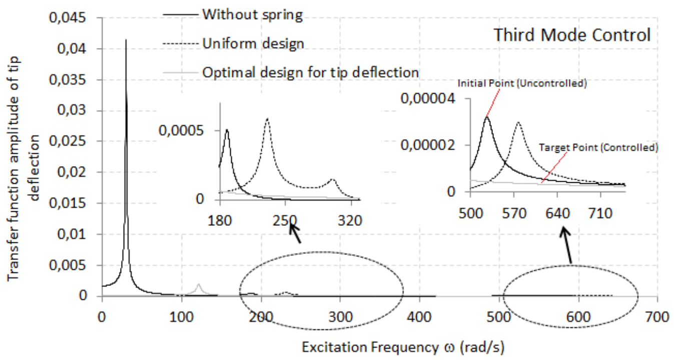
Fig. 7 Variation of the first order sensitivity of the amplitude of tip absolute acceleration with respect to redesign step number in the first-three modes



(a)



(b)

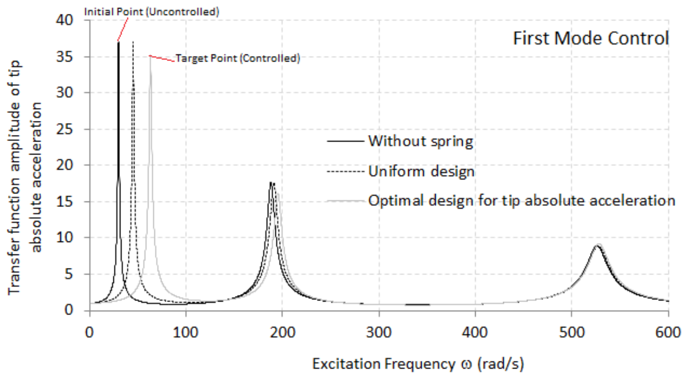


(c)

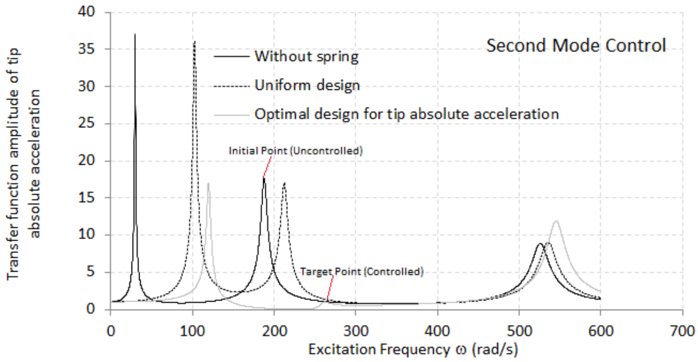
Fig. 8 Variation of transfer function amplitude of tip deflection with respect to excitation frequency

according to excitation frequency is plotted in Fig. 8(a) for three designs. The black line shows the case when no springs exist. When there are no springs, the tip deflection transfer function amplitude reduces to a minimum level after the optimal design is performed at the excitation frequency where first resonance occurs.

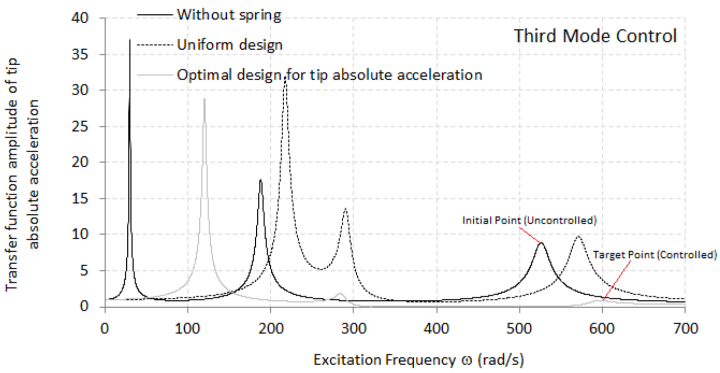
Target point in Fig. 8(a) shows the resonance point in optimal design. It is observed that optimal design for tip deflection reduces objective function at the resonance point better compared



(a)



(b)



(c)

Fig. 9 Variation of transfer function amplitude of tip absolute acceleration with respect to excitation frequency

to uniform design. In addition; frequency behaviours at the first mode for the control of tip deflection regarding optimal design and other designs at the second and third resonance regions are shown in Fig. 8(a). While in the second resonance region optimal design shows a better performance compared to other designs, in the third resonance region it is observed that all three designs show similar behaviours.

In Fig. 9(a), optimal design for tip absolute acceleration, uniform design and design without springs are plotted for change of transfer function amplitude of tip absolute acceleration with the excitation frequency. It is observed that optimal design evaluated for acceleration minimization reduces acceleration behaviour more compared to both uniform design and design without spring at the first resonance region. In Fig. 9 (a), it can be seen that optimal design evaluated at the first mode also shows a better performance compared to other designs at the second resonance region. In the same figure, it is observed that in the third resonance region the frequency behaviour of optimal design is slightly larger than the others.

At the second stage; optimal location and number of springs are calculated by minimizing both absolute acceleration and tip deflection of cantilever beam at the second mode. Simple algorithm shown in Section 3.5 is used in order to minimize transfer function amplitude of tip deflection at the second mode. In order that partial differential equations of objective function, f_1 converge to zero; $\overline{K}$ value selected at the first mode is changed as $\overline{K}$=2.1946 10^7 N/m.

As shown in Fig. 2(b), optimal spring distribution which will minimize transfer function amplitude of tip deflection at the second mode is found. For this case; all of the springs are located at the tip of cantilever beam, as it was the case at the first mode (k_1=k_2=k_3=k_4=k_5=0 and k_6=2.1946 10^7 N/m). Simple algorithm in Section 3.6 is used, in order to find spring distribution which will minimize tip acceleration transfer function amplitude as shown in Fig. 2(c) (k_1=k_2=k_3=k_4=k_5=0 and k_6=1.96784 10^7 N/m). Even though, locations of spring distribution are the same for both design which minimizes tip deflection and design which minimizes tip absolute acceleration, there is a difference in the total springs coefficient.

When the algorithm in Section 3.6 is used, if the value of objective function increases with respect to the previous step (in accordance with the constraint at Step 5 in Section 3.6) at any stage of design, the process should be stopped. During the minimization of acceleration, the process is stopped at Step 270 since the value of objective function, f_2 increased compared to the objective function value at the previous step. The calculated values of k_j until that step are taken as optimal values.

If the algorithm was also applied after the 270th step, the objective function, f_2 will increase and will diverge from the minimum value. Accordingly, in the obtained optimal designs regarding the control of second mode; even though locations of springs are the same, there is a difference in their stiffness coefficients. As total spring coefficient values increase, tip deflection response will continuously decrease. However; the same situation does not occur for the absolute acceleration. Even though increase in total stiffness constant $\overline{K}$ decreases the objective function, f_2 down to a certain point, it increases the objective function after that point. Because of this situation, the constraint given in Step 5 of Section 3.6 is placed in the algorithm. This causes a difference in $\overline{K}$ values of optimal design in each one of two objective functions.

In Fig. 3(b), the variation of f_1 with respect to redesign step number is plotted for the minimization of both of two objective functions. In addition; the variation of this function can be seen for the case of uniform spring design in Fig. 3(b). The transfer function value of absolute acceleration up to 270th design step for all three design cases are plotted in Fig. 4(b). It is observed that both of the optimal designs show a better performance compared to uniform design. In Fig. 5(b), it is shown that optimal designs increase second natural circular frequency, ω_2 more than uniform design.

Fig. 6(b) shows the change in first order sensitivities of f_1 with respect to design variables for the control of second mode. It is observed that convergence is satisfied at the end of design. The change in first order sensitivities of f_2 with respect to design variables can be seen in Fig. 7(b). It can be seen that convergence is also satisfied during the optimization process which occurs until the 270th step.

For the case of second mode control, if the amplitude of transfer function of tip deflection is investigated for excitation frequency in Fig. 8(b), it is observed that the target point value at optimal case is significantly reduced. In the same figure, the optimal design amplitude is larger than other design response amplitudes at the third resonance region. Fig. 9(b) shows the frequency behaviour of transfer function amplitude of absolute acceleration in different designs for the case of second mode control. If the second resonance region is investigated, it is seen that optimal design behaviour shows a significantly better performance compared to other designs. If the first and third resonance regions are examined in the same figure, it can be said that optimal design shows a good performance in the first resonance region, while it shows a bad performance in the third resonance region.

In the first two modes of cantilever beam; after being controlled by the springs, in the third stage the optimal distribution of springs, which minimizes f_1 and f_2 evaluated at the third mode, are investigated. Figs. 2(d) and 2(e) show optimal placement of spring constants for minimization of f_1 and f_2. In the third mode; when f_1 is minimized, the optimal spring constants are found to be k_3=1.45608 10^6N/m and k_4=1.0775 10^8 N/m. The stiffness constants of other springs are found to be zero. At this stage; it can be seen that total spring constant value is increased, in order to control the third mode. At the end of design, most of the total stiffness is added to the spring at the 4th node; while a small part of it is added to the spring at the 3rd node. The optimal spring design which minimizes f_2 in the third mode is shown in Fig. 2(e). According to optimal design result which minimizes f_2, it is found that total spring stiffness, $\overline{K}$=9.6465 10^7 N/m should be added to the spring at the 4th node. When this design is compared with the optimal design based on tip deflection, total $\overline{K}$ is added only to the 4th node. In addition; total spring constant, $\overline{K}$ is smaller than the total spring constant for tip deflection minimization.

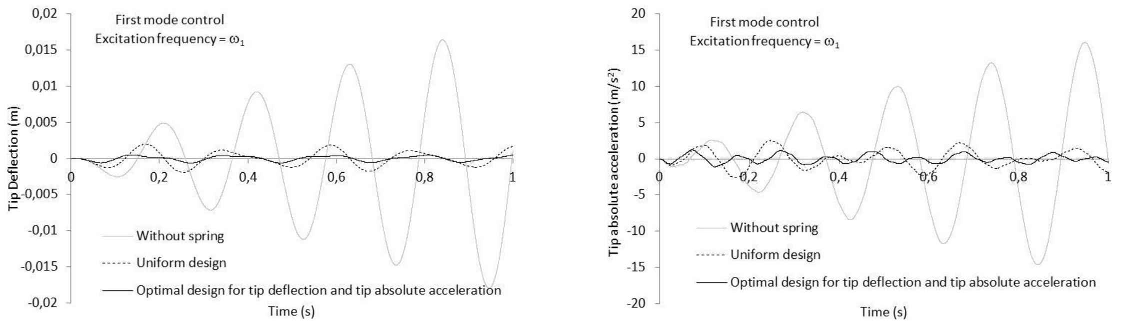
In the numerical examples, the total stiffness $\overline{K}$ changes according to both types of the objective functions and the mode number of the beam. Increase in total stiffness constant $\overline{K}$ generally decreases the objective function until convergence is satisfied. The objective functions increase after that point, if the total stiffness is increased. In this point, when the convergence is satisfied, the algorithm should be stopped. This situation causes various in $\overline{K}$ values of optimal design in each one of two objective functions and in each one of the different mode.

In Figs. 3(c) and 4(c), the change of f_1 and f_2 according to the design steps at all three designs are shown for the third mode control case. It is observed in Fig. 5(c) that optimal designs increase third natural circular frequency, ω_3 better compared to uniform design.

Figs. 6(c) and 7(c) show the variation of first order sensitivities of f_1 and f_2 with respect to design variables during third mode control. It can be seen that convergence is satisfied for both of the optimizations.

Fig. 8(c) shows the change in f_1 with respect to excitation frequency optimal design and other designs calculated for third mode displacement control. When the third resonance region is examined, the resonance point for the case of without spring decreases significantly after being controlled regarding f_1. Optimal design evaluated for the third mode shows a good performance in the second resonance region, while it has a bad performance in the first resonance region for f_1.

The change of f_2 with respect to excitation frequency can be seen in Fig. 9 (c) for the control of third mode. It is observed that optimal design shows a significantly better performance than design with no spring and uniform design.



(a) (b)

Fig. 10 Time history of the tip deflection of the beam under the sinusoidal base excitation for the first mode control

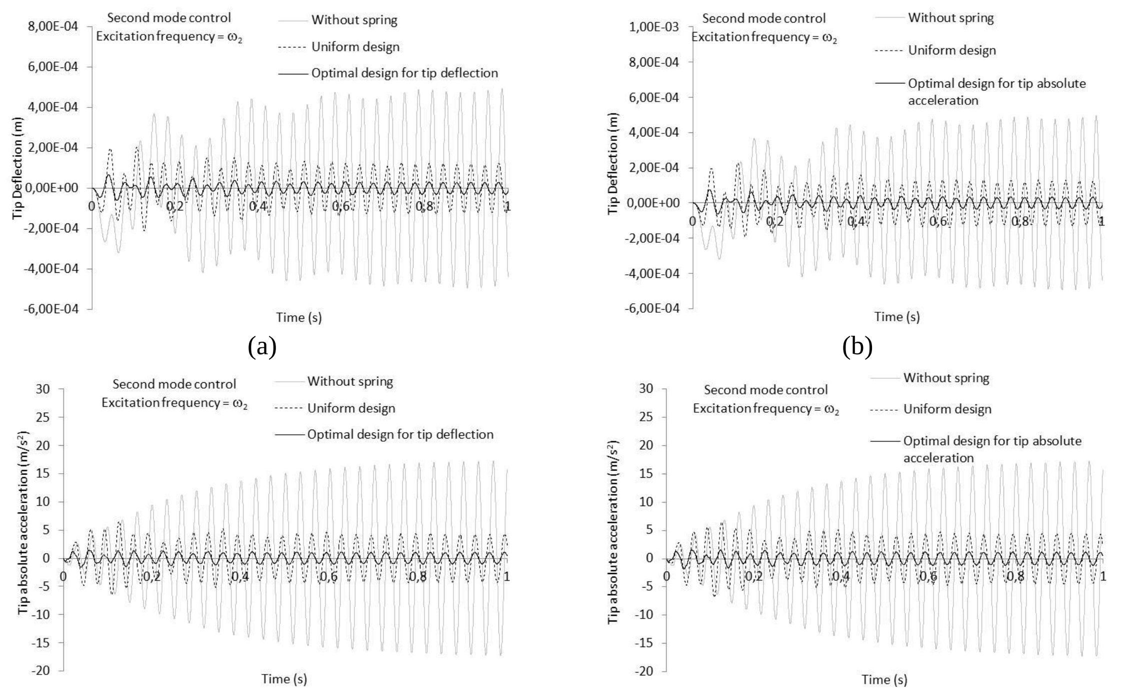


Fig. 11 Time history of the tip deflection of the beam under the sinusoidal base excitation for the second mode control

In this paper, optimum locations and sizes of springs supporting a cantilever beam are investigated for the first three modes to minimize the transfer function amplitude of tip deflection and tip absolute acceleration. At the optimization stages applied for the control of first three modes; f_1, f_2, first three natural circular frequencies and their first order sensitivities with respect to design variables are provided. In addition; the calculated optimal design is compared with design with no spring and uniform design cases considering excitation frequency parameter for the first three modes.

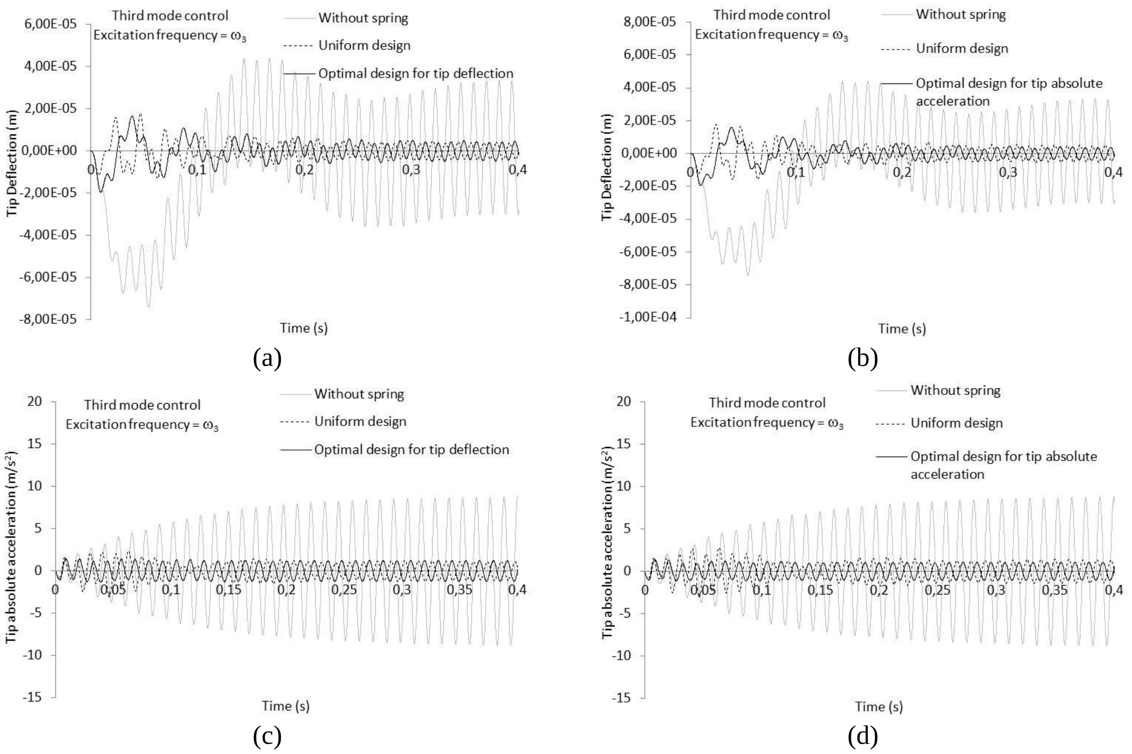


Fig. 12 Time history of the tip deflection of the beam under the sinusoidal base excitation for the third mode control

In addition; time history analyses are performed under a sinusoidal support movement to test the calculated optimum spring designs. Initially, tip deflection and absolute acceleration of optimal designs for the control of first mode and other designs are examined under base acceleration, $\ddot{u}_g = \sin \omega_1 t$ which defines the resonance case. In Figs. 10(a) and 10(b), it is seen that optimal designs which are calculated both for displacement and acceleration, and are equal to each other show a significantly good performance at resonance situation both for tip deflection and tip absolute acceleration. The optimal designs calculated for the second mode are tested using time history analyses in Figs. 11 (a) - (d) for the selected base excitation, $\ddot{u}_g = \sin \omega_2 t$. It is observed that optimal design reduces the behaviour of both tip deflection and tip absolute acceleration considerably. Optimal designs at the third mode control are tested by time history analyses under the selected base excitation, $\ddot{u}_g = \sin \omega_3 t$, and it is observed in Figs. 12 (a)-(d) that they show a remarkably good performance for both tip deflection and tip absolute acceleration.

The resonance presents the worst case in terms of structural response in the dynamic analyses. The engineering design problems are generally based the resonance cases. In this study, three excitation frequencies, which is equal the first three natural frequencies of the beam are selected. The proposed method can be applied to obtain optimal design for the any other modes. One of the structural modes can be used to obtain the optimal design. Then the optimal design obtained can be tested using harmonic loadings according to that mode.

5. Conclusions

Support conditions of beams change their dynamic responses which are encountered for many engineering problems. In this paper; optimal locations and sizes of springs which support a cantilever Timoshenko beam for the first three modes are investigated. In the optimization problem; transfer function amplitudes of both tip deflection and tip absolute acceleration are selected as the objective functions. Objective functions are minimized in order to find the optimal spring distribution. While SDSA method proposed by Takewaki (1998) is used for the minimization of tip deflection transfer function amplitude at the first mode, a simple first order approximation is proposed in this paper for the minimization of tip deflection in the second and third modes. Moreover, a simple first order algorithm is proposed to minimize the transfer function of tip absolute acceleration. Some sensitivity equations needed for optimization are derived. Optimal designs on a cantilever Timoshenko beam are examined and numerical results are plotted. Optimal designs calculated for two different objective functions for the first three modes of beam with 12 degrees of freedom are found. Both frequency responses and time responses of the calculated optimal designs are investigated, and it is shown that they give a good performance. The following conclusions can be drawn according to the numerical analyses results:

• In the first mode, the spring designs, which minimize both tip deflection and tip absolute acceleration, are equivalent regarding both spring locations and their sizes.

• In the second mode, even though designs which minimize both of the defined objective functions have the same support location for springs, they show a difference at the total spring stiffness coefficients. The total stiffness, $\overline{K}$ value, which is calculated for the optimization of tip absolute acceleration, is less.

• In the third mode, for the calculated design to minimize tip deflection while total stiffness, $\overline{K}$ is allocated at the third and fourth nodes (mostly at the fourth node, while a small part of it is allocated at the third node.), for the design which minimize tip absolute acceleration $\overline{K}$ is located only at the fourth node. In addition; total stiffness, $\overline{K}$ values are different for each one of two optimizations.

• The calculated optimal designs and defined uniform designs are also investigated in terms of frequency behaviours and time responses. Especially at the resonance cases, the behaviours are examined for the first three modes, and it is shown that optimal designs reduce the defined dynamic behaviours effectively.

In this paper; the optimal design of springs referring to any mode behaviour, spring locations and sizes which minimize tip deflection and tip absolute acceleration for cantilever beams are investigated. The methods taken from the literature and the proposed methods in this paper are for general use; and they can be applied on other structural systems.

References

Akesson, B. and Olhof, N. (1988), “Minimum stiffness of optimally located supports for maximum value of beam eigenfrequencies”, *J. Sound. Vib.*, **120**(3), 457-463.

Albaracin, J.M., Zannier, L. and Gross, R.O. (2004), “Some observations in the dynamics of beams with intermediate supports”, *J. Sound. Vib.*, **271**, 475-480.

Aydin, E. and Boduroglu, M.H. (2008), “Optimal placement of steel diagonal braces for upgrading the seismic capacity of existing structures and its comparison with optimal dampers”, *J. Construct. Steel Res.*,

64, 72-86.

Aydin, E., Boduroglu, M.H. and Guney, D. (2007), "Optimal damper distribution for seismic rehabilitation of planar building structures", *Engineering Structures*, **29**, 176-185.

Bojczuk, D. and Mroz, Z. (1998), "On optimal design of supports in beam and frame structures", *Struct. Optim.*, **16**, 47-57.

Bojczuk, D. and Mroz, Z. (1998), "Optimal design of trusses with account for topology variation", *Mech. Struct. Mach.*, **26**(1), 21-40.

Buhl, T. (2001), "Simultaneous topology optimization of structure and supports", *Struct. Multidiscip. Optim.*, **23**, 336-346.

Ching, H.C. and Gene, J.W.H. (1992), "Eigenvalue sensitivity analysis of planar frames with variable joint and support locations", *AIAA J.*, **30**(8) 2138-3147.

Cimellaro, G.P. (2007), "Simultaneous stiffness-damping optimization of structures with respect to acceleration, displacement and base shear", *Eng. Struct.*, **29**, 2853-2870.

Dems, K. and Turant, J. (1997), "Sensitivity analysis and optimal design of elastic hinges and supports in beam and frame structures", *Mech. Struct. Mach.*, **25**(4), 417-443.

Fayyah, M.M. and Razak, H.A. (2012), "Condition assessment of elastic bearing supports using vibration data", *Construct. Build. Mater.*, **30**, 616-628.

Frishwell, M.I. and Wang, D. (2007), "The minimum support stiffness required to raise the fundamental natural frequency of plate structures", *J. Sound. Vib.*, **301**, 665-677.

Garstecki, A. and Mroz, Z. (1987), "Optimal design of supports of elastic structures subject to loads and initial distortions", *Mech. Struct. Mach*., **15**, 47-68.

Imam, M.H. and Al-Shihri, M. (1996), "Optimum topology of structural supports", *Comput. Struct.*, **61**(1), 147-154.

Jihang, Z. and Weighong, Z. (2006), "Maximization of structural natural frequency with optimal support layout", *Struct. Multidisc. Optim.*, **31**, 462-469.

Kong, J. (2009), "Vibration of isotropic and composite plates using computed shape function and its application to elastic support optimization", *J. Sound. Vib.*, **326**, 671-686.

Lin, H.Y. (2010), "An exact solution for free vibrations of a non-uniform beam carrying multiple elastic supported rigid bars", *Struct. Eng. Mech.*, **34**(4), 399-416.

Liu, Z.S., Hu, H.C. and Wang, D.J. (1996), "New method for deriving eigenvalue rate with respect to support locations", *AIAA J.*, **34**, 864-866.

Marcelin, J.L. (2001), "Genetic search applied to selecting support position in machining of mechanical parts", *Int. J. Adv. Manufact. Technol.*, **17**, 344-347.

Mroz, Z. and Lekszycki, T. (1982), "Optimal support reaction in elastic frame structures", *Comput. Struct.*, **14**, 179-185.

Mroz, Z. and Haftka, R.T. (1994), "Design sensitivity of non-linear structures in regular and critical states", *Int. J. Solid. Struct.*, **31**, 2071-2098.

Mroz, Z. and Rozvany, G.I.N. (1975), "Optimal design of structures with variable support positions", *J. Opt. Theory Appl.*, **15,** 85-101.

Olhof, N. and Akessson, B. (1991), "Minimum stiffness of optimally located supports for maximum value of column buckling loads", *Struct. Multidiscip. Optim.*, **3**, 163-175.

Olhof, N. and Taylor, J.E. (1998), "Designing continuous columns for minimum total cost of material and interior supports", *J. Struct. Mech.*, **6**(4), 367-382.

Prager, W. and Rozvany, G.I.N. (1975), "Plastic design of beams; optimal locations of supports and steps in yield moment", *Int. J. Mech. Sci*, **17**, 627-631.

Przemieniecki, J.S. (1968), *Theory of Matrix Structural Analysis*, McGraw-Hill, Newyork.

Rozvany, G.I.N. (1975), "Analytical treatment of some extended problems in structural optimization", *J. Struct. Mech.*, **1**, 359-385.

Sinha, J.K. and Frishwell, M.I. (2001), "The location of spring supports from measured vibration data", *J. Sound. Vib.*, **244**(1), 137-153.

Son, J.H. and Kwak, B.M. (1993), "Optimization of boundary conditions for maximum fundamental

frequency of vibrating structures", *AIAA J.*, **31**(12), 2351-2357.
Szelag, D. and Mroz, Z. (1978), "Optimal design of elastic beams with unspecified support positions", *ZAMM*, **58**, 501-510.
Takewaki, I. (1998), "Optimal damper positioning in beams for minimum dynamic compliance", *Comput. Method. Appl. Mech. Eng.*, **156**, 363-373.
Wang, B.P. (1993), "Eigenvalue sensitivity with respect to location of internal stiffness and mass attachment", *AIAA J.*, **31**, 791-794.
Wang, B.P. and Chen, J.L. (1996), "Application of genetic algorithm for the support location optimization of beams", *Comput. Struct.*, **58**(4), 597-800.
Wang, C.Y. (2003), "Minimum stiffness of an internal elastic support to maximize the fundamental frequency of a vibrating beam", *J. Sound. Vib.*, **259**(1), 229-232.
Wang, D. (2004), "Optimization of support positions to minimize the maximal deflection of structures", *Int. J. Solid. Struct.*, **41**, 7445-7458.
Wang, D. (2006), "Optimal design of structural support positions for minimizing maximal bending moment", *Finite Elem. Anal. Des.*, **43**, 95-102.
Wang, D., Frishwell, M.I. and Lei, Y. (2006), "Maximizing the natural frequency of a beam with an intermediate elastic support", *J. Sound. Vib.*, **291**, 1229-1238.
Wang, D., Jiang, J.S. and Zhang, W.H. (2004), "Optimization of support positions to maximize the fundamental frequency of structures", *Int. J. Numer. Method. Eng.*, **61**, 1584-1602.
Wang, D., Yang, Z.C. and Yu Z.G. (2010), "Minimum stiffness location of point support for control of fundamental natural frequency of rectangular plate by Rayleigh-Ritz method", *J. Sound. Vib.*, **329**, 2792-2808.
Won, K.M. and Park, Y.S. (1998), "Optimal support positions for a structure to maximize its fundamental natural frequency", *J. Sound. Vib.*, **213**(5), 801-812.
Zhu, J.H. and Zhang, W.H., (2010), "Integrated layout design of supports and structures", *Comput. Method. Appl. Mech. Eng.*, **199**, 557-569.