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Stochastic interpolation of earthquake ground motions under spectral uncertainties

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Abstract. Closed-form solutions are analytically derived for stochastic properties of earthquake ground motion fields, which are conditioned by an observed time series at certain observation sites and are characterized by spectra with uncertainties. The theoretical framework presented here can estimate not only the expectations of such simulated earthquake ground motions, but also the prediction errors which offer important information for the field of engineering. Before these derivations are made, the theory of conditional random fields is summarized for convenience in this study. Furthermore, a method for stochastic interpolation of power spectra is explained.

Key words: conditional random fields; power spectrum; spectral uncertainties; stochastic interpolation; numerical simulation; conditional probability density function; estimation errors; Fourier coefficients.

1. Introduction

The theories or methodologies regarding random fields with the observed data as conditions, which we call "conditional random fields" (CRF), have been investigated from various standpoints by many researchers (for example, Kawakami, *et al.* 1989, 1992, Borgman 1990, Ditlevsen 1991, Vanmarcke and Fenton 1991, Kameda and Morikawa 1992, 1994). The majority of these methodologies, however, requires a priori information about the spectral characteristics of random fields for their formulations. According to the circumstances, the requirements for the formulations may restrict the applicability of the methodology to real phenomena. Thus, Morikawa and Kameda (1996) have proposed a method for the stochastical interpolation of power spectra using the observed earthquake ground motions. An earthquake ground motion field, conditioned by a time series and recorded at certain observation sites, can be simulated with their results. The following question then arises: How trustworthy are the simulated ground motions?

It is important to offer answers to this question, because such information is often used as a basis for decision making in various engineering problems. The purpose of this study is to analytically derive the stochastic properties of CRF with spectral uncertainties. Before we discuss

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them in detail, in the following sections, the theory of CRF is summarized within the limits of the information needed for this study. Then, we explain a method for the stochastic interpolation of power spectra using the observed earthquak ground motions.

2. Summary of the theory of conditional random fields

The theory of conditional random fields deals with a stochastic time-space field that is conditioned by deterministic time functions observed at certain discrete locations in space (Kameda and Morikawa 1992, 1994, Morikawa and Kameda 1993). The theory is summarized in this section within the limits of the information needed for this study. For a more detailed description and relevant studies, the reader is referred to a recent publication by Kameda and Morikawa (1994).

2.1. Formulation

The conditional random field in this study is represented in terms of a multi-variate stochastic process that is conditioned by a set of deterministic time functions. This type of representation facilitates the analytical treatment of probabilistic conditioning. It is assumed that the stochastic processes, before they are conditioned by deterministic time functions, are zero-mean stationary Gaussian processes.

The multi-variate conditioned stochastic process is represented by $U_j(t|u_i(t);\ i=1,2,\ldots,m)$; $j=m+1,\ldots,n$ at site j. Here, $u_i(t)$ stands for the deterministic time function specified at site i. For simplicity, $U_j(t|u_i(t);\ i=1,2,\ldots,m)$ will be denoted by $U_j(t|cnd.)$, in which the word "*cnd.*" means that the quantity under discussion is conditioned by $u_i(t)$; $i=1,2,\ldots,m$. This term will be used in the same context for other quantities. Without loss of generality, we hereafter assume that $m=n-1$ and discuss the conditioned stochastic process $U_n(t|cnd.)$. This is because its stochastic characters are independent of the site numbers and can be treated just as if it were a deterministic time function, once its stochastic characters have been clarified.

Stationary Gaussian processes such as $U_i(t)$; $i=1,2,\ldots,n$ can generally be expanded in a Fourier series as

$$U_i(t)=\sum_k (A_{ik}\cos\omega_k t+B_{ik}\sin\omega_k t) \qquad (i=1,2,\ldots,n). \tag{1}$$

Fourier coefficients A_{ik} and B_{ik} at frequency $\omega_k=2\pi f_k\,(i=1,2,\ldots,n$, for a fixed $k)$ are mutually independent zero-mean Gaussian random variables, and their covariance matrix, Λ_k, is repesented by S_{iik} and S_{ijk} in which $S_{iik}=S_{ii}(\omega_k)$ is a power spectral density function associated with $U_i(t)$ at frequency ω_k and $S_{ijk}=S_{ij}(\omega_k)$ is a cross spectral density function between $U_i(t)$ and $U_j(t)$ at frequency ω_k (Kameda and Morikawa 1992).

The deterministic time functions used for conditioning, $u_i(t)$; $i=1,2,\ldots,n-1$, are treated as realized values in the random field. They can be represented in a similar way to Eq. (1) as

$$u_i(t)=\sum_k (\tilde{a}_{ik}\cos\omega_k t+\tilde{b}_{ik}\sin\omega_k t) \qquad (i=1,2,\ldots,n-1), \tag{2}$$

where $\tilde{a}_{ik}$ and $\tilde{b}_{ik}$ denote realized values of Fourier coefficients corresponding to A_{ik} and B_{ik}, respectively.

2.2. Analytical derivation of basic parameters

The $2n$-dimensional joint Gaussian probability density function (PDF) is derived for Fourier coefficients A_{ik} and B_{ik} ($i=1, 2, ..., n$) at frequency ω_k by means of their covariance matrix Λ_k. The 2-dimensional conditional joint PDF can then be obtained for Fourier coefficients A_{nk} and B_{nk} conditioned by realized values $\tilde{a}_{ik}$ and $\tilde{b}_{ik}$ ($i=1, 2, ..., n-1$).

It has been analytically proven that conditioned Fourier coefficients A_{nk} and B_{nk} are mutually independent Gaussian variables (Kameda and Morikawa 1992). Modifying the results derived by Kameda and Morikawa (1992), with respect to conditional mean values $\langle A_{nk}|cnd.\rangle$ and $\langle B_{nk}|cnd.\rangle$, the following relations can be obtained between the conditional mean values and power spectra S_{nnk} at site n:

$$\langle A_{nk}|cnd.\rangle = \alpha_{A_k}\sqrt{S_{nnk}} \tag{3a}$$

$$\langle B_{nk}|cnd.\rangle = \alpha_{B_k}\sqrt{S_{nnk}}, \tag{3b}$$

where $\alpha_{A_{nk}}$ and $\alpha_{B_{nk}}$ are defined as functions of $\tilde{a}_{ik}$, $\tilde{b}_{ik}$, and S_{ijk} ($i=1, 2, ..., n-1, j=1, 2, ..., n$). In the same way, conditional variances $\sigma^2_{A_{nk}|cnd.}$ and $\sigma^2_{B_{nk}|cnd.}$ are written as

$$\sigma^2_{A_{nk}|cnd.} = \sigma^2_{B_{nk}|cnd.} = \sigma^2_{nk|cnd.} = \alpha_k S_{nnk}, \tag{4}$$

where α_k depends only upon S_{ijk}. A detailed derivation of Eqs. (3) and (4) is shown in Appendix A.

From Eqs. (1), (3), and (4), the conditional mean value and the variance of conditioned stochastic process $U_n(t|cnd.)$ yield, respectively,

$$\mu_{U_n|cnd.}(t) = \sum_k \{\sqrt{S_{nnk}}(\alpha_{A_k} \cos\omega_k t + \alpha_{B_k} \sin\omega_k t)\} \tag{5}$$

$$\sigma^2_{U_n|cnd.} = \sum_k \alpha_k S_{nnk}. \tag{6}$$

3. Stochastic interpolation of ground motion spectra

A method for stochastic interpolation of ground motion spectra, based on the work of Morikawa and Kameda (1996), is summarized here considering the assumption that the ground structure is weakly inhomogeneous.

In order to consider the variety of ground structures, the stochastic spectra in this method are divided into terms reflecting the deterministic feature of the ground structure and the randomness from the deterministic term. Thus, power spectrum $S_{ii}(f)$ at site i ($i=1, 2, ..., n$) is represented by $S_{ii}(f)=S^r_{ii}(f)\,S^{\tilde{f}}_{ii}(f)$, where $S^r_{ii}(f)=S^s_{ii}(f)/S^s_{11}(f)$, $S^s_{ii}(f)$ denotes the deterministic site effects which are indepndent of events, and $S^{\tilde{f}}_{ii}(f)$ is a random component with different stochastic properties for every event.

If $U_i(t)$ is a stationary Gaussian process with zero-mean, its "raw" power spectrum follows the chi-squared distribution with two freedoms. However, since power spectrum $S_{ii}(f)$ is ordinarily estimated in smoothed values for the "raw" power spectrum, its probability distribution may be a χ^2 distribution with many freedoms. Thus, it is substituted it with the log-normal distribution in order to treat the power spectra as stochastic processes in the frequency domain. The marginal PDF of power spectrum S_{iik} is then represented as

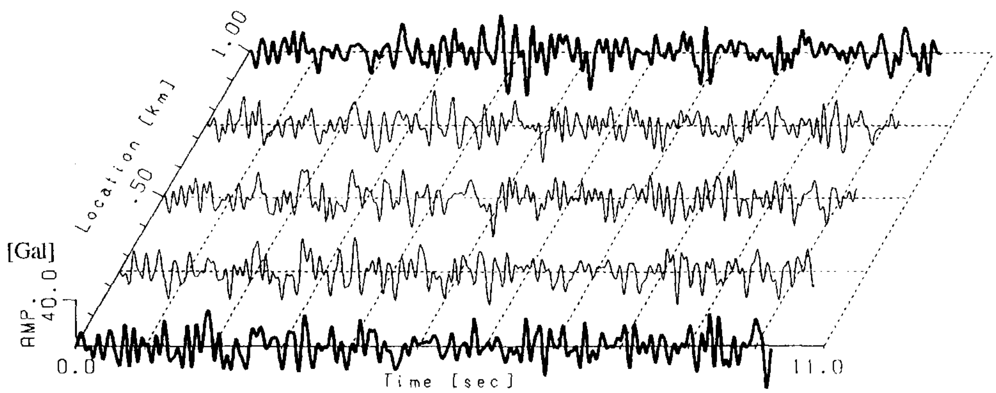


Fig. 1 An example of the psudo-earthquake ground motion field.

$$f_{S_{iik}}(s_{iik}) = \frac{1}{\sqrt{2\pi}\zeta_{ik}} \exp\left[-\frac{1}{2}\left(\frac{\ln s_{iik} - \lambda_{ik}}{\zeta_{ik}}\right)^2\right] \qquad (i=1,\ 2,\ \dots,\ n). \tag{7}$$

Hereafter, we treat the stationary Gaussian processes derived from the logarithm of $S_{ii}(f)$ as

$$\ln S_{ii}(f) = \ln S^r_{ii}(f) + \ln S^{\bar{r}}_{ii}(f) \qquad (i=1,\ 2,\ \dots,\ n). \tag{8}$$

On the one hand, $\ln S^r_{nn}(f)$ is determined at site n where there is no information on the ground motion, using the linear interpolation of observed values at site i ($i=1, 2, \dots, n-1$). This is due to the supposition that $S^r_{ii}(f)$ is a deterministic value changing gradually in the study area. On the other hand, the stochastic properties of $\ln S^{\bar{r}}_{nn}(f)$, conditioned by observed values, are immediately obtained through the theory of CRF.

From the above procedure, we can estimate site effect $S^r_{nn}(f)$ and random component $S^{\bar{r}}_{nn}(f)$ at site n even though we have no a priori information with respect to the spectral characteristics. Therefore, the conditional properties of $S_{nn}(f)$ will be deermined by $\lambda_{nk|cnd}$ and $\zeta_{nk|cnd}$, combining $S^r_{nn}(f)$ and $S^{\bar{r}}_{nn}(f)$ as follows:

$$\lambda_{nk|cnd.} = \ln \hat{S}^r_{nn}(f_k) + E[\ln S^{\bar{r}}_{nn}(f_k)|cnd.], \tag{9a}$$

$$\zeta^2_{nk|cnd.} = \mathrm{Var}[\ln S^{\bar{r}}_{nn}(f_k)|cnd.], \tag{9b}$$

where $\hat{S}^r_{nn}(f)$ is an estimated value of $S^r_{nn}(f)$. It should be noted that in this method the prediction errors for the power spectra can be estimated using Eq. (9).

4. Conditioned stochastic processes with random spectra

At first, Fig. 1 shows an example of the psudo-earthquake ground motion field simulatd under the power spectra obtained in the previous section using the simulation technique of conditional random fields.

Essentially, the cross spectra using the observed time series must be identified before the simulation of the wave fields. For the sake of convenience, however, a certain cross spectrum is used as a target in this simulation, because sufficient investigations have not been conducted to identify the cross spectra, as mentioned in a previous section. The simulated wave field shown in this figure may include at least two estimation errors: one caused by the estimation of the random power spectra and the other caused by the estimation of wave fields. In this section, we discuss errors in the estimation of the wave fields with random power spectra, leaving the way to identify cross spectra to future studies.

From various points of view, some researchers have established a methodology for the derivation of the estimation errors of the conditional random fields obtained through the spectra given deterministically as a priori information (Kawakami and Ono 1992, Vanmarcke and Fenton 1991, Kameda and Morikawa 1992, 1994). However, such a methodology is not sufficient for considering random fields with stochastic spectra. Thus, we extend straightforwardly the theory of conditional random fields and present the theoretical framework of the fields with stochastic spectra. In cases where we know deterministically the spectra at site n, for which observed data is not available, the conditional mean and variance of conditioned stochastic process $U_n(t|cnd.)$ are represented by Eqs. (5) and (6), as shown in Section 2. In the case of $U_n(t|cnd.)$ with stochastic spectra, $\mu_{U_n|cnd.}(t)$ and $\sigma^2_{U_n|cnd.}$ in Eqs. (5) and (6), respectively, must be treated as random variables in order to be dependent on the probabilistic distribution of power spectrum S_{nnk} introduced in Eq. (7). Their probabilistic distributions are derived analytically.

The PDFs of conditional mean $\langle A_{nk}|cnd.\rangle$ and conditional variance $\sigma^2_{nk|cnd.}$ of the Fourier coefficient are discussed. The PDF of $\langle B_{nk}|cnd.\rangle$ may be derived by replacing *A with B*. Incorporating Eqs. (3) and (4) into Eq. (7) and making the change in variables for the PDF, the following equations are obtained:

$$f_{\mu_{Ank}}(\mu_{Ank})=\frac{1}{\sqrt{2\pi}\frac{\zeta_{nk}}{2}}\frac{1}{\mu_{Ank}}\cdot\exp\left[-\frac{1}{2}\left(\frac{\ln\left(\frac{\mu_{Ank}}{\alpha_{Ak}}\right)-\frac{\lambda_{nk}}{2}}{\frac{\zeta_{nk}}{2}}\right)^2\right] \tag{10a}$$

$$f_{\upsilon_{nk}}(\upsilon_{nk})=\frac{1}{\sqrt{2\pi}\zeta_{nk}}\frac{1}{\upsilon_{nk}}\cdot\exp\left[-\frac{1}{2}\left(\frac{\ln\left(\frac{\upsilon_{nk}}{\alpha_k}\right)-\lambda_{nk}}{\zeta_{nk}}\right)^2\right], \tag{10b}$$

where $\mu_{Ank}\equiv\langle A_{nk}|cnd.\rangle$ and $\upsilon_{nk}\equiv\sigma^2_{nk|cnd.}$. From the moment of the first or second order in Eq. (10), the expected values and variances are derived as

$$E\left[\mu_{Ank}\right]=\frac{\alpha_{Ak}}{\exp\left[\frac{\zeta^2_{nk}}{8}\right]}\sqrt{E\left[S_{nnk}\right]} \tag{11a}$$

$$\mathrm{Var}\left[\mu_{Ank}\right]=\alpha^2_{Ak}\,e^{\lambda_{nk}}\left(e^{\frac{\zeta^2_{nk}}{2}}-e^{\frac{\zeta^2_{nk}}{4}}\right) \tag{11b}$$

$$E\left[\upsilon_{nk}\right]=\alpha_k\,E\left[S_{nnk}\right] \tag{11c}$$

$$\mathrm{Var}\left[\upsilon_{nk}\right]=\alpha^2_k\,\mathrm{Var}\left[S_{nnk}\right]. \tag{11d}$$

As the next step, the expectations and variances for each mean $\mu_{U_n|cnd.}(t)$ and variance $\sigma^2_{U_n|cnd.}$ of the conditional stochastic process $U_n(t|cnd.)$ are discussed, using Eq. (11), at the site where observed data is not available. The data is derived from the central limit theorem and variances are Gaussian variables because μ_{Ank} and υ_{nk} can be represented by a linear combination of the harmonic components shown in Eqs. (5) and (6). From Eqs. (5), (11a), and (11b), conditional mean value $\mu_{U_n|cnd.}(t)$ of conditional stochastic processes $U_n(t|cnd.)$ is a non-stationary Gaussian process with the following time-varying expectation and variance:

$$E\left[\mu_{U_n|cnd.}(t)\right]=\sum_k exp\left[\frac{\lambda_{nk}}{2}+\frac{\zeta^2_{nk}}{8}\right]\cdot\{\alpha_{A_k}\ \cos\omega_k t+\alpha_{B_n}\ \sin\omega_k t\} \tag{12a}$$

$$\mathrm{Var}\left[\mu_{U_n|cnd.}(t)\right]=\sum_k e^{\lambda_{nk}}(e^{\frac{\zeta_{nk}}{2}}-e^{\frac{\zeta_{nk}}{4}})\cdot\{\alpha^2_{A_k}\ \cos^2\ \omega_k t+\alpha^2_{B_k}+\alpha^2_{B_k}\ \sin^2\ \omega_k t\}. \tag{12b}$$

From Eqs. (6), (11c), and (11d), conditional variances $\sigma^2_{U_n|cnd.}$ of $U_n(t|cnd.)$ are Gaussian variables with the following expectation and variance:

$$E\left[\sigma^2_{U_n|cnd.}\right]=\sum_k \alpha_k\ \exp\left[\frac{\zeta^2_{nk}}{2}\right] \tag{13a}$$

$$\mathrm{Var}\left[\sigma^2_{U_n|cnd.}\right]=\sum_k\ \alpha^2_k\,(e^{\zeta^2_{nk}}-1)\cdot\exp\ \left[2\lambda_{nk}+\zeta^2_{nk}\right]. \tag{13b}$$

Furthermore, the probabilistic distribution of standard deviation $\sigma_{U_n|cnd.}$ of $U_n(t|cnd.)$ can be derived using Eq. (13) as

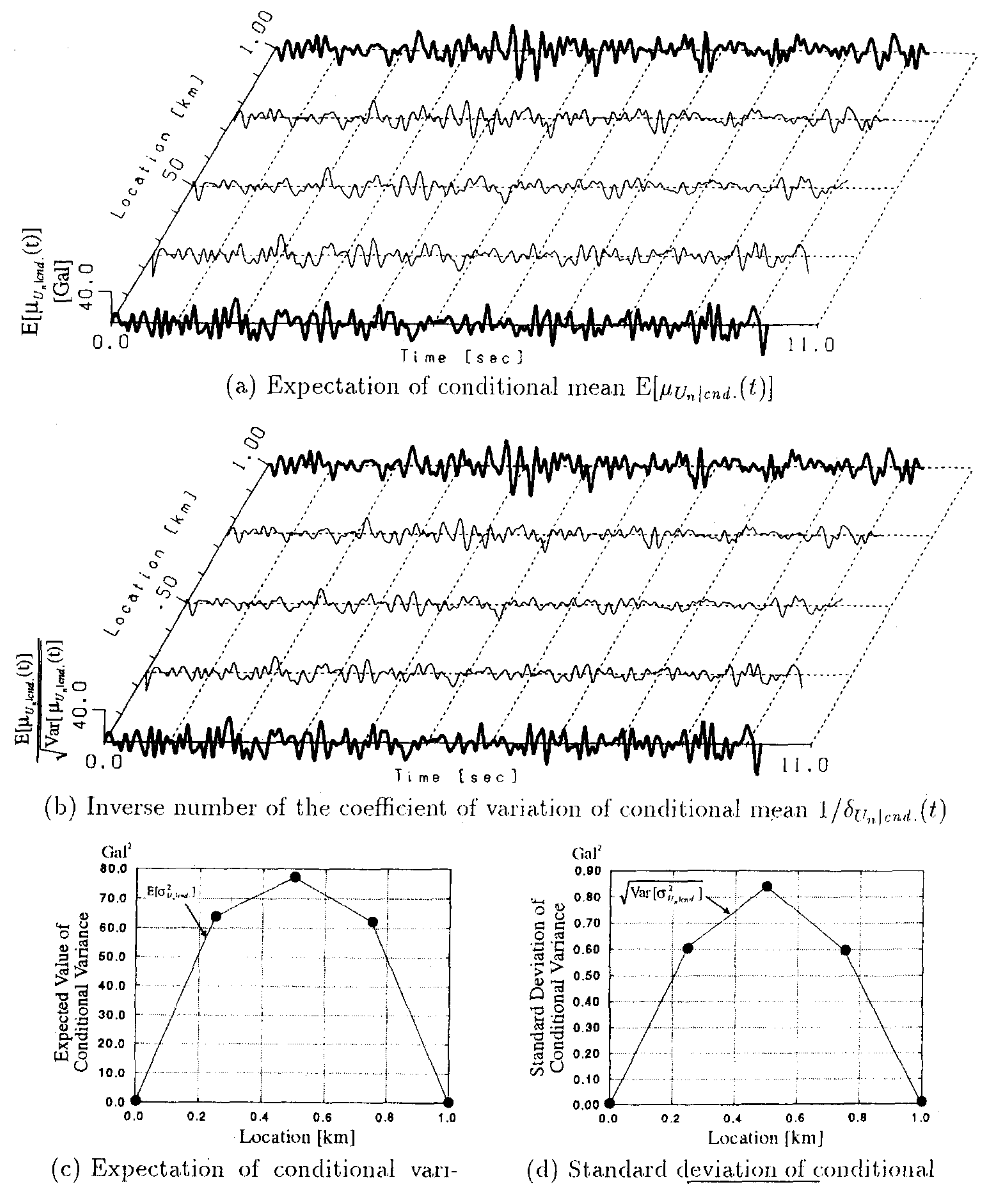
$$f_{\eta_{U_n}}(\eta_{U_n})=\frac{2\eta_{U_n}}{\sqrt{2\pi}\mathrm{Var}[\sigma^2_{U_n|cnd.}]}\cdot\exp\left[-\frac{1}{2}\left(\frac{\eta^2_{U_n}-E\left[\sigma^2_{U_n|cnd.}\right]}{\mathrm{Var}[\sigma^2_{U_n|cnd.}]}\right)^2\right], \tag{14}$$

where $\eta_{U_n}\equiv\sigma_{U_n|cnd.}$. If it is necessary to explicitly describe the moment of η_{U_n} of the first or second order, it can be done (see Appendix B). However, in the case of a numerical calculation of the expectation and variance of η_{U_n} by means of a digital computer, it may be prompter to use a numerical integration instead of the moment of η_{U_n} which is analytically derived.

For cases in which S_{nnk} is given deterministically as $S_{nnk}=\hat{S}_{nnk}$, it has been proven that mean value $\mu_{U_n|cnd.}(t)$ and variance $\sigma^2_{U_n|cnd.}$, which are represented by Eqs. (5) and (6), respectively, for the conditioned stochastic process at site n, $U_n(t|cnd.)$, coincide with $E\left[\mu_{U_n|cnd.}(t)\right]$ in Eq. (12a) $E\left[\sigma^2_{U_n|cnd.}\right]$ in Eq. (13a), respectively, and variances $\mathrm{Var}[\mu_{U_n|cnd.}(t)]$ and $\mathrm{Var}[\sigma^2_{U_n|cnd.}]$ are identified with zero (see Appendix C). This means that no contradiction exists between the theory of conditional random fields and the extended one.

In order to visualize the above analytical results, we numerically calculate the conditional mean values and variances using the set of data presented in Fig. 1. Fig. 2 shows an example of the expectations and uncertainties of the conditional mean values and variances for the conditioned stochastic processes. In this figure, the uncertainties of $\mu_{U_n|cnd.}(t)$ are represented by on inverse number of the coefficient of variation, $1/\delta_{U_n|cnd.}(t)$. The expectation of conditional mean values $E\left[\mu_{U_n|cnd.}(t)\right]$ and $1/\delta_{U_n|cnd.}(t)$ converge to zero with the distance between the observation site and the estimating site. The tendencies of the expectations are similar to those of the conditional mean values calculated under the 'deterministic power spectra.' Furthermore, the values for $1/\delta_{U_n|cnd.}(t)$ show that the errors in estimation for $\mu_{U_n|cnd.}(t)$ are relatively the largest at the midmost site. The expectation and standard deviation of the conditional variance, $\sigma^2_{U_n|cnd.}$, increase with the distance from the observation site.

Note that in this calculation, the standard deviation of the conditional mean values and variances, namely, $\sqrt{\mathrm{Var}[\mu_{U_n|cnd.}(t)]}$ and $\sqrt{\mathrm{Var}\left[\sigma^2_{U_n|cnd.}\right]}$, are very small in comparison to expectations $E\left[\mu_{U_n|cnd.}(t)\right]$ and $E\left[\sigma^2_{U_n|cnd.}\right]$. $\mathrm{Var}[\mu_{U_n|cnd.}(t)]$ and $\mathrm{Var}[\sigma^2_{U_n|cnd.}]$ depend on the coherence (in quefrency domain) of the power spectra between any two sites. This suggests that we can estimate



(a) Expectation of conditional mean $E[\mu_{U_n|cnd.}(t)]$

(b) Inverse number of the coefficient of variation of conditional mean $1/\delta_{U_n|cnd.}(t)$

(c) Expectation of conditional variance $E[\sigma^2_{U_n|cnd.}]$

(d) Standard deviation of conditional variance $\sqrt{\mathrm{Var}[\sigma^2_{U_n|cnd.}]}$

Fig. 2 Expectations and uncertainties of the conditional mean values and variances for the conditioned stochastic processes.

wave fields with satisfactory accuracy even though the estimation errors caused by the spectra are not considered in cases where a high coherency in the quefrency domain is identified. Further investigation into this problem is needed, however, from the viewpoint of quantitative analysis.

5. Conclusions

The theory of conditional random fields and a method for stochastic interpolation of power spectra were summarized. The basic framework of stochastic properties was then analytically derived for conditional random fields with spectral uncertainties. From this, we discussed quantitatively the prediction errors, which are the uncertainties of estimated values for simulated earthquake ground motion fields. Furthermore, the derivation was justified using numerical calculations.

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Appendixes

A. Derivation of Eqs. (3) and (4)

We show a representation of the conditional mean and variance of the stochastic process at site n, which is conditioned by the deterministic time functions observed at $n-1$ sites. These values have been derived by Kameda and Morikawa (1992) using the elements of the inverse matrices of covariance matrices for Fourier coefficients. Based on the results shown in the reference (Kameda and Morikawa 1992), Eqs. (3) and (4) are derived from the calculation of inverse matrices.

A.1. Notations

The parameters and notations used throughout this section are defined.

$S_{ppk} \in \boldsymbol{IR}$	Power spectrum of a stochastic process at site p.
$S_{ppk} \in \boldsymbol{C}$	Cross spectrum between stochastic pocesses at site p and site q; $S_{pq}(w)=K_{pq}(w)+jQ_{pq}(w)=\sqrt{S_{pp}(w)S_{qq}(w)\,coh_{pq}(w)}\,e^{j\theta_{pq}(w)}$ (where $K_{pq}(w)$, $Q_{pq}(w) \in \boldsymbol{IR}$, $j=\sqrt{-1}$).
${}^{IR}\Lambda_k \in \boldsymbol{IR}^{2n}$	Covariance matrix of Fourier coefficients at frequency ω_k in real number.
$\Lambda_k \in \boldsymbol{C}^n$	Covariance matrix of Fourier coefficients at frequency ω_k in complex number.
$Z_{pk} \in \boldsymbol{C}$	Fourier coefficient in complex number at frequency ω_k and site p; $Z_{pk}=A_{pk}+jB_{pk}$ (where A_{pk}, $B_{pk} \in \boldsymbol{IR}$)
${}^t\boldsymbol{X}$	Transposed matrix of $\boldsymbol{X}$.
$[X]_{pq}$	(p, q) element of matrix X.
Z^*	Complex conjugate of Z.
$\boldsymbol{R}[Z]$	Real part of Z.
$\boldsymbol{F}[Z]$	Imaginary part of Z.
$\langle Z \rangle$	Expected value of Z.
σ^2_Z	Variance of Z.

A.2. Complex representation of conditional means and variances

At first, the conditional mean and variance of the stochastic process conditioned by the $n-1$

deterministic time functions, derived by Kameda and Morikawa (1992), are rewirtten concisely using complex numbers and matrices. Next, they are represented as a function of power spectrum S_{nnk} at site n.

The covariance matrix for Fourier coefficients A_{pk} and B_{pk} is represented by

$$\frac{{}^{IR}\Lambda_k}{\Delta w} = \begin{bmatrix} S_{11k} & 0 & \cdots & K_{1nk} & -Q_{1nk} \\ 0 & S_{11k} & \cdots & Q_{1nk} & K_{1nk} \\ K_{12k} & Q_{12k} & \cdots & K_{2nk} & -Q_{2nk} \\ -Q_{12k} & K_{12k} & \cdots & Q_{2nk} & K_{2nk} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ K_{1nk} & Q_{1nk} & \cdots & S_{nnk} & 0 \\ -Q_{1nk} & K_{1nk} & \cdots & 0 & S_{nnk} \end{bmatrix}. \tag{15}$$

Extending the definition of power and cross spectra by means of 2×2 matrices such as

$$\boldsymbol{S}_{ppk} = \begin{bmatrix} S_{ppk} & 0 \\ 0 & S_{ppk} \end{bmatrix}, \qquad \boldsymbol{S}_{pqk} = \begin{bmatrix} K_{pqk} & -Q_{pqk} \\ Q_{pqk} & K_{pqk} \end{bmatrix}, \tag{16}$$

unit matrix $\boldsymbol{E}$ and imaginary unit $\boldsymbol{J} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ are introduced. Then, Eq. (16) can be rewritten as follows:

$$\boldsymbol{S}_{ppk} = S_{ppk}\boldsymbol{E} \tag{17a}$$

$$\boldsymbol{S}_{pqk} = K_{pqk}\boldsymbol{E} + Q_{pqk}\boldsymbol{J}. \tag{17b}$$

Substituting Eq. (16) into Eq. (15), the covariance matrix is reduced to

$$\frac{\Lambda_k}{\Delta w} = \begin{bmatrix} S_{11k} & S_{12k} & S_{13k} & \cdots & S_{1nk} \\ S^*_{12k} & S_{22k} & S_{23k} & \cdots & S_{2nk} \\ S^*_{13k} & S^*_{23k} & S_{33k} & \cdots & S_{3nk} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ S^*_{1nk} & S^*_{2nk} & S^*_{3nk} & \cdots & S_{nnk} \end{bmatrix} \tag{18}$$

Hereafter, since the 2×2 submatricies are treated as scalar, $\boldsymbol{S}_{ppk}$ and $\boldsymbol{S}_{pqk}$ are represented as S_{ppk} and S_{pqk}, respectively. From Eq. (18), it is observed that Λ_k is a Hermitian matrix of order n.

Let's define complex Fourier coefficients $Z_{pk} = A_{ik}\boldsymbol{E} + B_{pk}\boldsymbol{J}$ ($p = 1, 2, \ldots, n$) by means of real Fourier coefficients A_{pk} and B_{pk}. The realized values of Fourier coefficients are then $\tilde{z}_{pk} = \tilde{a}_{pk}\boldsymbol{E} + \tilde{b}_{pk}\boldsymbol{J}$ ($p = 1, 2, \ldots, n-1$). Under the above complex representation, we rewrite the conditional means and variances of the Fourier coefficients conditioned by $n-1$ deterministic time functions, which are shown in the reference (Kameda and Morikawa 1992), as follows:

$$\langle Z_{nk} | cnd. \rangle = -\frac{\sum_{p=1}^{n-1} [\Lambda_k^{-1}]_{pn} \cdot \tilde{z}_{pk}}{[\Lambda_k^{-1}]_{nn}} \tag{19a}$$

$$\sigma^2_{Z_{nk}|cnd.} \equiv \sigma^2_{nk|cnd.} = \frac{1}{[\Lambda_k^{-1}]_{nn}}. \tag{19b}$$

In order to repersent Eq. (19) as a function of S_{nnk}, inverse matrix Λ_k^{-1} is further modified. Dividing Λ_k in Eq. (18) into four submatrices as

$$\Lambda_k = \left[\begin{array}{c|c} \Lambda_{11k} & \Lambda_{12k} \\ \hline {}^t\Lambda^*_{12k} & \Lambda_{22k} \end{array}\right] \begin{array}{l} \}\, n-1 \\ \}\, 1 \end{array} \qquad \begin{array}{cc} n-1 & 1 \end{array} \tag{20}$$

we obtain the well-known representation for inverse matrix Λ_k^{-1}, namely,

$$Q_k \Lambda_k^{-1} = \left[\begin{array}{c|c} Q_k \Lambda_{11k}^{-1} + {}^t(P_k^*) P_k & -P_k \\ \hline -{}^t(P_k^*) & 1 \end{array}\right], \tag{21}$$

where

$$P_k = \Lambda_{11k}^{-1} \Lambda_{12k}$$

$$Q_k = \Lambda_{22k} - {}^t(\Lambda_{12k}^*)\ \Lambda_{11k}^{-1} \Lambda_{12k} \tag{22}$$

For the Fourier coefficients in complex number, we define two column vectors ${}^t\boldsymbol{Z}_k = [Z_{1k}\ Z_{2k}\ \cdots\ Z_{nk}]$ and ${}^t\tilde{\boldsymbol{z}}_k = [\tilde{z}_{1k}\ \tilde{z}_{2k}\ \cdots\ \tilde{z}_{n-1\,k}]$. Noting that Q_k is scalar, substituting Eq. (21) and $\tilde{z}_k$ into Eq. (19) yields

$$\langle Z_{nk} | cnd. \rangle = {}^t(\tilde{z}_k^*)\ \Lambda_{11k}^{-1}\ \Lambda_{12k} \tag{23a}$$

$$\begin{aligned} \sigma^2_{nk|cnd.} &= Q_k \\ &= \Lambda_{22k} - {}^t(\Lambda^*_{12k})\ \Lambda_{11k}^{-1}\ \Lambda_{12k}. \end{aligned} \tag{23b}$$

Eq. (23) corresponds to the extension of the equation derived by Hoshiya and Kuwana (1993) to a complex representation.

A.3. Conditional mean values

Eq. (23a) is again represented in real number to derive the conditional mean of the conditioned stochastic process at site n as a function of power spectrum S_{nnk}. Expanding the right-hand side of Eq. (23a), its real and imaginary parts become, respectively,

$$\langle A_{nk} | cnd. \rangle = \sum_{p,q=1}^{n-1} \{(\tilde{a}_{pk} K_{qnk} + \tilde{b}_{pk} Q_{qnk})\ R[\Lambda_{11k}^{-1}]_{pq} + (-\tilde{a}_{pk} Q_{qnk} + \tilde{b}_{pk} K_{qnk})\ I[\Lambda_{11k}^{-1}]_{pq}\}$$

$$\langle B_{nk} | cnd. \rangle = \sum_{p,q=1}^{n-1} \{(\tilde{a}_{pk} Q_{qnk} - \tilde{b}_{pk} K_{qnk})\ R[\Lambda_{11k}^{-1}]_{pq} + (\tilde{a}_{pk} K_{qnk} + \tilde{b}_{pk} Q_{qnk})\ I[\Lambda_{11k}^{-1}]_{pq}\}. \tag{24}$$

Replacing K_{qnk} and Q_{qnk} with power spectra S_{qqk}, coherence function coh_{qnk}, and phase lag function θ_{qnk}, the following forms can be obtained:

$$\langle A_{nk}|cnd.\rangle = \alpha_{Ak}\sqrt{S_{nnk}}$$
$$\langle B_{nk}|cnd.\rangle = \alpha_{Bk}\sqrt{S_{nnk}}, \tag{25}$$

where

$$\alpha_{Ak} = \sum_{p,q=1}^{n-1}\sqrt{S_{qqk}\,coh_{qnk}}\,/\{(\tilde{a}_{pk}\ \cos\theta_{qnk} + \tilde{b}_{pk}\ \sin\theta_{qnk})\ R\,[\Lambda^{-1}_{11k}]_{pq} + (-\tilde{a}_{pk}\ \sin\theta_{qnk} + \tilde{b}_{pk}\ \cos\theta_{qnk})\ I\,[\Lambda^{-1}_{11k}]_{pq}\}$$
$$\alpha_{Bk} = \sum_{p,q=1}^{n-1}\sqrt{S_{qqk}\,coh_{qnk}}\,/\{(\tilde{a}_{pk}\ \sin\theta_{qnk} - \tilde{b}_{pk}\ \cos\theta_{qnk})\ R\,[\Lambda^{-1}_{11k}]_{pq} + (\tilde{a}_{pk}\ \cos\theta_{qnk} + \tilde{b}_{pk}\ \sin\theta_{qnk})\ I\,[\Lambda^{-1}_{11k}]_{pq}\}. \tag{26}$$

A.4. Conditional variances

The expansion of Eq. (23b) yields only the real part as

$$\sigma^2_{nk|cnd.} = S_{nnk} - \sum_{p,q=1}^{n-1}\{(K_{pnk}K_{qnk} + Q_{pnk}Q_{qnk})\ R\,[\Lambda^{-1}_{11k}]_{pq} + (-K_{pnk}Q_{qnk} + Q_{pnk}K_{qnk})\ I\,[\Lambda^{-1}_{11k}]_{pq}\}. \tag{27}$$

Arranging Eq. (27) as a function of power spectrum S_{nnk} at site n, we obtain the following form:

$$\sigma^2_{nk|cnd.} = \alpha_k S_{nnk}, \tag{28}$$

where

$$\alpha_k = 1 - \sum_{p,q=1}^{n-1}\sqrt{S_{ppk}S_{qqk}coh_{pnk}\,coh_{qnk}}\cdot\{R\,[\Lambda^{-1}_{11k}]_{pq}\ cos(\theta_{pnk} - \theta_{qnk}) + I\,[\Lambda^{-1}_{11k}]_{pq}\ \sin(\theta_{pnk} - \theta_{qnk})\}. \tag{29}$$

B. 1st and 2nd order moments of the standard deviation of $U_n(t|cnd.)$

The 1st and 2nd order moments of the standard deviation of the conditioned stochastic processes can be analytically derived with spectral uncertainties, which are introduced as η_{U_n} in Eq. (14) as follows (Gradshteyn, *et al.* 1990):

$$E\,[(\eta_{U_n})^\kappa] = \frac{2^{\varepsilon_\kappa}\Gamma(\nu_\kappa)}{\sqrt{2\pi}\beta^{\nu_\kappa}\mathrm{Var}[\sigma^2_{\hat{U}n|cnd.}]}\cdot D_{-\nu_\kappa}\left(\frac{\gamma}{\sqrt{2\beta}}\right)\cdot\exp\left[\frac{\gamma^2}{8\beta} - \frac{(E\,[\sigma^2_{\hat{U}_n|cnd.}])^2}{2\mathrm{Var}[\sigma^2_{\hat{U}n|cnd.}]}\right] \tag{30}$$

where $\kappa = 1,\ 2,\ \beta^{-1} = 2\ \mathrm{Var}[\sigma^2_{\hat{U}_n|cnd.}],\ \gamma = -\dfrac{E\,[\sigma^2_{\hat{U}n|cnd.}]}{\mathrm{Var}[\sigma^2_{\hat{U}_n|cnd.}]},\ \varepsilon_1 = 1/4,\ \varepsilon_2 = -1,\ \nu_1 = 3/2$, and $\nu_2 = 2$. $\Gamma(\cdot)$ denotes the gamma function and $D_\nu(\cdot)$ denotes the parabolic cylinder function of order ν.

C. Relation between stochastic processes with 'stochastic' spectra and 'fixed' spectra

In this section, we show that mean value $\mu_{U_n|cnd.}(t)$ in Eq. (5) and variance $\sigma^2_{\hat{U}_n|cnd.}$ in Eq. (6) of conditioned stochastic process $U_n(t|cnd.)$ at site n coincide with $E\,[\mu_{\hat{U}_n|cnd.}(t)]$ in Eq. (12a) and $E\,[\sigma^2_{\hat{U}_n|cnd.}]$ in Eq. (13a), respectively, when power spectrum $S_{nnk} \equiv \hat{S}_{nnk}$ at site n is deterministically known.

In this case, the probability density function of S_{nnk}, determined by Eq. (7) with the δ function, can be replaced as follows:

$$f_{S_{nnk}}(s_{nnk})=\delta(s_{nnk}-\hat{S}_{nnk}). \tag{31}$$

We derive the expected value and variance of the conditional mean, and the expected value and variance of the conditional variance for conditioned stochastic processes on the basis of this equation.

C.1. Conditional mean

Sicne the conditional means of Fourier coefficients A_{nk} of the conditioned stochastic process are determined by Eq. (3), the probability density functions of $\langle A_{nk}|cnd.\rangle$, corresponding to Eq. (10a), can be obtained as

$$f\mu_{A_{nk}}(\mu_{A_{nk}})=\frac{2\mu_{A_{nk}}}{\alpha_{A_k}}\cdot\delta\left(\frac{\mu_{A_{nk}}}{\alpha_{A_k}}-\hat{S}_{nnk}\right). \tag{32}$$

Using this equation, the 1st and 2nd order moments for $\mu_{A_{nk}}$ are reduced to $\alpha_{A_k}\hat{S}_{nnk}$ and $\alpha^2_{A_k}\hat{S}_{nnk}$, respectively, and therefore, $E[\mu_{A_{nk}}]=\alpha_{A_k}\hat{S}_{nnk}$ and $\mathrm{Var}[\mu_{A_{nk}}]=0$. This means that $E[\mu_{U_n|cnd.}(t)]$ coincides with the equation which is set at $S_{nnk}=\hat{S}_{nnk}$ in Eq. (5).

C.2. Conditional variance

Since the conditional variances of the Fourier coefficients of the conditioned stochastic process are determined by Eq. (4), the probability density function of $\sigma^2_{nk|cnd.}$, corresponding to Eq. (10b), is

$$f_{\upsilon_{nk}}(\upsilon_{nk})=\frac{1}{\alpha_k}\cdot\delta\left(\frac{\upsilon_{nk}}{\alpha_k}-\hat{S}_{nnk}\right). \tag{33}$$

The 1st and 2nd order moments for υ_{nk} yield $\alpha_k\hat{S}_{nnk}$ and $\alpha^2_k\hat{S}^2_{nnk}$, respectively, from Eq. (33). Therefore, $E[\sigma_{\upsilon_{nk}}]=\alpha_k\hat{S}_{nnk}$ and $\mathrm{Var}[\upsilon_{nk}]=0$. It is observed that substituting $\hat{S}_{nnk}$ into S_{nnk} in Eq. (6) yields $E[\sigma^2_{U_n|cnd.}]$.

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