Structural Engineering and Mechanics, Vol. 5, No. 4 (1997) 415-431
DOI: http://dx.doi.org/10.12989/sem.1997.5.4.415

Stability and minimum bracing for stepped columns with semirigid connections: Classical elastic approach

J. Dario Aristizabal-Ochoa†

School of Mines, National University, A.A. 75267, Medellin, Colombia

Abstract. Stability equations that evaluate the elastic critical axial load of stepped columns under extreme and intermediate concentrated axial loads in any type of construction with sidesway totally inhibited, partially inhibited, and uninhibited are derived in a classical manner. These equations can be utilized in the stability analysis of framed structures (totally braced, partially braced, and unbraced) with stepped columns with rigid, semirigid, and simple connetions. The proposed column classification and the corresponding stability equations overcome the limitations of current methods which are based on a classification of braced and unbraced columns. The proposed stability equations include the effects of: 1) semirigid connections; 2) step variation in the column cross section at the point of application of the intermediate axial load; and 3) lateral and rotational restraints at the intermediate connection and at the column ends. The proposed method consists in determining the eigenvalue of a 2×2 matrix for a braced column at the two ends and of a 3×3 matrix for a partially braced or unbraced column. The stability analysis can be carried out directly with the help of a pocket calculator. The proposed method is general and can be extended to multi-stepped columns. Various examples are inclued to demonstrate the effectiveness of the proposed method and to verify that the calculated results are exact. Definite minimum bracing criteria for single stepped columns is also presented.

Key words: buckling; bracing; building codes; columns; construction type; computer application; frames; loads; stability

1. Introduction

The stability analyses of prismatic columns under extreme axial loads including a uniformly distributed axial loading and the effects of semirigid connections have been presented by the writer recently (1994a-c). Approximate and nonparadoxical approaches that include closed-formulas to evaluate the effective length K-factor for columns in framed structures of any type of construction were presented. The closed-form formulas were derived and then utilized in the design of steel and reinforced concrete columns using current codes (AISC-LRFD 1986, AISC-WDS 1990, ACI-1989 revised 1992). In addition, a complete set of classical stability equations for prismatic columns with semirigid connections and their application to plane frames have been presented by the writer (1996). However, columns having intermediate concentrated load or/and stepped cross sections require special treatment. Anderson and Woodward (1972) and Castiglioni (1986) have treated the stability problem of stepped steel columns in a simplified manner. Similarly, Shrivastava (1980) has treated the problem of a prismatic column under varying axial load. Buckling equations for stepped columns under an intermediate axial load

† 125-Year Generation Professor

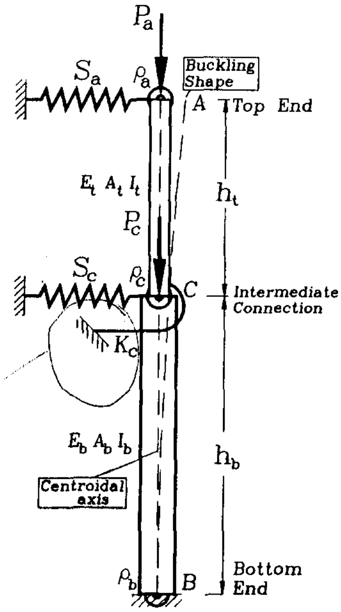


Fig. 1 Single stepped column under end and intermediate axial loads with semirigid connections. (Structural Model).

in any type of construction (rigid, simple, and semirigid frames) are not available in the technical literature, particularly, from the classical stability analysis point of view.

The main objective of this pubication is to present such analysis and the complete set of stability equations for single stepped columns with semirigid connections and sideway uninhibited, partially inhibited, and totally inhibited subjected to an intermediate axial load in addition to extreme axial loads. The buckling analysis consists in determining the lowest eigenvalue of a 2×2 matrix for a braced column at both ends and of a 3×3 matrix for a partially braced or unbraced column. This analysis can be carried out directly with the help of a pocket calculator. Minimum bracing criteria for stepped columns under end and intermediate axial loads are also presented. Several examples are included that demonstrate the effectiveness and exactness of the proposed stability equations and bracing criteria.

2. Structural model

2.1. Assumptions

Consider the stepped column that connects points A and B as shown by Fig. 1. This consists of column segments AC and CB with semirigid connections at the extreme ends A and B, and at the intermediate joint C. It is assumed that: 1) column segments AC and CB are made of homogeneous linear elastic materials with mechanical and geometric properties E_t I_t A_t h_t and E_b I_b A_b h_b (where E=elastic modulus, I=moment of inertia, A=cross sectional area, and h=span). The subindices t and b indicate the top and bottom segments, respectively; 2) the centroidal axis of each segment is a straight line with both axes lineup; 3) column AB is subjected simultaneously to a top-end axial loading P_a at A, and to an intermediate concentrated loading P_c at C with both loads applied along the common centroidal axis; 4) the column's lateral sways are partially inhibited by displacement springs S_a and S_c located at A and C, respectively, and an external rotational spring S_θ located at C. The rotational fixity factors at the ends of column A and B. and at the intermediate joint C are assumed to be ρ_a, ρ_b, and ρ_c, respectively. For ideally rigid connections, the rotational fixity factors are equal to one ($\rho = 1.0$); whereas, for ideally hinged connections, these factors are equal to zero ($\rho = 0$). In real connections (i.e., semirigid) the fixity factors vary between one and zero ($0 < \rho < 1.0$). A complete discussion on the rotational fixity factors are presented by the writer (1994a-c) and Cunningham (1990).

2.2. Proposed stability equations

2.2.1. Stability criteria

In a frame with sidesway uninhibited or partially inhibited every column is defined as having reached its critical load when sidesway buckling of the frame occurs, with the distribution of axial loads among the columns P_a and P_c being as specified. Similarly, in a frame with sidesway inhibited every column is defined as having reached its critical load when at least one of the columns of the frame buckles first, with the load distribution among the columns as specified.

The stability analysis presented in this paper consists in determining the set of critical loads $(P_a)_{cr}$ and $(P_c)_{cr}$ that makes column AB buckle (Fig. 1). Both loads can be determined by making the determinant of the stiffness matrix of the column $[K]$ given by Eq. (1) equal to zero. This matrix includes the second-order effects caused by the applied pattern of axial loads P_a and P_c.

$$[K] = \begin{bmatrix} K_{11} & \text{Symmetric} & \\ K_{21} & K_{22} & \\ K_{31} & K_{32} & K_{33} \end{bmatrix} \tag{1}$$

Where the stiffness coefficients k_{ij} are given by Eqs. (2)-(7):

$$K_{11} = \frac{3u_t^2\rho_c(1-\rho_a) + 9\rho_a\rho_c(1-u_t/\mathrm{Tan}\,u_t)}{D_t}\,\frac{EI_t}{h_t} + \frac{3u_b^2(1-\rho_b) + 9\rho_b(1-u_b/\mathrm{Tan}\,u_b)}{D_b}\,\frac{EI_b}{h_b} + S_\theta \tag{2}$$

$$K_{21} = \frac{3u_t^2\rho_c(1-\rho_a) + 9\rho_a\rho_c u_t(1-\mathrm{Cos}\,u_t)/\mathrm{Sin}\,u_t}{D_t}\,\frac{EI_t}{h_t^2} + \frac{3u_b^2(1-\rho_b) + 9\rho_b u_b(1-\mathrm{Cos}\,u_b)/\mathrm{Sin}\,u_b}{D_b}\,\frac{EI_b}{h_b^2} \tag{3}$$

$$K_{31} = \frac{3u_t^2\rho_c(1-\rho_a) + 9\rho_a\rho_c u_t(1-\mathrm{Cos}\,u_t)/\mathrm{Sin}\,u_t}{D_t}\,\frac{EI_t}{h_t^2} \tag{4}$$

$$K_{22} = u_t^2\left[\frac{3(\rho_a + \rho_c - 2\rho_a\rho_c) + 9\rho_a\rho_c\dfrac{\mathrm{Tan}(u_t/2)}{u_t/2}}{D_t} - 1\right]\frac{EI_t}{h_t^3}$$

$$+u_b^2\left[\frac{3(1-\rho_b)+9\rho_b\dfrac{\mathrm{Tan}(u_b/2)}{u_b/2}}{D_b}-1\right]\frac{EI_b}{h_b^3}+S_c \tag{5}$$

$$K_{32}=-u_t^2\left[\frac{3(\rho_a+\rho_c-2\rho_a\rho_c)+9\rho_a\rho_c\dfrac{\mathrm{Tan}(u_t/2)}{u_t/2}}{D_t}-1\right]\frac{EI_t}{h_t^3} \tag{6}$$

$$K_{33}=u_t^2\left[\frac{3(\rho_a+\rho_c-2\rho_a\rho_c)+9\rho_a\rho_c\dfrac{\mathrm{Tan}(u_t/2)}{u_t/2}}{D_t}-1\right]\frac{EI_t}{h_t^3}+S_a \tag{7}$$

and

$$D_t=u_t^2(1-\rho_a)(1-\rho_c)+3(\rho_a+\rho_c-2\rho_a\rho_c)(1-u_t/\mathrm{Tan}\,u_t)+9\rho_a\rho_c\left[\frac{\mathrm{Tan}(u_t/2)}{u_t/2}-1\right] \tag{8}$$

$$D_b=3(1-\rho_b)(1-u_b/\mathrm{Tan}\,u_b)+9\rho_b\left[\frac{\mathrm{Tan}(u_b/2)}{u_b/2}-1\right] \tag{9}$$

$$u_t=\sqrt{\frac{(P_a)_{cr}}{(EI)_t/h_t^2}} \tag{10a}$$

$$u_b=\sqrt{\frac{(P_a)_{cr}+(P_c)_{cr}}{(EI)_b/h_b^2}} \tag{10b}$$

where

E	Young's modulus
I_t and I_b	moments if inertia of columns AC and CB, respectively
h_t and h_b	lengths of columns AC and CB, respectively
$(P_a)_{cr}$	total compressive critical load applied at A
$(P_a)_{cr}+(P_c)_{cr}$	total compressive critical load at C
ρ_a and ρ_b	rotational fixity factors of column AB at top A and bottom B, respectively
ρ_c	rotational fixity factor of column AC at the intermediate joint C
S_a	lateral stiffness restraining column AB against sidesway at top A
S_c	lateral stiffness restraining column AB against sidesway at intermediate joint C
S_θ	stiffness of rotational spring restraining column AB at intermediate joint C.

The first and second rows and columns of matrix in Eq. (1) correspond to the rotational and lateral deflection at the intermediate joint C; the third row and column correspond to the lateral sidesway of top end A. The rotations at A and B are condensed out in this approach and are represented by the rotational fixity factors ρ_a and ρ_b, respectively. It is assumed that the axial deformations and shear distortions in both segments AC and CB are negligible compared to their bending deformations. The stability coefficients given by Eqs. (2)-(7) are derived in Appendix I in a classical manner.

In general the sidesway buckling of column AB (Fig. 1) is based on the lowest eigenvalue of the characteristic equation $|[K]|=0$. This requires the solution of a 3×3 determinant which might be carried out using a pocket calculator. The stability analysis of multi-stepped columns with more than one intermediate axial load can be carried out in similar fashion, except that the number of degrees of freedom and the size of the matrices will be larger (an increase of 2-DOF for every additional intermediate node). Then, a computer program would be required to solve the eigenvalue of multi-stepped columns. However, the analytical procedure described herein is similar.

Four different types of buckling modes are possible for a single stepped column as shown by Fig. 2a-d. These modes and the corresponding eigenvalue equations can be obtained from Eq. (1) as follows:

(1) When sidesways between A, B and C are totally inhibited

The stability equation for this particular case (Fig. 2a) is reduced to $K_{11}=0$ or simply

$$\frac{3u_t^2\rho_c(1-\rho_a)+9\rho_a\rho_c(1-u_t/\mathrm{Tan}\ u_t)}{D_t}\frac{EI_t}{h_t}+\frac{3u_b^2(1-\rho_b)+9\rho_b(1-u_b/\mathrm{Tan}\ u_b)}{D_b}\frac{EI_b}{h_b}+S_\theta=0 \quad (11)$$

The validity of Eq. (11) is cheked by Example 1.

(2) When sidesway between A and B is totally inhibited.

The stability equation for this particular case (Fig. 2b) is reduced to $K_{11}K_{22}-K_{12}^2=0$ or simply

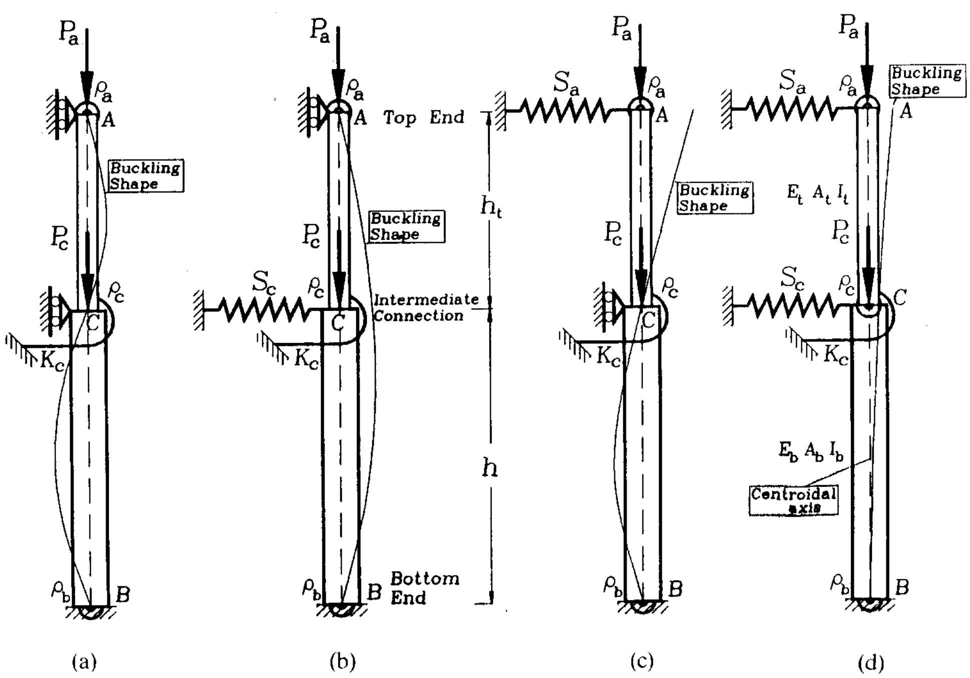


Fig. 2 Buckling modes of a single stepped column under end and intermediate axial loads with semirigid connections.

$$\left[\frac{3u_t^2\rho_c(1-\rho_a)+9\rho_a\rho_c(1-u_t/\mathrm{Tan}\ u_t)}{D_t}\frac{EI_t}{h_t}+\frac{3u_b^2(1-\rho_b)+9\rho_b(1-u_b/\mathrm{Tan}\ u_b)}{D_b}\frac{EI_b}{h_b}+S_\theta\right]\times$$
$$\left\{u_t^2\left[\frac{3(\rho_a+\rho_c-2\rho_a\rho_c)+9\rho_a\rho_c\dfrac{\mathrm{Tan}(u_t/2)}{u_t/2}}{D_t}-1\right]\frac{EI_t}{h_t^3}+u_b^2\left[\frac{3(1-\rho_b)+9\rho_b\dfrac{\mathrm{Tan}(u_b/2)}{u_b/2}}{D_b}-1\right]\frac{EI_b}{h_b^3}+S_c\right\}$$
$$-\left[\frac{3u_t^2\rho_c(1-\rho_a)+9\rho_a\rho_c u_t(1-\mathrm{Cos}\,u_t)/\sin u_t}{D_t}\frac{EI_t}{h_t^2}+\frac{3u_b^2(1-\rho_b)+9\rho_b u_b(1-\mathrm{Cos}\,u_b)/\sin u_b}{D_b}\frac{EI_b}{h_b^2}\right]^2=0 \tag{12}$$

The validity of formula Eq. (12) is checked by Example 2.

(3) When sidesway between C and B is totally inhibited

The stability equation for this particular case (Fig. 2c) is reduced to $K_{11}K_{33}-K_{31}^2=0$ or simply

$$\left[\frac{3u_t^2\rho_c(1-\rho_a)+9\rho_a\rho_c(1-u_t/\mathrm{Tan}\,u_t)}{D_t}\frac{EI_t}{h_t}+\frac{3u_b^2(1-\rho_b)+9\rho_b(1-u_b/\mathrm{Tan}\,u_b)}{D_b}\frac{EI_b}{h_b}+S_\theta\right]\times$$
$$\left\{u_t^2\left[\frac{3(\rho_a+\rho_c-2\rho_a\rho_c)+9\rho_a\rho_c\dfrac{\mathrm{Tan}(u_t/2)}{u_t/2}}{D_t}-1\right]\frac{EI_t}{h_t^3}+S_a\right\}$$
$$-\left[\frac{3u_t^2\rho_c(1-\rho_a)+9\rho_a\rho_c u_t(1-\cos u_t)/\sin u_t}{D_t}\frac{EI_t}{h_t^2}\right]^2=0 \tag{13}$$

The validity of Eq. (13) is checked by Example 3.

(4) Columns with sideway partially inhibited and uninhibited

For stepped columns with the lateral sway between A and B partially inhibited (i.e., when $S_a\neq 0$ and $S_c\neq 0$) or uninhibited (i.e., when $S_a=S_c=0$) as shown in Fig. 2d, the general sidesway buckling is based on the lowest eigenvalue of the characteristic equation $|[K]|=0$. This requires the solution of a 3×3 determinant which might be carried out using a pocket calculator or simply

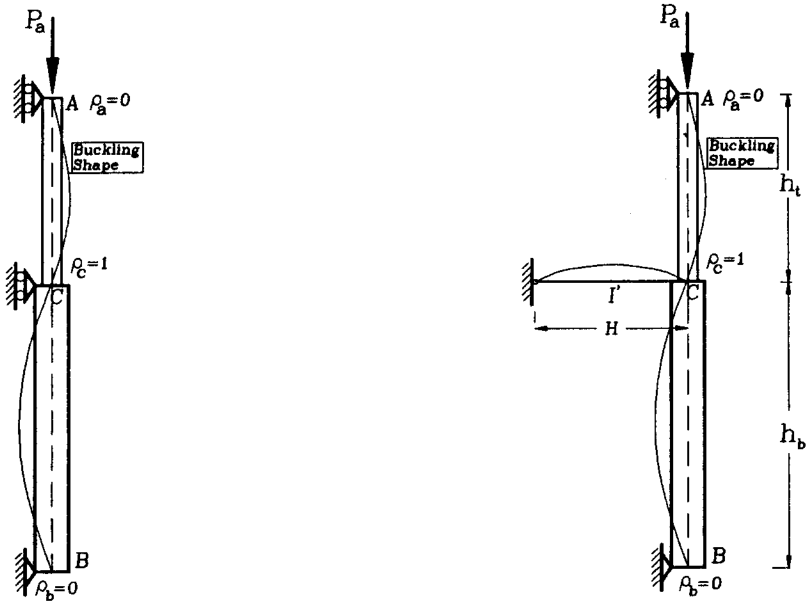
$$K_{11}(K_{22}K_{33}-K_{32}^2)-K_{21}(K_{21}K_{33}-K_{31}K_{32})+K_{31}(K_{21}K_{32}-K_{22}K_{31})=0 \tag{14}$$

The validity of Eq. (14) is checked by Example 4.

3. Verification study: Examples

Example 1

Eq. (11) was checked against solutions presented by Timoshenko and Gere (1961, pages 66-76) for the buckling of continuous beams. The case under consideration is a stepped column with hinged ends and compressed at A and B by P as shown in Fig. 3a (hinged at A: $\rho_a=0$;



(a) On three supports with hinged ends (b) On end supports and rigidly connected to a bar at C.

Fig. 3 Example 1: Stepped beam-column (after Timoshenko & Ger 1961, pp. 66-70).

rigid connection at C with no exterior restraints and no load: $\rho_c=1.0$, $S_\theta=0$, $P_c=0$; and hinged at B: $\rho_b=0$).

Substituting in Eq (11) $\rho_a=\rho_b=0$, and $\rho_c=1.0$, the eigenvalue equation becomes

$$\frac{u_t^2}{(1-u_t/\mathrm{Tan}\,u_t)}\frac{EI_t}{h_t}+\frac{u_b^2}{(1-u_b/\mathrm{Tan}\,u_b)}\frac{EI_b}{h_b}+S_\theta=0 \tag{15a}$$

when $S_\theta=0$ Eq. (15a) can be reduced to Eq. (15b) as follows

$$\frac{1}{u_t^2}(1-u_t/\mathrm{Tan}\,u_t)+\frac{1}{u_b^2}(1-u_b/\mathrm{Tan}\;u_b)\frac{h_bI_t}{h_tI_b}=0 \tag{15b}$$

Eq. (15b) is identical to the solution reported by Timoshenko and Gere (1961. Eq. (b), page 67).

An additional example was considered when $S_\theta=3EI'/H$ (provided by the transverse element located at midspan with a hinged far end) $h_t=h_b$, $I_t=I_b$ as shown in Fig. 3b. Then Eq. (15a) becomes

$$\frac{u_t^2}{(1-u_t/\mathrm{Tan}\,u_t)}\frac{EI_t}{h_t}+\frac{u_b^2}{(1-u_b/\mathrm{Tan}\,u_b)}\frac{EI_b}{h_b}+3EI'/H=0 \tag{15c}$$

Since $u_t=u_b$ (15c) is reduced to Eq. (15d)

$$\frac{3}{u_t^2}(1-u_t/\mathrm{Tan}\,u_t)=-\frac{2HI_t}{h_tI'} \tag{15d}$$

which is identical to the solution reported by Timoshenko and Gere (1961, bottom of page

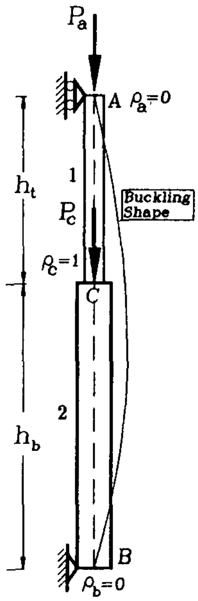


Fig. 4 Example 2: Simple supported stepped column (After Timoshenko & Gere 1961, p. 100).

69). Note that in Timoshenko's notation $u_t=2u_1$ and $u_b=2u_2$.

Example 2

Results from Eq. (12) were tested against tabulated values presented by Timoshenko and Gere (1961, Table 2-6, p. 100). The case under consideration is a stepped column with hinged supports at A and B and compressed at A by P_a and by P_c at midspan as shown in Fig. 4 (hinged at A: $\rho_a=0$; rigid connection at C with no exterior restraints: $\rho_c=1.0$, $S_c=S_\theta=0$; and hinged at B: $\rho_b=0$).

The values of the reduced length L of the column calculated from Eq. (12) are listed in Table 1. They are practically identical to those derived by Timoshenko and Gere, where $(P_a+P_c)_{cr}=\pi^2 EI_b L^2$.

Example 3

The validity of Eq. (13) was tested against solutions obtained utilizing the classical method of slope-deflection (Salmon and Johnson 1980) for a stepped column with hinged ends and compressed at A by P_a and by P_c at C as shown in Fig. 5 (free at A: $\rho_a=0$, $S_a=0$; braced at C with no exterior rotational restraint: $\rho_c=1.0$. $S_\theta=0$; and hinged at B: $\rho_b=0$).

Substituting in Eq. (13) $\rho_a=\rho_b=0$, $\rho_c=1.0$, and $S_\theta=0$ the eigenvalue equation becomes

$$\left[\frac{u_t^2}{(1-u_t/\mathrm{Tan}\,u_t)}\frac{EI_t}{h_t}+\frac{u_b^2}{(1-u_b/\mathrm{Tan}\,u_b)}\frac{EI_b}{h_b}\right]u_t^2\left[\frac{1}{(1-u_t/\mathrm{Tan}\,u_t)}-1\right]\frac{EI_t}{h_t^3}-\left[\frac{u_t^2(EI_t)/h_t^2}{(1-u_t/\mathrm{Tan}\,u_t)}\right]^2=0 \tag{16a}$$

Table 1 Calculated-versus-theoretical values of L/l for a stepped column (Problem 2: After Timoshenko & Gere 1961, Table 2-6, p. 100)

$\frac{(P_1+P_2)}{P_1}$	1.00		1.25		1.50		1.75		2.00		3.00	
I_2/I_1	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}	m_{Th}	m_{Cal}
1.00	1.00	1.00000	0.95	0.94904	0.91	0.91397	0.89	0.88847	0.87	0.86892	0.82	0.82257
1.25	1.06	1.06229	1.005	1.00505	0.97	0.96553	0.94	0.93662	0.91	0.91455	-na-	0.86187
1.50	1.12	1.12354	1.06	1.06045	1.02	1.01675	0.99	0.98470	0.96	0.96019	-na-	0.90149
1.75	1.18	1.18321	1.11	1.11467	1.07	1.06707	1.04	1.03208	1.01	1.00529	-na-	0.94088
2.00	1.24	1.23810	1.16	1.16745	1.12	1.11620	1.08	1.07846	1.05	1.04951	-na-	0.97976

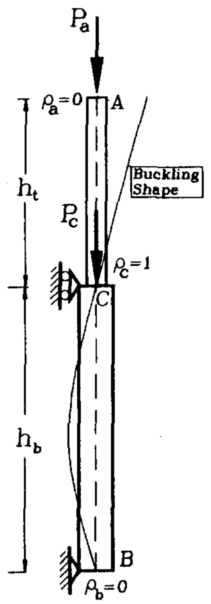


Fig. 5 Example 3: Stepped column with simple supports at C and B and top sidesway.

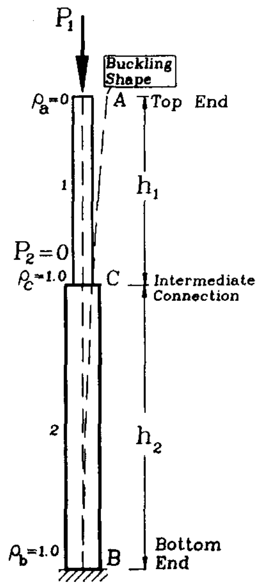


Fig. 6 Example 4: Stepped cantilever column (After Timoshenko & Gere 1961, p. 115).

$$\text{or} \quad \frac{u_t^4}{(1-u_t/\mathrm{Tan}\,u_t)^2}\left[u_t/\mathrm{Tan}\,u_t+\frac{u_b^2(1-u_t/\mathrm{Tan}\,u_t)}{u_t\mathrm{Tan}\,u_t(1-u_b/\mathrm{Tan}\,u_b)}\ \frac{I_b}{I_t}\ \frac{h_t}{h_b}-1\right]=0 \tag{16b}$$

from which

$$\left[u_t/\mathrm{Tan}\,u_t+\frac{u_b^2(1-u_t/\mathrm{Tan}\,u_t)}{u_t\mathrm{Tan}\,u_t(1-u_b/\mathrm{Tan}\,u_b)}\ \frac{I_b}{I_t}\ \frac{h_t}{h_b}-1\right]=0 \tag{16c}$$

For the particular case: $P_c=0$, $h_t=h_b$, $I_t=I_b$, the stability equation is reduced to $2u_t=\mathrm{Tan}\,u_t$, whose solution is $u_t=1.16556$ or $(P_a)_{cr}=\pi^2EI_t(2.695h_t)^2$. Using the slope-deflection method (Salmon and Johnson 1980, p. 840-842) a stability equation identical to Eq. (16c) can be obtained.

Example 4

Table 2 Calculated-versus-theoretical m-factor for a step-variable column (Problem 4: After Timoshenko & Gere 1961, Table 2-10, p. 115)

$h_2/(h_1+h_2)$	0.2		0.4		0.6		0.8	
I_1/I_2	m_{Thrl}	m_{Cal}	m_{Thrl}	m_{Cal}	m_{Thrl}	m_{Cal}	m_{Thrl}	m_{Cal}
0.01	0.15	0.15344	0.27	0.27052	0.60	0.59843	2.26	2.25706
0.1	1.47	1.46750	2.40	2.40063	4.50	4.49778	8.59	8.58799
0.2	2.80	2.79651	4.22	4.22180	6.69	6.69418	9.33	9.33015
0.4	5.09	5.08844	6.68	6.67739	8.51	8.50980	9.67	9.67421
0.6	6.98	6.97941	8.19	9.18500	9.24	9.24378	9.78	9.78394
0.8	8.55	8.55122	9.18	9.17672	9.63	9.63146	9.84	9.83755

Results from the 3×3 eigenvalue equation $[K]=0$ or Eq. (14) were tested against tabulated solutions presented by Timoshenko and Gere (1961, Table 2-10, p. 115) for a stepped cantilever column shown in Fig. 6 (free at A: $\rho_a=S_a=0$; rigid connection at C with no exterior restraints: $\rho_c=1.0$, $S_c=S_\theta=0$; and, perfectly fixed at B: $\rho_b=1$).

Table 2 shows that the values of m calculated from Eq. (14), which are listed with five significant figures, are practically identical to those by Timoshenko and Gere. Note that m is for the hinged-hinged column, and $m/4$ for the cantilever column, where m is used by Timoshenko and Gere in $P_{cr}=mEI_2/(h_1+h_2)^2$.

4. Partially braced columns and minimum lateral bracing

4.1. Partially braced column criterion

A partially-braced stepped column is one whose total critical load P_{cr} lies between the critical load obtained from Eqs. (11), (12) or (13) and that from Eq. (14) assuming that $S_a=S_c=S_\theta=0$ as follows:

$$\underset{\substack{\text{Braced Column}\\ \text{from Eqs. (11), (12) or (13)}}}{P_{\text{Critical}}} \geq \underset{\substack{\text{Partially Braced Column}\\ \text{from Eq. (14)}}}{P_{\text{Critical}}} \geq \underset{\substack{\text{Unbraced Column}\\ \text{From Eq. (14) with } S_a=S_c=S_\theta=0}}{P_{\text{Critical}}} \tag{17}$$

In addition, the upper limits on the critical loads given by Eqs. (11), (12) and Eq. (13) depend on what column's joints that are being braced (i.e., if the stepped column is fully braced at A and C simultaneously or only at A or at C, respectively). It is obvious that:

$$\underset{\text{obtained from Eq. (11)}}{P_{\text{Critical}}} \geq \underset{\text{obtained from Eq. (12)}}{P_{\text{Critical}}} \geq \underset{\text{obtained from Eq. (13)}}{P_{\text{Critical}}} \geq \underset{\text{obtained from Eq. (14)}}{P_{\text{Critical}}} \tag{18}$$

This cirterion is simple to apply and indicates that the total critical load P_{cr} of a partially braced column is less than that of the same column but with sidesway inhibited, as indicated by Eqs. (17) and (18).

4.2. Minimum bracing criterion

The minimum bracing required to convert a stepped column with sidesway uninhibited or partially inhibited into a braced column can be determined utilizing Eqs. (17) and (18) or by comparing Eqs. (11), (12) or (13) to (14) depending on which column's joints are braced, as follows:

$$\underset{\substack{\text{Braced Column at } A \text{ and } C\\ \text{obtained from Eq. (11)}}}{P_{\text{Critical}}} = \underset{\substack{\text{Partially Braced}\\ \text{obtained from Eq. (14)}}}{P_{\text{Critical}}} \tag{19a}$$

$$\underset{\substack{\text{Braced Column at } A\\ \text{obtained from Eq. (12)}}}{P_{\text{Critical}}} = \underset{\substack{\text{Partially Braced}\\ \text{obtained from Eq. (14)}}}{P_{\text{Critical}}} \tag{19b}$$

$$\underset{\substack{\text{Braced Column at } C\\ \text{obtained from Eq. (13)}}}{P_{\text{Critical}}} = \underset{\substack{\text{Partially Braced}\\ \text{obtained from Eq. (14)}}}{P_{\text{Critical}}} \tag{19c}$$

By combining Eqs. (11), (12) or (13) with (14) as indicated by Eqs. (19a)-(19c), the required S_a and S_c can be determined directly following the steps described below: 1) The end fixity factors ρ_a, ρ_b must determined for both conditions braced and unbraced, as shown in Example 5; 2) The u-factors for the desired braced conditions are calculated from the corresponding Eqs. (11), (12) or (13) utilizing, of course, the fixity factors ρ_a and ρ_b for the braced case; 3) The braced u-factors along with ρ_a and ρ_b for unbraced conditions previously calculated are substituted into Eq. (14) from which the required minimum bracings S_a and S_c can be calculated directly.

An example describing the calculation of $(S_a)_{min}$. and $(S_c)_{min}$ for a bent frame is presented next.

Example 5 Minimum lateral bracing for a bent frame

Utilizing the minimum bracing criteria and the steps just decribed, determine the lateral bracings required to convert the bent frame shown in Fig. 7a into a braced frame. The bracings that need to be analyzed are: 1) S_a along the top level (i.e., frame braced at A and A' only); 2) S_c at the intermediate connection (i.e., frame braced at C and C' only); and 3) S_a and S_c along the top and intermediate levels (i.e., frame braced at A and C). Assume that $\rho_b = \rho_c = 1$ in both columns.

Solution: Since the frame is symmetrical, columns AB and $A'B'$ are both identical with the same loads and boundary conditions. Therefore, the stability analysis can be reduced to that of a single column.

The first step is to find the fixity factors at the column ends for unbraced and braced conditions. Relationships between the rotational restraints and the fixity factors in framed structures are presented in Appendix 1. For this particular frame the fixity factors are as follows:

i) For Unbraced Conditions along AA' the frame would buckle in a anti-symmetric shape (Figs. 7b-c) with the beam providing rotational restraints at both ends A and A' of magnitude

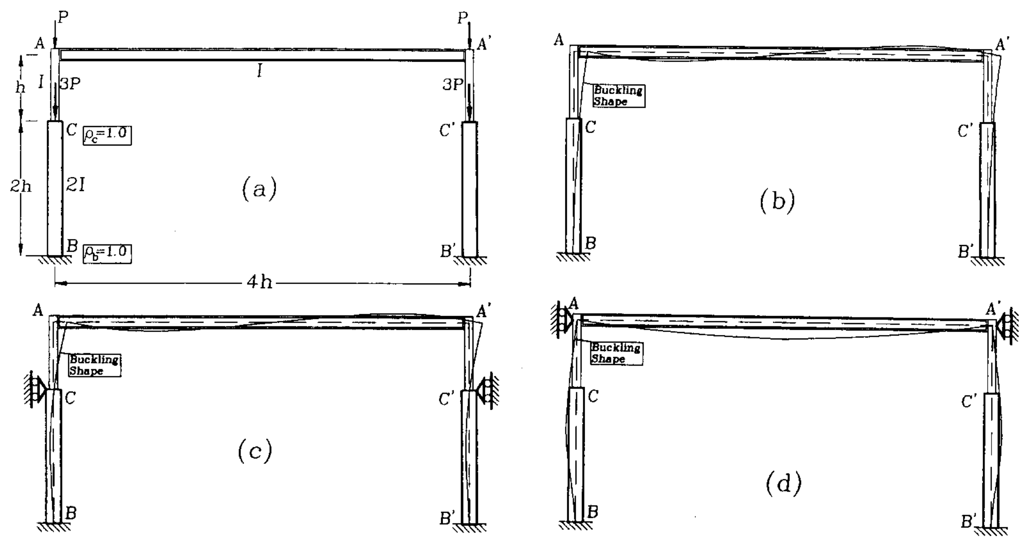


Fig. 7 Example 5: Minimum lateral bracings for a simple bent frame with stepped columns
(a) Structural model;
(b) Buckling mode with sidesway between A and B uninhibited;
(c) Buckling mode with sidesway between C and B totally inhibited;
(d) Buckling mode with sidesways between A and B totally inhibited.

$6EI/L = 1.5EI//h$ (since $L=4h$). Therefore, $\rho_a = 1/(1+3/1.5) = 1/3$, and $\rho_b = \rho_c = 1$.

ii) For Braced Conditions along AA' the frame would buckle in a symmetric shape (Fig. 7d) with the beam providing rotational restraints at both ends A and A' of magnitude $2EI/L = 0.5$ EI/h. Therefore, $\rho_a = 1/(1+3/0.5) = 1/7$, and $\rho_b = \rho_c = 1$.

The required bracings for column AB for each one of the requested cases are calculated as follows: 1) Braced at A only-. Taking into consideration that: $I_t = I$, $h_t = h$, $I_b = 2I$, $h_b = 2h$, $P_a = P$, $P_c = 3P$, $\rho_a = 1/7$, and $\rho_b = \rho_c = 1$, the solutions for braced conditions obtained from Eq. (12) are: $(P_a)_{cr} = 0.19072\ \pi^2 EI/h^2$ and $(P_a)_{cr} + (P_c)_{cr} = 0.76288\ \pi^2 EI/h^2$ (or $u_t = \pi\sqrt{0.19072} = 1.37198$ and $u_b = \pi\sqrt{1.52576} = 3.88055$). Now, the magnitude of S_a can obtained directly from Eq. (14) by substituting the values of $u_t = 1.37198$, $u_b = 3.88055$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$. The result was $\boldsymbol{S_a = 3.3818EI/h^3}$ **per column.**; 2) Braced at C only-. Taking into consideration that: $I_t = I$, $h_t = h$, $I_b = 2I$, $h_b = 2h$, $P_a = P$, $P_c = 3P$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$, the solutions for braced conditions obtained from Eq. (13) are: $(P_a)_{cr} = 0.20909\pi^2$ EI/h^2 and $(P_a)_{cr} + (P_c)_{cr} = 0.83637\pi^2 EI/h^2$ (or $u_t = = \pi\sqrt{0.209092} = 1.43654$ and $u_b = \pi\sqrt{1.67273} = 4.06315$). Now, the magnitude of S_c can obtained directly from Eq. (14) by substituting the values of $u_t = 1.43654$, $u_b = 4.06315$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$. The result was $\boldsymbol{S_c = 2.773225 \times 10^8 EI/h^3}$ **per column.**; 3) Braced at A and C simultaneously-. Taking into consideration that: $I_t = I$, $h_t = h$, $I_b = 2I$, $h_b = 2h$, $P_a = P$, $P_c = 3P$, $\rho_a = 1/7$, and $\rho_b = \rho_c = 1$, the solutions for braced conditions obtained from Eq. (11) are: $(P_a)_{cr} = 0.329338\pi^2 EI/h^2$ and $(P_a)_{cr} + (P_c)_{cr} = 1.31735\pi^2 EI/h^2$ (or $u_t = \pi\sqrt{0.329338} = 1.802898$ and $u_b = \pi\sqrt{2.634707} = 5.09936$). Now, the magnitude of S_a can obtained directly from the characteristic equation $|[K]| = 0$ by deleting the second row and column (or $K_{11}K_{33} - K_{31}^2 = 0$) and substituing the values of $u_t = 1.802898$, $u_b = 5.09936$, $\rho_a = 1/3$, and $\rho_b = \rho_c = 1$. The result was $S_a = 47.3354EI/h^3$ per column. Similarly, S_c can obtained directly from the characteristic equation $|[K]| = 0$ by deleting the third row and column (or $K_{11}K_{22} - K_{21}^2 = 0$) and substituting the values of $u_t = 1.802898$, $u_b = 5.09936$, $\rho_a = 1/7$, and $\rho_b = \rho_c = 1$. The result was $\boldsymbol{S_c = 1.0966774 \times}$

$10^8 EI/h^3$ per column.

It is interesting to note that: 1) the trend indicated by Eq. (18) for $(P_a)_{cr}+(P_c)_{cr}$ for this particular frame is $1.31735\pi^3 EI/h^2>0.83637\pi^2 EI/h^2>0.76288\pi^2 EI/h^2>0.16074\pi^2 EI/h^2$ (this last value corresponds to the unbraced frame); and 2) the magnitude of the required lateral bracing at A is relatively small compared to that required at C.

5. Summary and conclusions

The complete set of four stability equations by which the buckling loads of single stepped columns in any type of construction can be evaluated is presented in a classical manner. The proposed equations include the effects of: 1) semirigid connections at the column joints; 2) step variation in the column cross section at the point of application of the intermediate load; and 3) lateral and rotational restraints at the column ends and intermediate connection. The method is particularly applicable to the stability analysis of stepped columns of any type of construction with sidesway inhibited, partially inhibited, and uninhibited. To understand the four-way classification for a single stepped column and the corresponding stability equations, four examples are presented and the results compared to those using other methods. A verification study indicates that the calculated elastic buckling loads are exact. The proposed formulation consists in determining the eigenvalue of a 1×1 matrix for braced columns at three supports, of a 2×2 matrix for braced columns at two supports, and of a 3×3 matrix for unbraced columns. The eigenvalue calculations can be carried out with the help of a pocket calculator.

The proposed classification and the complete set of transcendental equations for stepped columns are more general than those from other methods. In addition, definite criteria are given to determine the minimum amount of lateral bracings required by stepped columns in framed structures to achieve any nonswaying buckling mode. The proposed algorithm can be extended to multi-stepped columns with semirigid connections.

Analytical studies indicted that the stability of stepped columns increases substantially with the magnitude of the lateral restraints and the fixity at the column base and at the intermediate connection. The degree of fixity at the top end has less influence on the overall stability of stepped columns with sidesway inhibited than in frames with sidesway uninhibited.

Appendix 1

Formulae derivation

The stiffness coefficients including the second-order effects for a single stepped column AB (Fig. 1) can be obtained by adding the stiffnesses of its segments AC with and CB as shown by Eqs. (1)-(7). The stiffness coefficients or stability functions of a single prismatic column with semirigid connections are derived below for quick reference.

Assumptions

Consider a prismatic element that connects points A and B as shown in Fig. 8a. The element AB is made up of the column itself $A'B'$, and two lumped flexural connectors AA' and BB' at the top

(a) Structural model (b) Degrees of freedom and end actions

Fig. 8 Prismatic column AB with flexural DOFs 1-4 including the fixity factors ρ_a and ρ_b.

and bottom ends, respectively. It is assumed that: 1) the column $A'B'$ is made of a homogeneous linear elastic material with a modulus of elasticity E; 2) the centroidal axis of the member is a straight line; 3) the column is loaded with an end axial load P along the centroidal axis of the cross section with a principal moment of inertia I and cross are A; and 4) deformation are small so that the principle of superposition can be applied.

The flexurual connectors AA' and BB' have stiffnesses κ_a and κ_b (whose units are in force-distance/radian), respectively. The ratios $R_a=\kappa_a/(EI/h)$ and $R_b=\kappa_b/(EI/h)$ are denoted as the stiffness indices of the flexural connections. Where I=column's moment of inertia about the principal axis in question, and h=column's height. These indices vary from zero (i.e., $R_a=R_b=0$) for simple connections (i.e., pinned) to infinity (i.e., $R_a=R_b=\infty$) for fully rustrained connections (i.e., rigid). It is important to note that the proposed algorithm can be utilized in the inelastic analysis of framed structures when the nonlinear behavior is concentraded at the connections. This can be carried out by updating the flexural stiffness of the connections AA' and BB' for each increment in a linear- incremental fashion. Gerstle (1988) has indicated lower and upper bounds for κ_a and κ_b. More recently, *Xu* and Grierson (1993) used these bounds in the design of frames with semirigid connections. For convenience the following two paramaters are introduced (Aristizabal-Ochoa 1994a):

$$\rho_a=\frac{1}{1+\dfrac{3}{R_a}} \tag{20a}$$

$$\rho_b=\frac{1}{1+\dfrac{3}{R_b}} \tag{20b}$$

where ρ_a and ρ_b are called the fixity factors. For hinged connections, both the fixity factor ρ and the rigidity index R are zero; but for rigid connections, the fixity factor is 1 and the rigidity index is infinity. Since the fixity factor can only vary from 0 to 1.0 (while the rigidity index R may vary form 0 to ∞), it is more convenient to use in the analysis of structures with semirigid connections (Cunningham 1990, Xu and Grierson 1993).

The relationships between the fixity factors ρ_a, ρ_b and the alignment charts ratios ψ_a and ψ_b (i.e.,

$\psi=\sum(EI/h)_c/\sum(EI/L)_g$ at the top and bottom ends, respectively) of a column in a symmetrical rigid frame with sidesway uninhibited or partially inhibited are: $\rho_a=2/(2+\psi_a)$, and $\rho_b=2/(2+\psi_b)$ (Aristizabal-Ochoa 1994a, p. 1276-1277). For symmetrical rigid frames with sidesway totally inhibted, the relationships are: $\rho_a=2/(2+3\psi_a)$, and $\rho_b=2/(2+3\psi_b)$. For unsymmetrical frames, the fixity factors can be determined using structural principles as shown by the writer (1994a-c).

Stiffness matrix

The classical stability equations for a prismatic column with rigid connections are formulated using the stiffness coefficients by Salmon and Johnson (1980, p. 837) as follows:

$$M_a=\frac{u\,\mathrm{Sin}\,u-u^2\mathrm{Cos}\,u}{2-2\,\mathrm{Cos}\,u-u\,\mathrm{Sin}\,u}\frac{EI}{h}\theta_a'+\frac{u^2-u\,\mathrm{Sin}\,u}{2-2\cos u-u\,\mathrm{Sin}\,u}\frac{EI}{h}\theta_b' \tag{21a}$$

$$M_b=\frac{u^2-u\,\mathrm{Sin}\,u}{2-2\,\mathrm{Cos}\,u-u\,\mathrm{Sin}\,u}\frac{EI}{h}\theta_a'+\frac{u\,\mathrm{Sin}\,u-u^2\mathrm{Cos}\,u}{2-2\,\mathrm{Cos}\,u-u\,\mathrm{Sin}\,u}\frac{EI}{h}\theta_b' \tag{21b}$$

or simply

$$M_a=r\frac{EI}{h}\theta_a'+s\frac{EI}{h}\theta_b' \tag{21c}$$

$$M_b=s\frac{EI}{h}\theta_a'+r\frac{EI}{h}\theta_b' \tag{21d}$$

where the functions $r=\dfrac{u\,\mathrm{Sin}\,u-u^2\mathrm{Cos}\,u}{2-2\,\mathrm{Cos}\,u-u\,\mathrm{Sin}\,u}$ and $s=\dfrac{u^2-u\,\mathrm{Sin}\,u}{2-2\mathrm{Cos}\,u-u\,\mathrm{Sin}\,u}$ are known as the classical stiffness coefficients; and θ' are the end slopes of member $A'B'$ measured with reference to the axis of the member (Fig. 8b). However, when member AB includes the two lumped flexural connectors AA' and BB' at the ends as shown by Fig. 8a, the stiffness matrix of member AB can be derived by the procedure explained below.

The four flexural degrees of freedom (DOF) of member AB are shown in Fig. 8b. DOF's 1 and 2 correspond to θ_a, θ_b, and DOF's 3 and 4 δ_a, and δ_b, respectively. For instance, the stiffness coefficients corresponding to a unit rotation at A: k_{11}, k_{21}, k_{31}, k_{41} (i.e., moments and shears forces at A and B necessary to have a unit rotation at A while B remains unchanged) are obtained from the following two end conditions:

1) At end A: $M_a=k_{11}$, $\theta_a=1$, and $\theta_a'=\theta_a-\dfrac{M_a}{\kappa_a}=1-\dfrac{k_{11}}{\kappa_a}$

2) At end B: $M_b=k_{21}$, $\theta_b=1$, and $\theta_b'=\theta_b-\dfrac{M_b}{\kappa_b}=0-\dfrac{k_{21}}{\kappa_b}$

when conditions are substituted into (21c-d), (22a-b) are obtained:

$$k_{11}=r\frac{EI}{h}\left(1-\frac{k_{11}}{\kappa_a}\right)-s\frac{EI}{h}\frac{k_{21}}{\kappa_b} \tag{22a}$$

$$k_{21}=s\frac{EI}{h}\left(1-\frac{k_{11}}{\kappa_a}\right)-r\frac{EI}{h}\frac{k_{21}}{\kappa_b} \tag{22b}$$

Now, taking into consideration that $R_a=\kappa_a/\left(\dfrac{EI}{h}\right)$ and $R_b=\kappa_b/\left(\dfrac{EI}{h}\right)$, then

$$k_{11}\left(1+\frac{r}{R_a}\right)=\frac{EI}{h}r-k_{21}\frac{s}{R_b} \tag{22c}$$

$$k_{12}\left(1+\frac{r}{R_b}\right)=\frac{EI}{h}s-k_{11}\frac{s}{R_a} \tag{22d}$$

Substituting Eq. (22d) into Eq. (22c) and using Eq. (20a-b) [i.e., $R_a=3\rho_a/(1-\rho_a)$ and $R_b=3\rho_b/(1-\rho_b)$], then k_{11} and k_{21} can be obtained as follows

$$k_{11}=\frac{3\rho_a(r^2-s^2)(1-\rho_b)+9\rho_a\rho_b r}{(r^2-s^2)(1-\rho_a)(1-\rho_b)+3r(\rho_a+\rho_b-2\rho_a\rho_b)+9\rho_a\rho_b}\frac{EI}{h} \tag{23}$$

$$k_{21}=\frac{9\rho_a\rho_b s}{(r^2-s^2)(1-\rho_a)(1-\rho_b)+3r(\rho_a+\rho_b-2\rho_a\rho_b)+9\rho_a\rho_b}\frac{EI}{h} \tag{24}$$

In terms of the u-factor and after tedious algebra reduction, k_{11} and k_{21} become:

$$k_{11}=\frac{3\rho_a(1-\rho_b)u^2+9\rho_a\rho_b(1-u/\text{Tan }u)}{(1-\rho_a)(1-\rho_b)u^2+3(\rho_a+\rho_b-2\rho_a\rho_b)(1-u/\text{Tan }u)+9\rho_a\rho_b[\text{Tan}(u/2)/(u/2)-1]}\frac{EI}{h} \tag{25}$$

$$k_{21}=\frac{9\rho_a\rho_b(u/\text{Sin }u-1)}{(1-\rho_a)(1-\rho_b)u^2+3(\rho_a+\rho_b-2\rho_a\rho_b)(1-u/\text{Tan }u)+9\rho_a\rho_b[\text{Tan}(u/2)/(u/2)-1]}\frac{EI}{h} \tag{26}$$

Now k_{31} and k_{41} can be obtained from static equilibrium conditions: $k_{31}=-k_{41}=\dfrac{k_{11}+k_{21}}{h}$ or

$$k_{31}=-k_{41}=\frac{3\rho_a(1-\rho_b)u^2+9\rho_a\rho_b u(1-\text{Cos }u)/\text{Sin }u)}{(1-\rho_a)(1-\rho_b)u^2+3(\rho_a+\rho_b-2\rho_a\rho_b)(1-u/\text{Tan }u)+9\rho_a\rho_b[\text{Tan}(u/2)/(u/2)-1]}\frac{EI}{h^2} \tag{27}$$

Similarly, the stiffness coefficients corresponding to DOF $\theta_b=1$ k_{22}, k_{32} and k_{42} can be obtained simply by exchanging ρ_a for ρ_b in Eqs. (25)-(27) as follows:

$$k_{22}=\frac{3\rho_b(1-\rho_a)u^2+9\rho_a\rho_b(1-u/\text{Tan }u)}{(1-\rho_a)(1-\rho_b)u^2+3(\rho_a+\rho_b-2\rho_a\rho_b)(1-u/\text{Tan }u)+9\rho_a\rho_b[\text{Tan}(u/2)/(u/2)-1]}\frac{EI}{h} \tag{28}$$

$$k_{31}=-k_{41}=\frac{3\rho_b(1-\rho_a)u^2+9\rho_a\rho_b(1-\text{Cos }u)/\text{Sin }u)}{(1-\rho_a)(1-\rho_b)u^2+3(\rho_a+\rho_b-2\rho_a\rho_b)(1-u/\text{Tan }u)+9\rho_a\rho_b[\text{Tan}(u/2)/(u/2)-1]}\frac{EI}{h^2} \tag{29}$$

k_{33}, k_{44} and k_{43} can be obtained from equilibrium: $k_{33}=k_{44}=-k_{43}=\dfrac{k_{31}+k_{32}-P}{h}$ as follows:

$$k_{33}=\left[\frac{3(\rho_a+\rho_b-2\rho_a\rho_b)+9\rho_a\rho_b\text{Tan}(u/2)/(u/2)}{(1-\rho_a)(1-\rho_b)u^2+3(\rho_a+\rho_b-2\rho_a\rho_b)(1-u/\text{Tan }u)+9\rho_a\rho_b[\text{Tan}(u/2)/(u/2)-1]}-1\right]\frac{EI}{h^2} \tag{30}$$

The stiffness coefficients given by Eqs. (25)-(30) can now be utilized in Eq. (31) to assemble the stiffness matrix for a single column with semirigid connections that includes the second-order effects caused by the end axial load P.

$$[k]=\begin{bmatrix} k_{11} & & & \\ k_{21} & k_{22} & & \\ k_{31} & k_{32} & k_{33} & \\ k_{41} & k_{42} & k_{43} & k_{44} \end{bmatrix} \tag{31}$$

Notations

The following symbols are used in this paper:

A_t and A_b cross area of columns AC and CB, respectively;

E	Young modulus;
I_t and I_b	moments of inertia of segment columns *AC* and *CB*, respectively;
h_t and h_b	lengths of segment columns *AC* and *CB*, respectively;
K_{ij}	stiffness coefficients for the 3-DOF stepped column given by Eq. (1);
k_{ij}	stiffness coefficient for a single prismatic element given by Eq. (31);
$(P_a)_{cr}$	compressive critical load at *A*;
$(P_a)_{cr}+(P_c)_{cr}$	total compressive critical load at *C* and *B*;
P	applied compression axial load to columns's ends;
P_a	applied compression axial load to columns segment *AC* at *A*;
P_c	applied compression axial load to column segment *CB* at *C*;
ρ_a and ρ_b	rotational fixity factors of column *AB* at top *A* and bottom *B*, respectively;
ρ_c	rotational fixity factor of column *AC* at the intermediate joint *C*;
S_a	lateral stiffness restraining column *AB* against sidesway at top *A*;
S_c	lateral stiffness restraining column *AB* against sidesway at intermediate Joint *C*;
S_θ	rotational stiffness restraining column *AB* externally at intermediate joint *C*;
r and s	classical stiffness coefficients for a beam-column.

References

ACI Committee 318, (1992), "Building code requirements for reinforced concrete and commentary", *ACI* 318-89/318R-89, Revised 1992, American Concrete Institute, Detroit, 347.

Anderson, J.P. and Woodware J.H. (1972), "Calculation of effective lengths and effective slenderness ratios of stepped columns," *Engineering Journal, AISC*, **9**(3), 157-166.

Aristizabal-Ochoa, J. Dario (1994a), "K-factor for columns in any type of construction: nonparadoxical approach," *J. Struct. Engrg., ASCE*, **120**(4), 1272-1290.

Aristizabal-Ochoa, J. Dario (1994b), "Slenderness K-factor for leaning columns," *J. Struct. Engrg., ASCE*, **120**(10), 2977-2991.

Aristizabal-Ochoa, J. Dario (1994c), "Stability of columns under uniform load with semirigid connections," *J. Struct. Engrg., ASCE*, **120**(11), 3212-3222.

Aristizabal-Ochoa, J. Dario (1996), "Braced, partially braced and unbraced columns; complete set of classical stability equations", *Structual Engineering and Mechanics*, **4**(4), July, 365-381.

Castiglioni, C.A. (1986), "Stepped columns: a simplified design method," *Engineering Journal, AISC*, **24**(1), 1-8.

Gerstle, K.H. (1988), "Effects of Connections on Frames." *Steel Beam-to Colukmn Building Connections*, W. F. Chen, Ed., Elsevier Science Publishers Ltd., New York, N. Y., 241-267.

Manual of Steel Construction, *Allowable Stress Design*. (1990). 9th Ed., Chicago, Il.

Manual of Steel Construction, *Load and Resistance Factor Design*. (1986). 1st Ed., Chicago, Il., American Institute of Steel Construction.

Manual of Steel Construction, (1993) *Load and Resistance Factor Design*. 1st Ed., Chicago, Il., Americal Institute of Steel Construction.

Salmon, C.G., and Johnson, J.E. (1980). *Steel Stuctures: Design and Behavior*, 2nd Edition, Harper and Row, Chapter 14.

Shrivastava, S.C. (1980), "Elastic brckling of a column under varying axial force," *Engineering Journal, AISC*, **18**(1), 19-21.

Timoshenko, S. and Gere. J. (1961). *Theory of Elastic Stability*, 2nd Ed., McGraw-Hill, Chapter II.

Xu, L., and Grierson, D.E. (1993). "Computer-automated design of semirigid steel frameworks." *Journal of Structural Engineering*, **119**, 1740-1760.