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Improving the eigenvalue using higher order elements without re-solving

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Abstract. High order finite element have a greater convergence rate than low order finite elements, and in general produce more accurate results. These elements have the disadvantage of being more computationally expensive and often require a longer time to solve the finite element analysis. High order elements have been used in this paper to obtain a new eigenvalue solution with out re-solving the new model. The optimisation of the eigenvalue via the differentiation of the Rayleigh quotient has shown that the additional nodes associated with the higher order elements can be condensed out and solved using the original finite element solution. The higher order elements can then be used to calculate an improved eigenvalue for the finite element analysis.

Key words: finite element; errors; eigenvalue improvement.

1. Introduction

There has been very few successful attempts to predict the error in the eigenvalue for a finite element analysis. Some work has been done by Friberg (1986) (1987), Cook (1991) and Fried (1971) using different approaches. However most of these techniques are not directly applicable to general finite element analysis.

The patch recovery technique is applied to a finite element analysis to improve the eigenvalue solution representing the natural frequency or buckling load factor of a structure. Work in this field has been successfully done by Stephen and Steven (1994) (1995). The improved eigenvalue is achieved by taking a patch around each element and interpolating a relatively high order polynomial for the distorted shape using the nodal displacements in the patch. The interpolated function is determined using a weighted least squares procedure and the order of the polynomial is chosen to be greater than that of the finite element shape function used in the elements of the patch.

Elements with higher order polynomial shape functions generally produce results that are more accurate than elements with low order shape functions (Cook, Malkus and Plesha 1989). A high order element requires additional nodes and larger number of equations to solve. It is common to use elements with low order polynomial shape functions as they are easier to define and the number of computations are reduced.

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Increasing the order of a polynomial in the elements of a finite element model requires adding additional nodes to the model. This will increase both the number of degrees of freedom of the system and also increase the bandwidth of the matrices to solve the problem.

The method produced in this paper calculates a new eigenvalue using the eigenvector solution from a finite element analysis with higher order elements than the original elements. The higher order elements have additional nodes that have their displacements determined from the original eigenvector.

2. Optimised eigenvalue

The eigenvalue λ, can be derived using the eigenvector $\{x\}$, and the Rayleigh quotient shown in Eq. (1):

$$\lambda = \frac{\{x\}^T[K]\{x\}}{\{x\}^T[M]\{x\}} \tag{1}$$

where $[K]$ is the elastic stiffness matrix and $[M]$ is either the mass matrix for natural frequency analysis or the geometric stiffness matrix for linear buckling analysis.

The elements with higher order shape functions will have nodes with no displacements from the original finite element analysis. To determine these displacements the eigenvalue, given by the Rayleigh quotient can be optimised for these displacements. The technique of optimising the eigenvalue has been used by Stephen and Steven (1995) to improve the eigenvector in certain applications. In this paper it will be used to improve the eigenvalue to give a more accurate solution.

To optimise the eigenvalue, the partial differential of the Rayleigh quotient must be equal to zero for each nodal displacement x_i. The differential can be written as Eq. (2):

$$\begin{aligned}\frac{\delta\lambda}{\delta x_i} &= \frac{\delta}{\delta x_i}\left[\frac{\{x\}^T[K]\{x\}}{\{x\}^T[M]\{x\}}\right] \\ &= \frac{\frac{\delta}{\delta x_i}[\{x\}^T[K]\{x\}]\,[\{x\}^T[M]\{x\}] - [\{x\}^T[K]\{x\}]\frac{\delta}{\delta x_i}[\{x\}^T[M]\{x\}]}{[\{x\}^T[M]\{x\}]^2}\end{aligned} \tag{2}$$

Looking at a portion of Eq. (2):

$$\frac{\delta}{\delta x_i}[\{x\}^T[K]\{x\}] = \left(\frac{\delta}{\delta x_i}\{x\}^T\right)[K]\{x\} + \{x\}^T[K]\left(\frac{\delta}{\delta x_i}\{x\}\right) \tag{3}$$

The displacements are linear in the vector form, hence:

$$\frac{\delta}{\delta x_i}\{x\}^T = \{0\ 0\ \cdots\ 0\ 1^i\ 0\ \cdots\ 0\} \tag{4}$$

Rewriting the differential shown in Eq. (3):

$$\frac{\delta}{\delta x_i}[\{x\}^T[K]\{x\}] = 2\{0\ 0\ \cdots\ 0\ 1^i\ 0\ \cdots\ 0\}[K]\{x\} \tag{5}$$

Similarly for the mass matrix (or the geometric stiffness matrix in the case of buckling):

$$\frac{\delta}{\delta x_i}[\{x\}^T[M]\{x\}]=2\{0\ 0\ \cdots\ 0\ 1^i\ 0\ \cdots\ 0\}[M]\{x\} \tag{6}$$

The product of the differentiated vector with the stiffness matrix and the full vector will be a scalar quantity that is dependant on the nodal displacements immediately surrounding the node that is differentiated. Rewriting the differential quantities in Eqs. (5) and (6) as:

$$\{0\ 0\ \cdots\ 0\ 1^i\ 0\ \cdots\ 0\}[K]\{x\}=([K]\{x\})_i \tag{7}$$

$$\{0\ 0\ \cdots\ 0\ 1^i\ 0\ \cdots\ 0\}[M]\{x\}=([M]\{x\})_i \tag{8}$$

which are one dimensional quantities.

Using the terminology in Eqs. (7) and (8) the differential of the eigenvalue can be written as:

$$\frac{\delta\lambda}{\delta x_i}=2\frac{([K]\{x\})_i[\{x\}^T[M]\{x\}]-[\{x\}^T[K]\{x\}]([M]\{x\})_i}{[\{x\}^T[M]\{x\}]^2} \tag{9}$$

The optimum of the eigenvalue will be when the partial differentials for all displacements are equal to zero. Looking at one displacement differential and setting it to zero, Eq. (9) can be written as:

$$0=([K]\{x\})_i[\{x\}^T[M]\{x\}]-[\{x\}^T[K]\{x\}]([M]\{x\})_i \tag{10}$$

$$0=([K]\{x\})_i-\frac{\{x\}^T[K]\{x\}}{\{x\}^T[M]\{x\}}([M]\{x\})_i \tag{11}$$

$$0=([K]\{x\})_i-\lambda([M]\{x\})_i \tag{12}$$

$$0=([[K]-\lambda[M]]\{x\})_i \tag{13}$$

The matrices inside the brackets in Eq. (13) multiplied by the eigenvector to the right will equal the zero vector. This is the characteristic eigen equation. Any other vector multiplied by these matrices will not equal zero. The eigenvalue and eigenvector must be the precise quantities to satisfy this equality. As the Rayleigh quotient is optimised, the eigenvalue and eigenvector are shown to be the optimal quantities.

The optimisation of the eigenvalue via the differentiation of the Rayleigh quotient has shown that the additional nodes as a result of the higher order elements can be calculated for each element individually.

3. Determining the new nodal displacements

Partitioning the eigen equation for the finite element model using the higher order terms, as shown in Eq. (14).

$$\left(\begin{bmatrix} K_{n,n} & K_{n,h} \\ K_{h,n} & K_{h,h} \end{bmatrix}-\lambda\begin{bmatrix} M_{n,n} & M_{n,h} \\ M_{h,n} & M_{h,h} \end{bmatrix}\right)\begin{Bmatrix} u_n \\ u_h \end{Bmatrix}=\begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{14}$$

where the subscript n refers to terms related to the original nodes in the original finite element model and the subscript h refers to terms related to the unknown nodes for the higher order elements.

The vector of known displacements is given by $\{u_n\}$, and the unknown displacements can be determined using the bottom row of the partitioned Eq. (14). Re-writing this equation. leads to the solution for the unknown displacements.

$$\{u_h\} = -([K_{h,h}] - \lambda[M_{h,h}])^{-1}([K_{h,n}] - \lambda[M_{h,n}])\{u_n\} \tag{15}$$

Choosing higher order elements that have the additional nodes with in the boundaries of the elements causes only one element to effect the unknown nodal displacement given in Eq. (15).

The optimum eigenvalue λ is not known in Eq. (15), hence the eigenvalue from the original finite element analysis is used to calculate the unknown nodal displacement associated with the additional nodes in the higher order element.

Once the additional nodal displacements have been calculated the eigenvalue can be calculated using the Rayleigh quotient given in Eq. (1). Unknown displacements can be calculated individually for each element and the numerator and denominator components for the Rayleigh quotient can be summed. Once all elements have been examined, the new eigenvalue can be determined be division of the numerator by the denominator.

4. Examples

4.1. Truss element

The standard truss finite element is a simple two node element. The elemental shape function for this element is given by:

$$N_n = \alpha_1 + \alpha_2 x \tag{16}$$

From the shape function the elastic stiffness, consistent mass and lumped mass matrices can be calculated for this element.

Elastic stiffness matrix:

$$[k] = \frac{EA}{h}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \tag{17}$$

Consistent mass matrix:

$$[m_C] = \frac{\rho h}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix} \tag{18}$$

Lumped mass matrix:

$$[m_L] = \frac{\rho h}{2}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{18}$$

For simplicity a two element finite element model was used to represent a one dimensional structure for a compression wave between two supports. This model is shown in Fig. 1.

There is only one degree of freedom in this model, hence the eigenvector solution is simply:

$$\{x\} = \{1\} \tag{20}$$

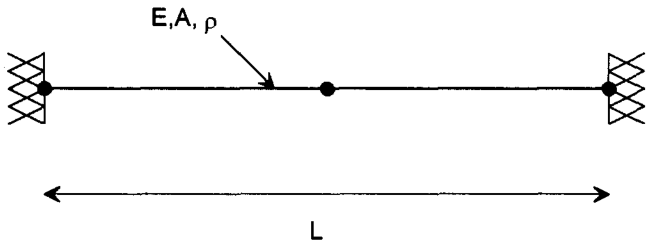

Fig. 1 One dimensional structure using truss elements.

The exact solution to this problem is known, and is:

$$\lambda^{EX} = \frac{\pi^2 EA}{\rho L^2} \tag{21}$$

The finite element analysis results using both the consistent and lumped mass matrices have been calculated and are shown following. As the exact solution is known, an error for the eigenvalue can be calculated.

Consistent mass matrix:

$$\lambda_C = \frac{12EA}{\rho L^2} \tag{22}$$

$$\varepsilon_C = +21.585\% \tag{23}$$

Lumped mass matrix:

$$\lambda_L = \frac{8EA}{\rho L^2} \tag{24}$$

$$\varepsilon_L = -18.943\% \tag{25}$$

The $+$ and $-$ signs indicate the error is above or below the exact solution respectively. The errors for this coarse model are highly significant in magnitude.

An element with a higher order polynomial shape function is a three node truss element. The finite element polynomial shape function for this element is:

$$N_h = \alpha_1 + \alpha_2 x + \alpha_3 x^2 \tag{26}$$

with a corresponding displacement vector:

$$\{u\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_h \end{Bmatrix} \tag{27}$$

Where u_1 and u_2 are the nodal displacements at the ends of the element. The nodal displacement u_h is for the additional node. If the additional node is in the middle of the element the elastic stiffness matrix becomes:

$$[k]=\frac{EA}{3h}\begin{bmatrix} 7 & 1 & -8 \\ 1 & 7 & -8 \\ -8 & -8 & 16 \end{bmatrix} \tag{28}$$

There are three types of mass matrices that can be calculated for this element.
Consistent mass matrix.

$$[m_C]=\frac{\rho h}{30}\begin{bmatrix} 4 & -1 & 2 \\ -1 & 4 & 2 \\ 2 & 2 & 16 \end{bmatrix} \tag{29}$$

Lumped mass matrix:

$$[m_L]=\frac{\rho h}{4}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \tag{30}$$

HRZ lumped mass matrix:

$$[m_H]=\frac{\rho h}{6}\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix} \tag{31}$$

For the two node truss element, the HRZ mass matrix is equal to the lumped mass matrix.

The eigenvalue and eigenvector solution for a two element model is using the three node truss element and for the various types of mass matrices has been calculated. The eigenvalue error is also calculated and shown is Appendix A. As expected, the eigenvalues are much more accurate for all three mass matrices used than the two node truss elements.

As there is only one extra degree of freedom in the three node truss element, the unknown quantity can be calculated in terms of the eigenvalue and end displacements of the two node truss element. This has been done using the three types of mass matrices available.

Consistent mass matrix:

$$u_h=\frac{40EA+\lambda\rho h^2}{8(10EA-\lambda\rho h^2)}(u_1+u_2) \tag{32}$$

Lumped mass matrix:

$$u_h=\frac{16EA}{(32EA-3\lambda\rho h^2)}(u_1+u_2) \tag{33}$$

HRZ mass matrix:

$$u_h=\frac{4EA}{(8EA-\lambda\rho h^2)}(u_1+u_2) \tag{34}$$

This produces a different relationship to the linear interpolation of the two node truss element shape function. However as h limits to zero all three relationships limit to the linear interpolation solution.

The nodal displacement for the additional nodes can be calculated. As there are two elements, there are two additional nodes to calculate. However, as the displacement of the structure is symmetric only one term needs to be calculated. The unknown nodal displacement is calculated

for each mass matrix type. Also the new eigenvalue and the eigenvalue error is also calculated for each mass matrix. The results are as follows:
Consistent mass matrix:

$$u_h = \frac{43}{56} \tag{35}$$

$$\lambda_C{}^* = \frac{6504EA}{647\rho L^2} = 10.0526\frac{EA}{\rho L^2} \tag{36}$$

$$\varepsilon_C{}^* = 1.854\% \tag{37}$$

Lumped mass matrix:

$$u_h = \frac{8}{13} \tag{38}$$

$$\lambda_L{}^* = \frac{2896EA}{297\rho L^2} = 9.75094\frac{EA}{\rho L^2} \tag{39}$$

$$\varepsilon_L{}^* = -1.203\% \tag{40}$$

HRZ mas matrix:

$$u_h = \frac{2}{3} \tag{41}$$

$$\lambda_H{}^* = \frac{248EA}{25\rho L^2} = 9.92000\frac{EA}{\rho L^2} \tag{42}$$

$$\varepsilon_H{}^* = -0.511\% \tag{43}$$

The eigenvalue in all three cases has been significantly improved from around 20% error to 1% error.

4.2. Beam element

The standard two node beam element has four degrees of freedom, being one translational and one rotational displacement at each node. Introducing an additional node in the centre of the beam element and letting this node have a translational displacement component only, the shape function will now be a quartic instead of the standard cubic function.

The new elastic stiffness matrix for a beam element with five degrees of freedom is:

$$[k_E] = \frac{EI}{5h^3}\begin{bmatrix} 316 & 94h & 196 & -34h & -512 \\ 94h & 36h^2 & 34h & -6h^2 & -128h \\ 196 & 34h & 316 & -94h & 512 \\ -34h & -6h^2 & -94h & 36h^2 & 128h \\ -512 & -128h & 512 & 128h & 1024 \end{bmatrix} \tag{44}$$

The geometric stiffness matrix:

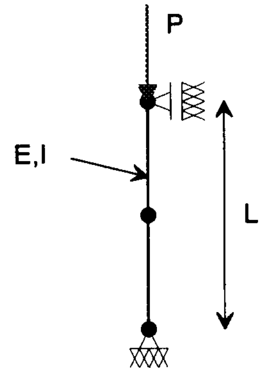

Fig. 2 Finite element model of a simply supported column.

$$[k_G]=\frac{P}{210h}\begin{bmatrix} 508 & 29h & 4 & 13h & -512 \\ 29h & 16h^2 & -13h & 5h^2 & -16h \\ 4 & -13h & 316 & -29h & 512 \\ 13h & 5h^2 & -29h & 16h^2 & 16h \\ -512 & -16h & 512 & 16h & 1024 \end{bmatrix} \tag{45}$$

Optimising the eigenvalue from the Rayleigh quotient using Eq. (15) for the unknown nodal displacement results in:

$$u_h=\frac{(1344EI-32\lambda Ph^2)(u_1+u_2)+(336EI-\lambda Ph^2)h(\theta_1-\theta_2)}{64(42EI-\lambda Ph^2)} \tag{46}$$

Limiting h to zero limits the additional nodal displacement to:

$$\lim_{h\to 0} u_h=\frac{1}{2}(u_1+u_2)+\frac{h}{8}(\theta_1-\theta_2) \tag{47}$$

which is the equation derived from the standard two node beam element shape function.

Examining a two element finite element model representing a pin-ended column buckling shown in Fig. 2, using two standard beam elements the eigenvalue for the buckling load is:

$$\lambda=\frac{16}{3}(13-2\sqrt{31})\frac{EI}{PL^2} \tag{48}$$

The exact buckling load is known:

$$\lambda^{EX}=\frac{\pi^2 EI}{PL^2} \tag{49}$$

Hence the finite element analysis eigenvalue error can be calculated as:

$$\varepsilon=+0.752\% \tag{50}$$

There is a total of four degrees of freedom in this model being two end rotations plus the middle node lateral displacement and rotation. The eigenvector solution for this problem is:

$$\{x\}=\left\{\begin{matrix} 1 \\ \dfrac{4+\sqrt{31}}{30}L \\ 0 \\ -1 \end{matrix}\right\} \tag{51}$$

Using three node beam elements, there is two additional nodes. As the displacement pattern is symmetric both the new nodes have a calculated displacement of:

$$u_h=\frac{7937+878\sqrt{31}}{960(37+4\sqrt{31})}L \tag{52}$$

Using this value in the calculation of the Rayleigh quotient using three node elements, the new eigenvalue becomes:

$$\lambda^*=9.8711\frac{EI}{PL^2} \tag{53}$$

and has an error or:

$$\varepsilon^*=+0.015\% \tag{54}$$

This is a significant improvement on the original eigenvalue from an error of 0.752% to 0.015%.

4.2. Portal frame

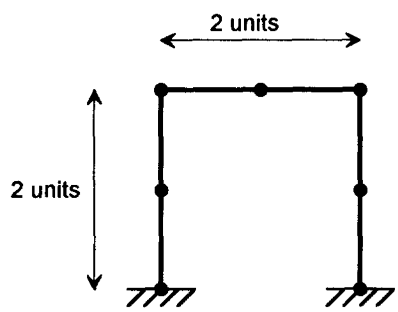

Fig. 3 Finite element model of a portal frame.

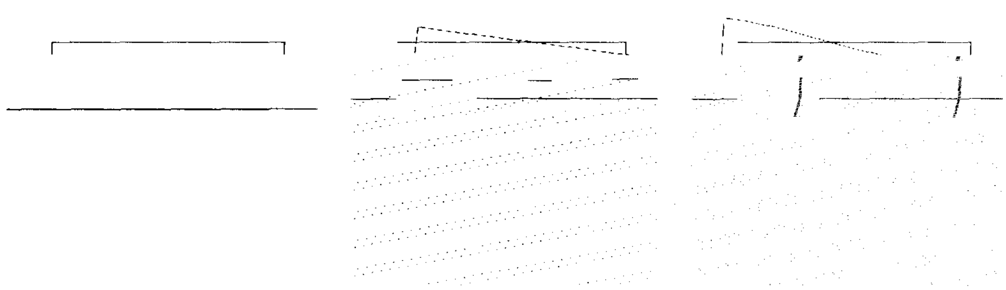

Fig. 4 Displacements for the vibrating portal frame for the first three modes.

A portal frame was analysed for the first three natural frequencies using combined beam and truss elements finite. A consistent mass matrix was used for this example. This frame had values of elastic modulus E, cross sectional area A, second moment of area I, and mass per unit length ρ all equal to one. The structure was analysed for a various number of mesh refinements to extrapolate the exact solution. One particular mesh for this structure is shown in Fig. 3. The displacements for the first three modes, along with the original shape are shown in Fig. 4.

Mode 1 is the extensional vibration of the columns of the portal frame. Mode 2 is a sway type displacement and Mode 3 is a combination of the extension or contraction of the columns with the sway of the portal frame.

The elastic stiffness and consistent mass matrices for the truss elements are shown in Eqs. (28) and (29) respectively. The additional nodal displacement can be calculated from these matrices and is shown in Eq. (32). The elastic stiffness matrix for a beam element with five degrees of freedom is shown in Eq. (44). The consistent mass matrix for this type of element is shown in Eq. (55).

$$[m_C]=\frac{\rho h}{1260}\begin{bmatrix} 260 & 20h & -46 & 7h & 80 \\ 20h & 2h^2 & -7h & h^2 & 8h \\ -46 & -7h & 260 & -20h & 80 \\ 7h & h^2 & -20h & 2h^2 & -8h \\ 80 & 8h & 80 & -8h & 512 \end{bmatrix} \tag{55}$$

From these matrices the additional nodal displacement for the beam element can be calculated as:

$$u_h=\frac{(16128EI+10\lambda h^4)(u_1+u_2)+(4032EI+\lambda h^4)(\theta_1-\theta_2)h}{64(504EI-\lambda h^4)} \tag{56}$$

This again limits to the displacement value predicted by the standard finite element shape function as h tends to zero.

Using this calculated displacement value with the elastic stiffness matrix for a five degrees of freedom beam element, and the three degrees of freedom truss element, the numerator of the Rayleigh quotient in Eq. (1) is similar as that for the standard four degrees of freedom beam element and standard two degrees of freedom truss element respectively. However the denominator terms for the Rayleigh quotient for the beam and truss elements are greatly different to the original finite element solution. The new eigenvalue results are very poor with this method.

The mass matrix using higher order elements does not relate to the mass matrix using the standard elements. To overcome this problem and derive an improved eigenvalue the new mass matrix must be scaled to relate to the original solution. An effective and simple way to achieve this is to calculate the Rayleigh quotient for the higher order elements as λ_h using the predicted displacement values from the higher order elements given by Eqs. (32) and (56) for the truss and beam elements respectively. Then second Rayleigh quotient λ_0 is calculated again using the interpolated displacements from the standard finite element shape functions. Scaling the mass matrix to relate it to the original solution can be done using the ratio of the two eigenvalues obtained. The improved eigenvalue for the finite element result can be calculated as:

$$\lambda^*=\lambda\left(\frac{\lambda_h}{\lambda_0}\right) \tag{57}$$

Table 1 Finite element analysis and improved eigenvalue errors for the portal frame example

Mode 1			
DOF	FEA Error	Error 1	Error 2
15	1.9825%	1.0368%	0.0998%
33	0.4860%	0.2431%	0.0009%
69	0.1026%	0.0415%	0.0195%
Mode 2			
DOF	FEA Error	Error 1	Error 2
15	2.4133%	0.9363%	0.5194%
33	0.5985%	0.2351%	0.1269%
69	0.1513%	0.0616%	0.0281%
Mode 3			
DOF	FEA Error	Error 1	Error 2
15	11.2972%	3.6034%	3.5585%
33	2.7272%	0.9275%	0.8407%
69	0.7063%	0.2634%	0.1776%

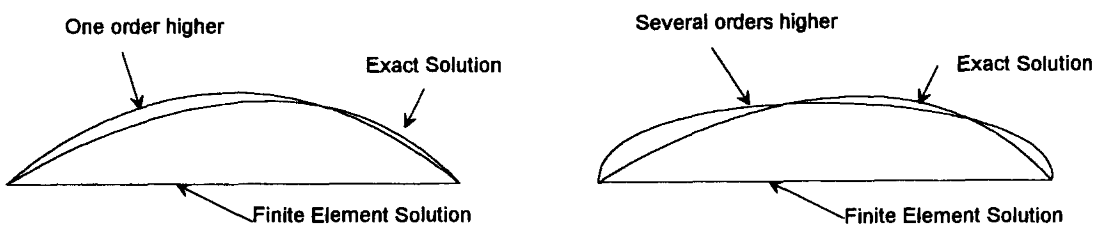

Fig. 5 Convergence of finite element functions.

However the consistent mass matrix is calculated from the square of the shape function, so it is better to have the new eigenvalue as:

$$\lambda^* = \lambda_0 \left(\frac{\lambda_B}{\lambda_A} \right)^2 \tag{58}$$

The finite element analysis errors along with the improved eigenvalue errors for both improvement techniques are tabulated following.

The displacement pattern for Mode 3 is much more complex than the vibrating shapes of Modes 1 and 2, hence the finite element analysis error is much greater for Mode 3. For the simple structure the finite element analysis errors are around 2 per cent for Modes 1 and 2 compared to 11 per cent for Mode 3. Similar relationships are encountered for the more refined meshes.

In all cases the improved eigenvalue is much better than the original finite element eigenvalue. The second method of the square of the ratio produces even better results than the linear ratio, and in some cases great improvements.

5. Comments

The examples in this paper have used only one order higher Lagrange polynomials to develop a new pair of elastic stiffness and mass matrices (or geometric stiffness matrix). Using functions of several orders higher than the basic ones does not always lead to an improved result. This can be seen visually with the aid of the previous Fig. 5.

The much higher order functions can develop waves or oscillations as they attempt to represent the exact solution, where as using only one order higher polynomials than the original finite element shape functions tends to produce shapes that better represent the exact solution. This has been found by Stephen and Stephen (1994) (1995) for a different technique and can justify the success with this method.

6. Conclusions

Higher order finite element have been used in this paper to obtain a new eigenvalue solution. The optimization of the eigenvalue via the differentiation of the Rayleigh quotient has shown that the additional nodes as a result of the higher order elements can be calculated for each element individually.

This paper shows that by using a finite element with a higher order polynomial shape function than the original solution an improved eigenvalue can be estimated with out resolving the finite element model for the increased complexity.

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Appendix A-Solutions for a three node truss element

The one dimensional structure used for the estimation of the natural frequency of a compression wave is modelled using two element truss elements with three nodes in each element. The structure is represented following in Fig. A1.

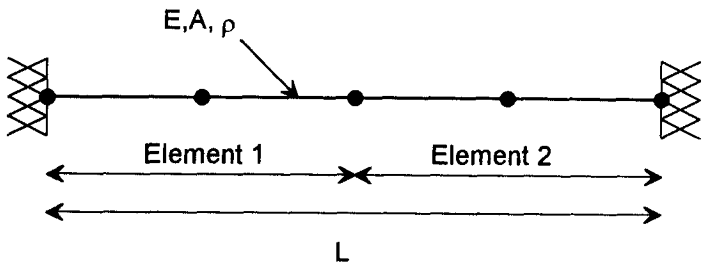

Fig. A1 One dimensional structure using truss elements of three nodes each.

The eigenvalue and eigenvector solution for a two element model is using the three node truss element and for the various types of mass matrices has been calculated. The eigenvalue error is also calculated and shown following.
Consistent mass matrix:

$$\lambda_C = \frac{16}{3}(13 - 2\sqrt{31})\frac{EA}{\rho L^2} \tag{A1}$$

$$\varepsilon_C = +0.752\% \tag{A2}$$

$$\{x_C\} = \begin{Bmatrix} 1 \\ \frac{4}{23}\left(2\sqrt{31} - 3\right) \\ 1 \end{Bmatrix} \tag{A3}$$

Standard lumped mass matrix:

$$\lambda_L = \frac{8}{3}(15 - \sqrt{129})\frac{EA}{\rho L^2} \tag{A4}$$

$$\varepsilon_L = -1.592\% \tag{A6}$$

$$\{x_L\}=\left\{\begin{array}{c}1\\ \frac{1}{8}\left(1+\sqrt{129}\right)\\ 1\end{array}\right\} \tag{A7}$$

HRZ lumped mass matrix:

$$\lambda_H=4(11-\sqrt{73})\frac{EA}{\rho L^2} \tag{A7}$$

$$\varepsilon_H=-0.462\% \tag{A8}$$

$$\{x_H\}=\left\{\begin{array}{c}1\\ \frac{1}{4}\left(\sqrt{73}-3\right)\\ 1\end{array}\right\} \tag{A9}$$

The errors for the eigenvalues using each type of mass matrix is quite small.