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Lateral buckling of thin-walled members with openings considering shear lag

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Abstract. The classical theory of thin-walled members is unable to reflect the shear lag phenomenon since it is based on the assumption of no shearing strains in the middle surface of the walls. In this paper, an energy equation for the lateral buckling of thin-walled members has been derived which includes the effects of torsion, warping and, especially, the shearing strains which reflect the shear lag phenomenon. A numerical analysis for the lateral buckling of thin-walled members with openings by using Galerkin's method of weighted residuals has been presented. The proposed numerical values and the predictions by experiment for the lateral buckling loads are to agree closely in the paper. The results from these comparisons show that the proposed method here is capable of predicting the lateral buckling of thin-walled members with openings. The fast convergence of the results indicates the numerical stability of the method. By the study, a very complex practical eigenvalue problem is transformed into a very simple one of solving only a linear equation with one variable.

Key words: lateral buckling; thin-walled member; shear lag.

1. Introduction

When a thin-walled member is transversely loaded in the plane of its greatest flexural rigidity, it may buckle laterally at a critical load, provided the flexural rigidity of thin-walled member in the plane of bending is large in comparison with the lateral bending rigidity. The lateral buckling is accompanies not only by some torsion of the member but also by some warping. Therefore, the buckling of thin-walled member differs from Euler's classical theory of lateral buckling and is much more complex than that. This lateral buckling is of important in the design of the member because thin-walled structures have widely been used in civil engineering, especially in tall buildings and bridges.

Although numerous investigations of static and dynamic problems of thin-walled members have been made during the past, there are a lot of problems, especially buckling problem, required to be further studied. In the previous analytical methods, the finite element method (Barsoum and Gallagher 1970, and Krajcinovic 1969) is now recognized as an effective tool for predicting buckling loads for thin-walled members. Initially, one-dimensional finite element models were used for lateral-torsional stability of beams. To provide for more generality, a three-dimensional assemblage of thin plate elements was developed (Jonson and Will 1974). However, this method requires large storage computer and much computing time, therefore, its application to practical design for complex structure is greatly limited.

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In recent years, many research efforts were all focused on developing some simple and accurate analytical method for thin-walled structures. In the classical analyses of the lateral buckling of thin-walled beam-columns (Vlasov 1961, Timoshenko and Gere 1961), the buckling is assumed to be dependent of the closed form solution of simply supported thin-walled member under equal and opposite end moments, modified by a so-called "moment modification factor" to allow for the nonuniform in-plane moment distribution. But, these analyses may result in unnecessary conservative design (Pandey and Sherbourne 1990). Among numerical simplified approaches, the energy method is an effective method (Thevendran and Shanmugam 1991).

It must be emphasized that the previous analyses are approximate because they neglect the deformation effect of the secondary shearing stresses due to warping restraint which reflect the shear lag phenomenon. Although the problem of shear lag in its manifestations has been recognized for several decades and has been studied in detail both analytically and experimentally for thin-walled closed member, relative few studies have been made on the effect of the shearing strains along the middle surface of the walls on the lateral buckling of thin-walled open member. The reason is that when the shearing strains are taken into account, the mathematical aspect of the problem becomes considerably complicated as it leads, for example, to an integro-differential equation in partial derivatives in the unknown warping function, for which no closed form solution is available (Mentrasti 1987).

If the effect of the shearing strains on the buckling is significant, a quick evaluation of the possible shearing strain effect is of importance to a practising engineer at the early stage of the design of thin-walled structures with open cross section. A computer run at this stage is neither feasible nor economical as even the cross section itself might be changed in further studies. In this paper, the writer developed a simplified approach to evaluate the effect of the shearing strains in thin-walled open members. To reduce the amount of numerical work in developing the approach, a simply supported thin-walled member subjected to a lateral point load at midspan, as shown in Fig. 1, is used to illustrate the approach developed to solve for the lateral buckling, in which s is a currilinear abscissa; t is the tickness of the wall and c is the shear center.

The present study is focused on establishing a simple numerical procedure, based on energy principle, to estimate the lateral buckling capacity of thin-walled members. A formula for the lateral buckling involving the warping, shear and torsional modes is derived using Galerkin's method of weighted residuals. The formula is believed to be adequate for practical designs and to be simple, as will be shown later in this paper. Because it is frequently necessary to cut openings in the webs of thin-walled members in buildings for the passage of service ducting

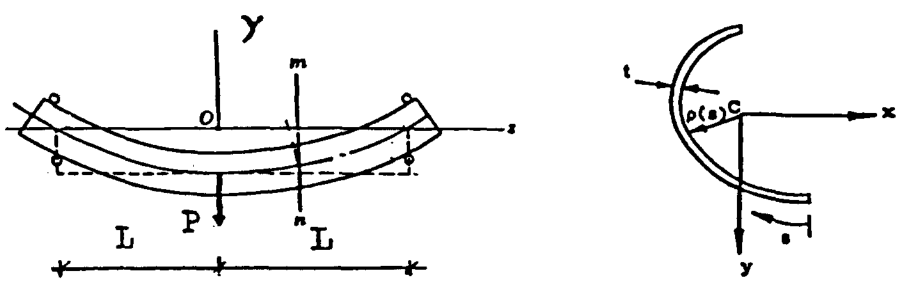

Fig. 1 Simply supported thin-walled member with coordinate system.

and piping, and in bridge structures for inspection purpose, considerable attention has been paid to this problem her. A typical *I*-beam with openings is taken as example to illustrate the application of the proposed method here. The numerically obtained values are compared with experimental results (Thevendran and Shanmugam 1991). A good agreement observed in this paper shows that the method is capable of predicting the lateral buckling of thin-walled members and the effect of openings on the lateral buckling. Though the formula for the lateral buckling load of thin-walled members is to deal with simply supported member, it is all suited to cantilevered thin-walled members subjected to a lateral point load at free end.

2. Energy equation for lateral buckling

Consider a prismatic thin-walled open member whose cross section is shown in Fig. 1, the present theory is based on the following three assumptions:

(1) The cross section can be regarded as rigid in its own plane. According to the Vlasov's assumption, the tangential displacement of any point on the centric line of the thin wall of the cross section can be expressed as

$$d(s,\ z)=\rho(s)\,\phi(z) \tag{1}$$

in which $\rho(s)$ is the distance from the shearing center of the cross section to the tangential line of the point s, and $\phi(z)$ is the twist angle of the cross section.

(2) The membrane stresses of σ_x and σ_y parallel to the x and y axes are much smaller than the longitudinal stress σ_z, and thus by Hooke's law, the longitudinal strain

$$\varepsilon_z \approx \sigma_z/E$$

in which E is the Young's modulus of elasticity.

(3) Kollbrunner and Hajdin's assumption for warping displacement is adopted, and thus the distribution of the warping displacement in the thin-walled member can be written as

$$w(s,\ z)=-\omega(s)\,\theta(z) \tag{2}$$

in which $\omega(s)$ is the sectorial coordinate with respect to point c; $\theta(z)$ is a function representing the distribution of the warping along the length of the member, and for open cross section

$$\omega(s)=\int_0^s \rho(s)\,ds; \tag{3}$$

2.1. Longitudinal normal strain

The longitudinal normal strain on the center line of the thin wall due to warping can be obtained by the kinematic equation of elasticity:

$$\varepsilon_w=\partial w/\partial z=-\omega\theta' \tag{4}$$

in which each prime denotes one derivative with respect to z.

The moment of M_x, applied to major axis, causes the section to twist and, when coupled with shears, causes an additional deflection u in the x-direction during buckling. The moment creates a longitudinal normal strain given by Youg and Trahair (1992)

$$\varepsilon_u = -xu''; \tag{5}$$

2.2. Shearing strains

Because the shear strains due to bending of the thin-walled member are neglected, the shearing strains of the thin-walled member consist of ones due to warping and uniform torsion. The shearing strains in the middle surface of the walls due to warping are

$$\gamma_{sz} = \partial d/\partial z + \partial w/\partial s = \rho\,\phi' - \partial\omega/\partial s\ \theta;$$

Substituting Eq. (3) into the above equation gives

$$\gamma_{sz} = \rho(\phi' - \theta); \tag{6}$$

When the warping is unrestricted, only the so-called St. Venant stresses and strains are present. Generally, the distribution of shearing strains in a thin-walled open member may be shown to be related to the rate of torsion by the expression

$$\gamma_{zs} = 2\nu\ \phi' \tag{7}$$

in which ν is the distance to any point in the cross section measured normally from its center line. Combining Eqs. (4), (5), (6) and (7) gives

$$\varepsilon = -xu'' - \omega\theta'; \tag{8}$$

$$\gamma = 2\nu\phi' + \rho(\phi' - \theta); \tag{9}$$

The expression for the strain energy stored in the thin-walled open member is

$$U = \frac{1}{2}\int_0^L \left[\int_{\sum s} (E\varepsilon^2 + G\gamma^2)t\,ds\right] dz \tag{10}$$

in which G is the shearing modulus of elasticity.

Substituting Eqs. (8) and (9) into Eq. (10) yields

$$\begin{aligned} U = \frac{1}{2}\int_0^L \{ & E[(\int_{\sum s} x^2 t\,ds)(u'')^2 + (\int_{\sum s} \omega^2 t\,ds)(\theta')^2 \\ & + 2(\int_{\sum s} x\omega t\,ds)\,u''\,\theta'] + G[(\int_{\sum s} \rho^2 t\,ds)(\phi' - \theta)^2 \\ & + (\int_{\sum s} 4\nu^2 t\,ds)(\phi')^2 + 2(\int_{\sum s} 2\nu\rho t\,ds)(\phi' - \theta)\,\phi']\}\,dz; \end{aligned} \tag{11}$$

The quantities in parentheses are various geometric properties of the cross section. In particular:

$I_y = \int_{\sum s} x^2 t\,ds$ the second moment of area about y-axis;

$I_w = \int_{\sum s} \omega^2 t\,ds$ the warping moment of inertia;

$J=\int_{\Sigma s} 4v^2 t\, ds$ the St. Venant torsional constant;

$I_p=\int_{\Sigma s} \rho^2 t\, ds$ the polar moment of inertial of the cross section about the center of twist.

The other terms are zero for the following reasons:

$\int_{\Sigma s} x\,\omega t\, ds=0$ because the warping displacements produce no net moment about y-axis (Gellin 1988);

$\int_{\Sigma s} 2v\rho t\, ds=0$ for a prismatic thin-walled open member;

As a result, Eq. (11) reduces as

$$U=\frac{1}{2}\int_0^L [EI_y(u'')^2+EI_w(\theta')^2+GJ(\phi')^2+GI_p(\phi'-\theta)^2]\, dz; \tag{12}$$

The potential energy of the loading system measured from the straight untwisted state is defined by

$$V=-\int_0^L\int_{\Sigma s} T_i \Delta_i t\, ds\, dz$$

in which T_i is a system of conservative surface forces acting on the member in the ***y-z*** plane, and Δ_i is the displacement components corresponding to the T_i. The sum of the strain energy of the member and its loading system can be expressed as

$$\Pi=U+V \tag{13}$$

When the member is in a state of equilibrium, the total potential energy Π of an elastic thin-walled member is stationary. Hence, the following equation must exist:

$$\delta\Pi=0;$$

Substituting Eq. (13) into the above equation gives

$$\delta\left\{\frac{1}{2}\int_0^L [EI_y(u'')^2+EI_w(\theta')^2+GJ(\phi')^2+GI_p(\phi'-\theta)^2]dz-\int_0^L\int_{\Sigma s} T_i \Delta_i t\, ds\, dz\right\}=0; \tag{14}$$

The present formula of energy equation for lateral buckling can be applied for prismatic thin-walled member with any kind of open cross section, for any loading system, and for general end boundary conditions.

3. Governing equation and boundary conditions for lateral buckling

The energy methods are not fundamentally different from the methods based on equilibrium, compatibility, etc., but they are often more convenient. Furthermore, if it is desired to develop a second-order theory, the energy theorems become very valuable tools. In this paper the minimum potential theorem is used to investigate the problem of the lateral buckling. For the sake of simplicity, a simply supported member of length $2L$ having a doubly symmetrical cross section and subjected to a lateral point load $\boldsymbol{P}$ at midspan is considered. Due to the member symmetry, half the member was analyzed.

When $\boldsymbol{P}$ is applied at the centroid of the cross section, the potential energy is given by Masur and Milbradt

$$V=-\int_0^L M_x \phi u'' \, dz \tag{15}$$

in which $\boldsymbol{M}_x$ is the internal moment about the $\boldsymbol{x}$-axis. For a simply supported beam subjected to lateral point load at midspan, the potential energy becomes

$$V=-P\int_0^L \phi u''(L-z)\, dz; \tag{16}$$

Substituting Eq. (16) into Eq. (14), we have

$$\delta\left\{\frac{1}{2}\int_0^L [EI_y(u'')^2+EI_w(\theta')^2+GJ(\phi')^2+GI_p(\phi'-\theta)^2-2P\phi u''(L-z)]\, dz\right\}=0; \tag{17}$$

For this case of a member subjected to no lateral forces and with both ends "simply supported", the lateral curvature of u'' in Eq. (17) can be eliminated by the following relation

$$EI_y u''-P(L-z)\phi=0, \tag{18}$$

to give the following potential energy expression:

$$\delta\left\{\frac{1}{2}\int_0^L [GJ_y(\phi')^2+EI_w(\theta')^2+GI_p(\phi'-\theta)^2-\frac{P^2}{EI_y}(L-z)^2\phi^2]\, dz\right]=0; \tag{19}$$

The first variation with respect to ϕ and θ, respectively, yields

$$\int_0^L [GJ\phi'\delta\phi'+GI_p(\phi'-\theta)\ \delta\phi'-\frac{P^2}{EI_y}(L-z)^2\phi\delta\phi]\, dz=0; \tag{20}$$

$$\int_0^L [EI_w\theta'\delta\theta'-GI_p(\phi'-\theta)\delta\theta]\, dz=0; \tag{21}$$

Noting that the processes of variation and differential can be permutable, the Eqs. (20) and (21), respectively, can be integrated by parts as follows;

$$[GJ\phi'\delta\phi+GI_p(\phi'-\theta)]\delta\phi\Big|_0^L -\int_0^L [GJ\phi''+GI_p(\phi'-\theta)'+\frac{P^2}{EI_y}(L-z)^2\phi]\,\delta\phi dz=0; \tag{22}$$

$$EI_w\theta'\delta\theta'\Big|_0^L-\int_0^L [EI_w\theta''+GI_p(\phi'-\theta)]\,\delta\theta dz=0; \tag{23}$$

To satisfy Eq. (14) for any arbitrary values of $\delta\phi$ and $\delta\theta$, the terms under the integral must vanish. This condition produces the following governing differential equations for the lateral buckling:

$$GJ\,\phi''+GI_p(\phi'-\theta)'+P^2(L-z)^2/EI_y\,\phi=0; \tag{24}$$

$$EI_w\,\theta''+GI_p(\phi'-\theta)=0; \tag{25}$$

And, the corresponding natural boundary conditions at $z=0$ and $z=L$, respectively, must be satisfied too

$$GJ\,\phi'+GI_p(\phi'-\theta)=0; \tag{26}$$

or $\phi=\text{Const.}$;

$$EI_w\,\theta'=0; \tag{27}$$

or $\theta=\text{Const.}$;

Eq. (24) gives

$$\theta'=\mu\phi''+P^2/(EI_y GI_p)\,(L-z)^2\,\phi; \tag{28}$$

Differentiating with respect to z twice gives

$$\theta''=\mu\phi'''+P^2[-2\,(L-z)\,\phi+(L-z)^2\,\phi']/(EI_y GI_p); \tag{29}$$

$$\theta'''=\mu\phi''''+P^2[2\phi-4(L-z)\,\phi'+(L-z)^2\,\phi'']/(EI_y GI_p) \tag{30}$$

in which $\mu=1+J/I_p$.

Eliminating $(\phi'-\theta)$ from Eqs. (24) and (25) gives

$$-GJ\,\phi''+EI_w\,\theta'''-P^2(L-z)^2/EI_y\,\phi=0; \tag{31}$$

Substituting Eq. (30) into the above equation yields

$$\mu\phi''''-K^2\phi''=P^2/(E^2I_wI_y)\,\{(L-z)^2\phi-\alpha^2[2\phi-4(L-z)\phi'+(L-z)^2\phi'']\} \tag{32}$$

in which $K^2=GJ/EI_w$; and $\alpha^2=EI_w/GI_p$.

Elimination $(\phi'-\theta)$ from Eqs. (25) and (26) yields

$$\theta''-K^2\phi'=0; \tag{33}$$

Substituting Eq. (29) into Eq. (33) gives

at $z=0$,

$$\mu\phi'''-K^2\phi'=-\alpha^2P^2/(E^2I_wI_y)(-2L\,\phi+L^2\phi'); \tag{34a}$$

at $z=L$,

$$\mu\phi'''-K^2\phi'=-0; \tag{34b}$$

Substituting Eq. (28) into Eq. (27) again we have

at $z=0$.

$$\mu\phi''=-\alpha^2P^2/(E^2I_wI_y)\,L^2\phi; \tag{35a}$$

at $z=L$,

$$\mu\phi''=-0; \tag{35b}$$

Eq. (32) is the differential equation governing the buckling of thin-walled member derived by the displacement variational method and Eqs. (34) and (35) are the corresponding boundary conditions.

When the member buckles laterally, the smallest such value that yields a non-trivial second solution for the member is known as the critical load called as P_{cr}.

Introducing nondimensional quantities

$$\xi=(L-z)/L;$$
$$\lambda^*=P_{cr}/P_e \tag{36}$$
$$(p_e)^2=(E^2 I_w I_y)/L^6, \tag{37}$$

Eq. (32) can be rewritten as

$$\mu\phi''''(\xi)-K^2L^2\phi''(\xi)=(\lambda^*)^2[\phi^2\phi-\alpha^2/L^2(2\phi+4\xi\phi'(\xi))+\xi^2\phi''(\xi)]; \tag{38}$$

And, the corresponding boundary conditions will be

(1) At the free end

$$\phi''=0; \tag{39a}$$
$$\mu\phi'''-K^2L^2\phi'=\alpha^2/L^2(\lambda^*)^2[2\phi+\phi']; \tag{39b}$$

(2) At the fixed end

$$\phi=0; \tag{40a}$$
$$\phi'=0; \tag{40b}$$

(3) At the simply supported end

$$\phi=0; \tag{41a}$$
$$\phi''=0; \tag{41b}$$

(4) At the symmetrical location

$$\phi'=0; \tag{42a}$$
$$\mu\phi'''-K^2L^2\phi'=\alpha^2/L^2(\lambda^*)^2[2\phi+\phi']; \tag{42b}$$

From Eqs. (36) and (37) we have

$$P_{cr}=\lambda^*(E^2I_yI_w)^{0.5}/L^3; \tag{43}$$

Eq. (38) is an eigenvalue problem of the governing differential equation and can usually be solved by numerical methods.

4. Numerical solution for lateral buckling equation

Of the available techniques, the Galerkin's method of weighted residuals is selected to solve the present problem. This method is proved to be convenient (Wang 1991).

Let the eigenvalue problem be defined by

$$M_{2m}(\psi)=\lambda N_{2n}(\psi) \qquad (m>n) \tag{44}$$

in a bounded domain D, which subjects to $\boldsymbol{m}$ boundary conditions at each boundary point, and their forms are:

$$A_i(\psi)=0 \tag{45a}$$

$$\bar{A}_i(\psi)=\lambda\bar{D}_i(\psi) \tag{45b}$$

in which M_{2m} and N_{2n} are linear, self-adjoint, differential operators of the $\boldsymbol{2m}$ and $\boldsymbol{2n}$ orders, respectively. A trial solution is assumed in the form

$$\psi_r=\sum_r C_j\eta_j \qquad j=1,\ 2,\ \cdots,\ r \tag{46}$$

in which the coefficients C_j are undetermined parameters, and values of η_j are independent known functions. So, it is chosen to satisfy at least the homogeneous boundary conditions of Eq. (45a).

Substituting Eq. (46) into Eq. (44) will give rise to residuals within domain D

$$R_1=-M_{2m}(\psi_r)+\lambda N_{2n}(\psi_r) \tag{47a}$$

and on the boundary $\boldsymbol{s}$

$$R_2=\sum[-\bar{A}_i(\psi_i)+\lambda\bar{D}_i(\psi_i)] \tag{47b}$$

Galerkin's method requires that the weighted averages of the residuals corresponding to the weighting functions η_j are identical with the orthogonality conditions (Crandall 1956)

$$\int_D R_1\eta_j dD+\sum\int_s R_2\eta_j ds=0 \qquad j=1,\ 2,\ \cdots,\ r \tag{48}$$

and provides a set of $\boldsymbol{r}$ algebraic equations for the determination of the unknown parameters C_j. These equations may then be expressed in the desired matrix form of

$$[A]\{C\}=\lambda[B]\{C\} \tag{49}$$

in which $\{C\}$ is a column vector of the unknown coefficients C_j, and $[A]$ and $[B]$ are the square matrices composed of elements a_{jk} and b_{jk} given by

$$a_{jk}=\int_D \eta_j M_{2m}(\eta_k)\,dD+\sum\int_s \eta_j\bar{A}_i(\eta_k)\,ds \tag{50a}$$

$$b_{jk}=\int_D \eta_j N_{2n}(\eta_k)\,dD+\sum\int_s \eta_j\bar{D}_i(\eta_k)\,ds \tag{50b}$$

In a self-adjoint eigenvalue problem, the matrices $[A]$ and $[B]$ will always be symmetric. Further, $[A]$ and $[B]$ will be of positive symmetric form if the operators M_{2m} or N_{2n}, respectively, are positive definite. In the present case, the generalized coordinate η_j is chosen from the trigonometric family $\mathbf{sin}\ m\pi\xi\ /\mathbf{2}$, which satisfied the homogeneous boundary conditions

$$\phi(0)=0 \text{ and } \phi'(1)=0;$$

The solution of Eq. (38) can be expressed for convenience as

$$\phi(\zeta)=\psi_r=\sum_r C_j\eta_j=\sum_r C_j\sin\frac{m\pi\zeta}{2}\ ; \tag{51}$$

To solve Eq. (38) using the method of weighted residuals the equation is rewritten in the form of Eq. (44)

$$M_4(\phi)=\lambda N_2(\phi); \tag{52}$$

$$\bar{A}_0(0)=\lambda \bar{D}_0(0); \tag{53a}$$

$$A_1(1)=\lambda \bar{D}_1(1) \tag{53b}$$

in which $M_4(\phi)=\mu\phi''''(\xi)-K^2L^2\phi''(\xi)$;

$$N_2(\phi)=\xi^2\phi(\xi)-\alpha^2[2\phi+4\xi\phi'(\xi)+\xi^2\phi''(\xi)]/L^2;$$
$$\bar{A}_0(0)=\phi''(0); \qquad \bar{D}_0(0)=0;$$
$$\bar{A}_1(1)=\mu\phi'''(1)-K^2L^2\phi'(1); \quad \bar{D}_1(1)=\alpha^2[2\phi(1)+\phi'(1)]/L^2;$$

Substituting Eq. (51) into Eq. (50) gives

$$a_{jk}=\int_0^1 \eta_j M_4(\eta_k)\,d\zeta+\sum_{\zeta=0}\eta_j\bar{A}_\zeta(\eta_k) \tag{54a}$$

$$b_{jk}=\int_0^1 \eta_j N_2(\eta_k)\,d\zeta+\sum_{\zeta=0}\eta_j\bar{D}_\zeta(\eta_k) \tag{54b}$$

From Eq. (49), λ is determined to be the smallest positive root satisfying

$$\det\,|[A]-\lambda[B]|=0 \tag{55}$$

Taking advantage of the condition of orthogonality

$$\int_0^1 \eta_j(\zeta)\,\eta_k(\zeta)\,a\zeta=|\frac{1}{2}\delta_{jk}| \tag{56}$$

It can be shown that

$$\begin{aligned} a_{jk}=&\int_0^1 \sin\frac{n\pi\zeta}{2}\left[\mu\frac{n^4\pi^4}{2^4}+K^2L^2\frac{n^2\pi^2}{2^2}\right]\sin\frac{n\pi\zeta}{2}\,d\zeta \\ &+\sin\frac{m\pi\zeta}{2}\left(-\mu\frac{n^3\pi^3}{2^3}-K^2L^2\frac{n\pi}{2}\right)\cos\frac{n\pi\zeta}{2}\Bigg|_{\zeta=1} \\ &+\sin\frac{m\pi\zeta}{2}\left(-\mu\frac{n^2\pi^2}{2^2}\right)\sin\frac{n\pi\zeta}{2}\Bigg|_{\zeta=0} \end{aligned} \tag{57}$$

Hence, if $j\neq k$,

$$a_{jk}=0, \tag{58}$$

and if $j=k$

$$a_{jk}=(\mu m^4\pi^4/2^4+K^2L^2m^2\pi^2/2^2)/2 \tag{59}$$

in which $m=2j-1$, and $n=2k-1$. The matrix $[A]$ is a diagonal matrix. A similar calculation yields b_{jk}, then if $j=k$

$$b_{jk}=[1/6+1/(m^2\pi^2)](1+\alpha^2/L^2m^2\pi^2/2^2), \tag{60}$$

and if $j \neq k$

$$b_{jk}=2^2/\pi^2[(-1)^{1/2(m-n)}/(m-n)^2-(-1)^{1/2(m+n)}/(m+n)^2](1+\alpha^2/L^2m^2\pi^2/2^2)$$
$$-2m\alpha^2/L^2[(-1)^{1/2(m+n)}/(m+n)+(-1)^{1/2(m+n)}/(m-n)]+2\alpha^2/L^2; \quad (61)$$

The matrix characteristic equation may be set up and solved for λ. The accuracy of the solution is determined by the terms of the sine series taken. From the convergence study in the next section of the present paper, one can see that for a satisfactory accuracy, it is sufficient to take only the first term of the sine series.

5. Comparison with experimental results

The following example has been investigated by Thevendran and Shanmugam (1991). And, it can be observed later in the next section that the proposed formula for the lateral buckling load in this paper is very simple.

Example:

The specimens of simply supported ***I***-beam are of length $\boldsymbol{2L}=940$ mm, web thickness $t_w=6$ mm, overall web depth $\boldsymbol{d}=75$ mm, flange thickness $t_f=10$ mm, and flange width $\boldsymbol{b}=23.5$ mm; The six locations for the openings over the left half of the span are indicated in Fig. 2.

The locations are numbered from the support to midspan, and are symmetric about midspan. The openings are either rectangular or circular, with their number and sizes varied. Specimens with the following sets of openings are studied:

1. No opening;
2. One rectangular opening;
3. Three rectangular openings;
4. Six rectangular openings;
5. Three circular opening;

For the sets 2, 3, and 4, two different sizes are considred, 62.5 mm×50 mm and 62.5 mm×25 mm; For 5, two different diameters, 38 mm and 25 mm, are considered.

The Test is carried out using specimens from plexiglass sheets having average value of Young's modulus $\boldsymbol{E}=2{,}860$ N/mm^2 and Poison's ratio $\nu=0.36$.

Since the presence of openings may be regarded as roughly similar to a reduction in the

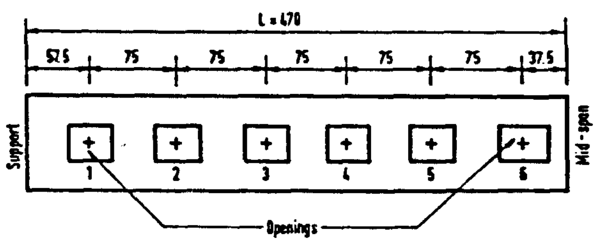

Fig. 2 Locations of openings.

effective thickness, a possible approximate evaluation of the buckling load for a member with openings may be derived by replacing the thickness by an effective thickness which takes account of the loss of material. This may be achieved by considering the horizontal portion of the member containing the openings to act as a segment with a reduced thickness, by "diffusing" the area of web sections between openings throughout the whole length of the member. In that case, the reduced coefficient of rigidity due to openings becomes (Coull and Alvarez 1980)

$$\alpha_1=(1-b_0/d)+[1-N a_0/(2L)]^3 b_0/d \tag{62}$$

in which $\boldsymbol{N}$ is the number of rectangular openings; $\boldsymbol{a_0}$ is the length of opening; $\boldsymbol{b_0}$ is the depth of opening, and $\boldsymbol{d}$ is the overall web depth.

For members with circular openings, a corresponding approximate formula is very complicated in view of the nonregularity of the web areas between openings. Since the effective width concept is relatively crude, the derivation of an accurate formula is not warranted, and a simple solution may be achieved by replacing the circular opening by an equivalent polygonal opening. The simplest approach is to replace the real circular opening by an octagonal opening, the polygon being circumscribed in a circle of diameter $\boldsymbol{D_0}$. The reduced coefficient then becomes (Coull and Alvarez 1980).

$$\alpha_2=(1-D_0/d)+0.172[N(D_0)^2/(d\,2L)]+(D_0/d)\,[1-ND_0/(2L)] \tag{63}$$

For $\boldsymbol{I}$-beam, EI_y, GJ_d, and EI_w (Timoshenko and Gooder 1951) are given by

$$EI_y=E/12\,[2b^3 t_f+\alpha_i d\,(t_w)^3] \tag{64a}$$

$$GJ_d=G/3\,[2k_f b(t_f)^3+\alpha_i k_w d(t_w)^3] \tag{64b}$$

$$EI_w=E/24\ [t_f b^3\ \ (d+t_f)^2] \tag{64c}$$

in which $\boldsymbol{i}$ is either 1 or 2; $\boldsymbol{G}$ is the shear modulus, which is given by

$$G=E/[2(1+\nu)];$$

k_f and k_w in Eq. (64b) are constants given by

$$k_i=1-0.63\ r_i \tag{65}$$

in which $\boldsymbol{i}$ is either $\boldsymbol{f}$ or $\boldsymbol{w}$, and $r_f=t_f/b$, $r_w=t_w/d$.

The numerically computed lateral buckling capacity of the member is compared with those obtained from experimental results (Thevendran and Shanmugam 1991) in Table 1.

From the Table 1, it can be seem that the lateral buckling loads of $\boldsymbol{P_{cr}}$ obtained by using the method of weighted residuals are close to the experimental results (Thevendran and Shanmugam 1991). And, the relative differences in the results between the two methods are all under 3.8%.

6. Convergence of the solutions

As mentioned previously, the lateral buckling mode is expressed as a trigonometric series. To study its convergence, the value of the lateral buckling load was examined by taking the first one term, or the first two terms, and subsequently the first three terms in the series. The

Table 1 Comparison of results (N)

Specimen	Size of opening (mm)	Location of opening	P_{cr} by experiment	P_{cr1} by proposed M.	P_{cr}/P_{cr}
SI0RA	–	–	707.8	681.0	0.962
SI1RC	62.5×50.	6	688.7	673.3	0.978
SI1RF	62.5×25.	6	698.7	677.2	0.969
SI3RA	62.5×50.	1, 3, 6	667.1	659.7	0.989
SI3RB	62.5×25.	1, 3, 6	684.7	670.9	0.980
SI6RA	62.5×50.	1-6	637.7	644.0	1.010
SI6RB	62.5×25.	1-6	674.9	664.3	0.984
SI3CA	D=38.	1, 3, 6	696.5	677.9	0.973
SI3CB	D=25.	1, 3, 6	700.9	679.6	0.970

Table 2 Effect of number of terms taken on values of P_{cr} (N)

Specimen	The first one term	The first two terms	The first three terms
SI0RA	680.9928742	680.9926769	680.9926732
SI1SC	673.3127423	673.3125414	673.3125376
SI1RF	677.1856725	677.1854736	677.1854698
SI3RA	659.7131684	659.7129603	659.7129564
SI3RB	670.8727556	670.8725534	670.8725497
SI6RA	644.0022958	644.0020789	644.0020749
SI6RB	664.2474457	664.2472401	664.2472362
SI3CA	677.8739932	677.8737946	677.8737909
SI3CB	679.6137844	679.6135867	679.6135829

differences in the results for the three different numbers of terms in the series as shown in the Table 2 almost disappear. Therefore, in an engineering design for the lateral stability of thin-walled member it is sufficient to use only the first one term in the Fourier expansion for the buckling mode. Hence, $m=1$, Eq. (55) reduces to:

$$(\lambda^*)^2=a_{11}/b_{11}. \tag{66}$$

Substituting the above equation into Eq. (43) gives

$$P_{cr}=\beta_1\{EI_y GJ[1+\mu\pi^2 EI_w/(a^2 GJ)]\}^{0.5}/a^2 \tag{67}$$

in which $\beta_1=2\pi/[2(1/6+1/\pi^2)(1+\alpha^2/L^2\pi^2/2^2)]^{0.5}$ is a constant.

It is very simple and convenient to use.

7. Effect of shearing strains on lateral buckling

If the shear strains along the middle surface of the walls are neglected, Eq. (38) will be reduced as

$$\phi''''(\xi)-K^2L^2\phi''=(\lambda^*)^2\xi^2\phi, \tag{68}$$

Table 3. Effect of shearing strains on values of P_{cr}(N)

R	20	10	5	2.5
γ_{zs} is neglected	2928.731	15057.136	97353.978	728322.428
γ_{zs} is considered	2926.676	14956.936	93624.837	622308.838
Differences	0.1%	1.%	4.%	15.%

The λ^* can be solved by the same calculation as the above section, we have

$$(\lambda^*)^2 = a_{11}/b_{11}$$

in which

$$a_{11} = (\pi^4/2^4 + K^2 L^2 \pi^2/2^2)/2; \tag{70}$$

$$b_{11} = 1/6 + 1/\pi^2; \tag{71}$$

The same example as the previous section with one rectangular opening is selected to demonstrate the effect. The different results considering the shearing strains in the middle surface of the walls are compared with the ones neglecting the shearing strains in the Table 3. in which R is the ratio of height of the member to width. From the example shown in the Table 3, it can be seen that the effect of the shearing strains in the middle surface of the walls on the lateral buckling increases significantly as R decreases.

8. Conclusions

From the numerical example above, the following some conclusions can be drawn:

(1) An energy equation for the lateral buckling of thin-walled open members has been derived in which the effects of torsion, warping and, especially, the shearing strains in the middle surface of the walls are taken into account. A numerical analysis for the lateral buckling of simply supported thin-walled member by using Galerkin's method of weighted residuals has been presented in this paper.

(2) The proposed numerical values and the predictions by experiment for the lateral buckling loads are to agree closely in this paper. The results from these comparisons show that the proposed method in this paper is capable of predicting the lateral buckling of simply supported thin-walled membe with openings.

(3) The fast convergence of the results indicates the numerical stability of the method.

(4) The method presented in this paper yields the formula for computing the lateral buckling load that is very simple and straightforward. By the study, a very complex practical eigenvalue problem is transformed into a very simple one of solving only a linear equation with one variable.

(5) The same method can be applied for other types of load and end constraint conditions.

(6) The effect of the shearing strains in the middle surface of the walls on the lateral buckling increases significantly as R decreases.

References

Barsoum, R. S. and Gallagher, R. H. (1970), "Finite element analysis of torsional and torsional-flexural stability problems", *Int. J. for Numerical Methods in Engrg.*, **2**(3).

Coull, A. and Alvarez, M.C. (1980), "Effect of openings on lateral buckling of beams", *J. of the Struct. Div., ASCE*, **106**(12), 2553-2560.

Crandall, S. (1956), *Engineering Analysis*, McGraw Hill Book Co., New York, N.Y.

Gellin, S. and Lee, G. C. (1988), "Finite element analysis of thin-walled structures: Finite elements available for the analysis of non-curved thin-walled structures", *Elsevier Applied Science*, 1-30. London and New York., 1988.

Jonson, C. P. and Will, K. M. (1974), "Beam buckling by finite element procedure", *J. of the Struct. Div., ASCE*, **100**(3), 669-685.

Kollbrunner, C. F. and Hajdin, H., "Wöibkraft-torsion dünnwardiger stäbe mit geschlossenem profil", *Schweized Stahlbauverband Mitteilungen der Technischen Kommision*, Heft 32.

Krajcinovic, D. (1969), "A consisted discrete element technique for thin-walled assemblages", *Int. J. of Solid and Structures*, **5**(7).

Masur, E. F. and Milbradt, K. P., "Collapse strength of redundant beams after lateral buckling", *J. Appl. Mech., Trans., ASME*, **24**(2), 283-288.

Mentrasti, L. (1987), "Torsion of closed cross-section thin-walled beams: The influence of shearing strain", *J. Thin-Walled Structures*, **5**, 277-305.

Pandey, M. D. and Sherbourne, N. S. (1990), "Elastic, lateral torsional stability of beams: general considerations", *J. of Struct. Engrg., ASCE*, **116**(2), 317-335.

Thevendran, V. and Shanmugam, N.E. (1991), "Lateral buckling of doubly symmetric beams containing openings", *J. of Engrg. Mech., ASCE*, **117**(7), 1427-1441.

Timoshenko, S. P. and Gooder, J. N. (1951), *Theory of Elasticity*, McGraw-Hill Book Co., New York, N.Y.

Timoshenko, S. P. and Gree, J. M. (1961), *Theory of Elastic Stability*, 2nd Ed., McGraw Hill Book Co., New York, N.Y.

Vlasov, V. Z. (1961), *Thin-Walled Elastic Beams*, 2nd Ed., Israel Program for Scientific Translation, Jerusalem, Israel.

Wang, Quanfeng (1991), "Stability of shear-wall building using method of weighted residuals", *J. of Engrg. Mech., ASCE*, **117**(3), 700-706.

Youg, L. P. and Trahair, N. S. (1992), "Energy equation for beam lateral buckling", *J. of Struct. Engrg., ASCE*, **118**(6), 1462-1479.