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Torsional analysis for multiple box cells using softened truss model

Daili Yang† and Chung C. Fu‡

Department of Civil Engineering, University of Maryland, U.S.A.

Abstract. A new torsional analysis method for multiple cell box based on the Softened Truss Model Theory was developed. This softened truss model unifies shear and torsion to address the problem associated with a torque applied on a box. The model should be very useful for the analysis of a reinforced concrete box under torque, especially for the bridge superstructure with multiple cell box sections.

Key words: bridge analysis; torsion; softened truss model; multiple cell; box; reinforced concrete.

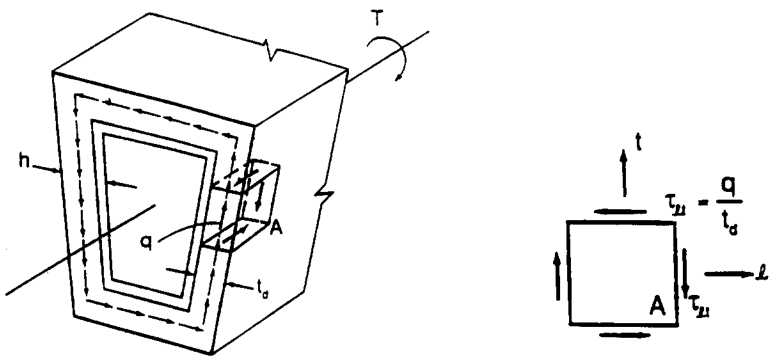
1. Introduction

Concrete is a material that has a very high compressive strength but weak tensile strength. When concrete is used in a structure to carry loads, the tensile regions are expected to crack and, therefore, must be reinforced with materials of high tensile strength, such as steel. The concept of utilizing concrete to resist compression and steel reinforcement to carry tension gave rise to the Truss model. In this model, concrete compression struts and steel tension ties form a truss that is capable of resisting applied loads. In the recently published AASHTO LRFD specifications (1994), truss modeling is also considered as an alternate method for reinforced or prestressed concrete structures.

By combining equilibrium, compatibility, and the softened constitutive laws of concrete, Hsu and his colleagues (1988) (1990) (1991a) (1991b) (1993) developed a theory that can predict with good accuracy the behavior of various types of structures subjected to shear. By including an additional equilibrium equation and four additional compatibility equations, the theory became applicable to torsion. This theory unified shear and torsion, and was called the softened truss model. However, up to now the theory has been applied to only a single cell section subjected to pure torsion.

Based on the Softtened Truss Model Theory, a new method has been developed by the authors to address torsion, especially for reinforced concrete box girder bridge superstructures with multiple cell sections.

† Research Assistant and Ph.D. Candidate
‡ Associate Professor, Ph.D., P.E., Director of the Bridge Engineering Software & Tech. Ctr.



(A) ELEMENT IN SHEAR FLOW ZONE (B) SHEAR STRESS ON ELEMENT A

Fig. 1 Hollow box subjected to torsion (Hsu 1993).

2. Previous research for single cell

In this section, a set of equations is given for solving single cell torsion. These equations are adopted directly from Hsu (1988).

A reinforced concrete prismatic member is subjected to an external torque T as shown in Fig. 1a. This external torque is resisted by an internal torque formed by the circulatory shear flow $\dot{q}$ along the periphery of the cross section. This shear flow q occupies a zone, called the shear flow zone, which has a thickness denoted t_d. This thickness t_d is a variable determined from the equilibrium and compatibility conditions. It is not the same as the given wall thickness h of a hollow member. Element A in the shear flow zone (Fig. 1a) is subjected to a shear stress $\tau_{lt}=q/t_d$ as shown in Fig. 1b.

3. Governing equations

According to the Unified Theory (Hsu 1993), the governing equations for equilibrium and compatibility are shown as follows

4. Equilibrium equations

$$\sigma_l=\sigma_d\cos^2\alpha+\sigma_r\sin^2\alpha+\rho_l f_l \tag{1}$$

$$\sigma_t=\sigma_d\sin^2\alpha+\sigma_r\cos^2\alpha+\rho_t f_t \tag{2}$$

$$\tau_{lt}=(-\sigma_d+\sigma_r)\sin\alpha\cos\alpha \tag{3}$$

$$T=\tau_{lt}(2A_0 t_d) \tag{4}$$

where

σ_l, σ_t and τ_{lt}	three stress components of an element
σ_d, σ_t	concrete stresses in d and r directions, respectively
α	angle between l and d axes
f_l, f_t	stresses in steel in the l and t directions, respectively
ρ_l, ρ_t	steel ratio in the l and t directions, respectively
T	external torque
A_0	cross-sectional area bounded by the center line of the shear flow zone
t_d	shear flow zone thickness

5. Compatibility equations

$$\varepsilon_l = \varepsilon_d \cos^2\alpha + \varepsilon_r \sin^2\alpha \tag{5}$$

$$\varepsilon_t = \varepsilon_d \sin^2\alpha + \varepsilon_r \cos^2\alpha \tag{6}$$

$$\frac{\gamma_{lt}}{2} = (-\varepsilon_d + \varepsilon_r)\sin\alpha\cos\alpha \tag{7}$$

$$\theta = \frac{p_0}{2A_0}\gamma_{lt} \tag{8}$$

$$\psi = \theta \sin 2\alpha \tag{9}$$

$$t_d = \frac{\varepsilon_{ds}}{\psi} \tag{10}$$

$$\varepsilon_d = \frac{-\varepsilon_{ds}}{2} \tag{11}$$

where

ε_l, ε_t, and γ_{lt}	three strain components in the l-t coordinate system
ε_d and ε_r	strains in the d and l directions, respectively
θ	angle of twist of a member
p_0	perimeter of the center line of shear flow zone
ψ	bending curvature of concrete struts
ε_{ds}	maximum strain of concrete struts

6. Constitutive laws of materials

6.1. Concrete struts

$$\sigma_d = k_1 \zeta f_c' \tag{12}$$

$$k_1 = {}_1(\varepsilon_{ds}, \zeta) \tag{13}$$

$$\zeta = {}_2(\varepsilon_d, \varepsilon_r) \tag{14}$$

$$\sigma_r = 0$$

6.2. Mild steel

$$f_l = {}_3(\varepsilon_l) \tag{15}$$

$$f_t = {}_4(\varepsilon_t) \tag{16}$$

where

k_1 coefficient defined as the ratio of the average compressive stress σ_d to the softened compressive peak stress $\sigma_p = \zeta f_c'$ in concrete struts
ζ softening coefficient
f_c' concrete strength

7. Selected constitutive equations

For the treatment of torsion, the constitutive equations, Eqs. (13) to (16), will be selected in the following text. The simple elastic-perfectly plastic stress-strain relationship of bare mild-steel bars was assumed, because the tensile stress of concrete has been neglected.

7.1. Concrete struts

$$k_1 = \frac{\varepsilon_{ds}}{\zeta \varepsilon_0}\left(1 - \frac{1}{3}\frac{\varepsilon_{ds}}{\zeta \varepsilon_0}\right) \qquad \frac{\varepsilon_{ds}}{\varepsilon_p} \le 1 \tag{13a}$$

$$k_1 = \left[1 - \frac{\zeta^2}{(2-\zeta)^2}\right]\left(1 - \frac{1}{3}\frac{\zeta \varepsilon_0}{\varepsilon_{ds}}\right) + \frac{\zeta^2}{(2-\zeta)^2}\frac{\varepsilon_{ds}}{\zeta \varepsilon_0}\left(1 - \frac{1}{3}\frac{\varepsilon_{ds}}{\zeta \varepsilon_0}\right) \qquad \frac{\varepsilon_{ds}}{\varepsilon_p} > 1 \tag{13b}$$

$$\zeta = \frac{0.9}{\sqrt{1 + 600\varepsilon_r}} \tag{14a}$$

7.2. Mild steel

$$f_l = E_s \varepsilon_l \qquad \varepsilon_l < \varepsilon_{ly} \tag{15a}$$

$$f_l = f_{ly} \qquad \varepsilon_l \ge \varepsilon_{ly} \tag{15b}$$

$$f_t = E_s \varepsilon_t \qquad \varepsilon_t < \varepsilon_{ty} \tag{16a}$$

$$f_t = f_{ty} \qquad \varepsilon_t \ge \varepsilon_{ty} \tag{16b}$$

where

ε_0 strain at f_c' usually taken as 0.002
ε_p compressive peak strain in concrete struts
E_s modulus of elasticity of steel bars
ε_{ly}, ε_{ty} yield strains of longitudinal and transverse steel bars, respectively
f_{ly}, f_{ty} yield stresses of longitudinal and transverse steel bars, respectively

8. Additional equations

$$t_d=\frac{A_0}{P_0}\left[\frac{(-\varepsilon_d)(\varepsilon_r-\varepsilon_d)}{(\varepsilon_l-\varepsilon_d)(\varepsilon_t-\varepsilon_d)}\right] \tag{17}$$

$$\varepsilon_l=\varepsilon_d+\frac{A_0(-\varepsilon_d)(-\sigma_d)}{A_l f_l} \tag{18}$$

$$\varepsilon_t=\varepsilon_d+\frac{A_0 s(-\varepsilon_d)(-\sigma_d)}{P_0 A_t f_t} \tag{19}$$

$$A_0=A_c-\frac{1}{2}P_c t_d+t_d^2 \tag{20}$$

$$p_0=p_c-4t_d \tag{21}$$

where

A_c	cross-sectional area bounded by the outer perimeter of the concrete
p_c	perimeter of the outer concrete cross section

$$\varepsilon_r=\varepsilon_l+\varepsilon_t-\varepsilon_d \tag{22}$$

$$\tan^2\alpha=\frac{\varepsilon_l-\varepsilon_d}{\varepsilon_t-\varepsilon_d} \tag{23}$$

From Eqs. (9), (10) and (11), we can also get an additional equation

$$\varepsilon_d=-\frac{1}{2}t_d\theta\sin 2\alpha \tag{24}$$

9. Theoretical development for multiple box cells

In the previous section, a set of equations is given for solving cell torsion. In bridge engineering, a significant number of reinforced concrete bridges have multiple box cells, so a set of simultaneous equations is needed to analyze structural torsion for multiple box cells. Next, those equations for a single cell box in the previous section will be expanded into equations for multiple cells, and a solution method given (Yang and Dalili 1994).

10. Equations for multiple cell box

Assume a structural section has N cells. According to restraint condition $\theta=\theta_1=\theta_2=\cdots=\theta_N$, a set of simultaneous equations for cell I can be obtained from Eqs. (1) to (24) in previous section. It should be noted that for pure torsion, $\sigma_l=\sigma_t=\sigma_r=0$, and Eqs. (5), (6) and (10) are replaced by Eqs. (18), (19) and (17) respectively, so we get a set of new Eqs. (1') to (18') for for cell I as follows;

10.1. Equilibrium equations

$$\tau_{lti} = -\sigma_{di} \sin\alpha_i \cos\alpha_i \tag{1'}$$

$$T_i = \tau_{lti}(2A_{0i}t_{di}) \tag{2'}$$

10.2. Compatibility equations

$$\frac{\gamma_{1ti}}{2} = (-\varepsilon_{di} + \varepsilon_{ri}) \sin\alpha_i \cos\alpha_i \tag{3'}$$

$$\theta = \theta_i = \frac{p_{0i}}{2A_{0i}} \gamma_{1ti} \tag{4'}$$

$$\psi_i = \theta \sin 2\alpha_i \tag{5'}$$

10.3. Constitutive laws of materials

10.3.1. Concrete struts

$$\sigma_{di} = k_{1i} \zeta_i f_c' \tag{6'}$$

$$k_{1i} = \quad {}_1(\varepsilon_{dsi},\ \zeta_i) \tag{7'}$$

$$\zeta_i = \quad {}_2(\varepsilon_{di},\ \varepsilon_{ri}) \tag{8'}$$

10.3.2. Mild steel

$$f_{li} = \quad {}_3(\varepsilon_{li}) \tag{9'}$$

$$f_{ti} = \quad {}_4(\varepsilon_{li}) \tag{10'}$$

10.4. Selected constitutive equations

10.4.1. Concrete struts

$$k_{1i} = \frac{\varepsilon_{dsi}}{\zeta_i \varepsilon_0} \left(1 - \frac{1}{3} \frac{\varepsilon_{dsi}}{\zeta_i \varepsilon_0}\right) \qquad \frac{\varepsilon_{dsi}}{\varepsilon_p} \leq 1 \tag{7a'}$$

$$k_{1i} = \left[1 - \frac{\zeta_i^2}{(2-\zeta_i)^2}\right] \left(1 - \frac{1}{3} \frac{\zeta_i \varepsilon_0}{\varepsilon_{dsi}}\right) + \frac{\zeta_i^2}{(2-\zeta_i)^2} \frac{\varepsilon_{dsi}}{\zeta_i \varepsilon_0} \left(1 - \frac{1}{3} \frac{\varepsilon_{dsi}}{\zeta_i \varepsilon_0}\right) \qquad \frac{\varepsilon_{dsi}}{\varepsilon_p} > 1 \tag{7b'}$$

$$\zeta_i = \frac{0.9}{\sqrt{1 + 600\varepsilon_{ri}}} \tag{8a'}$$

10.4.2. Mild steel

$$f_{li}=E_{si}\,\varepsilon_{li} \qquad \varepsilon_{li}<\varepsilon_{ly} \tag{9a'}$$

$$f_{li}=f_{ly} \qquad \varepsilon_{li}\geq\varepsilon_{ly} \tag{9b'}$$

$$f_{ti}=E_s\,\varepsilon_{ti} \qquad \varepsilon_{ti}<\varepsilon_{ty} \tag{10a'}$$

$$f_{ti}=f_{ty} \qquad \varepsilon_{ti}\geq\varepsilon_{ty} \tag{10b'}$$

10.4.3. Derived equations

$$t_{di}=\frac{A_{0i}}{p_{0i}}\left[\frac{(-\varepsilon_{di})(\varepsilon_{ri}-\varepsilon_{di})}{(\varepsilon_{1i}-\varepsilon_{di})(\varepsilon_{ti}-\varepsilon_{di})}\right] \tag{11'}$$

$$\varepsilon_{1i}=\varepsilon_{di}+\frac{A_{0i}(-\varepsilon_{di})(-\sigma_{di})}{A_{1i}f_{1i}} \tag{12'}$$

$$\varepsilon_{ti}=\varepsilon_{di}+\frac{A_{0i}s\,(-\varepsilon_{di})(-\sigma_{di})}{p_{0i}A_{ti}f_{ti}} \tag{13'}$$

$$A_{0i}=A_{ci}-\frac{1}{2}p_{ci}t_{di}+t_{di}^2 \tag{14'}$$

$$p_{0i}=p_{ci}-4\,t_{di} \tag{15'}$$

$$\varepsilon_{ri}=\varepsilon_{1i}+\varepsilon_{ti}-\varepsilon_{di} \tag{16'}$$

$$\tan^2\alpha_i=\frac{\varepsilon_{1i}-\varepsilon_{di}}{\varepsilon_{ti}-\varepsilon_{di}} \tag{17'}$$

$$\varepsilon_{di}=-\frac{1}{2}t_{di}\,\theta\sin2\alpha_i \tag{18'}$$

11. Solution procedures

A solution procedure is proposed as shown in the flow chart in Fig. 2. The procedure is described as follows:

Step 1: Input number of cell N. Input values of longitudinal steel area A_{li}, and ransverse steel area A_{ti} in cell I ($I=1, 2, \cdots, N$)

Step 2: Assume initial values of strain in the d direction, ε_{di}; strain in the r direction, ε_{ri}; and the thickness of shear flow zone, t_{di}.

Step 3: Calculate the cross-sectional properties A_{0i} and p_{0i} by Eqs. **(14')** and **(15')**.

Step 4: Calculate the coefficients ζ_i and k_{1i}, and the average concrete stresses σ_{di} from Eqs. **(8')**, **(7')**, and **(6')**, respectively.

Step 5: Solve the strains and stresses in the longitudinal steel (ε_{li} and f_{li}) from Eqs. **(12')** and **(9')**, and those in the transverse steel (ε_{ti} and f_{ti}) from Eqs. **(13')** and **(10')**

Step 6: Calculate the strain ε_{ri} from **(16')** and t_{di} from Eq. **(11')**. If both ε_{ri} and t_{di} for cell I are the same as assumed, the values obtained for all the strains are correct. If ε_{ri}

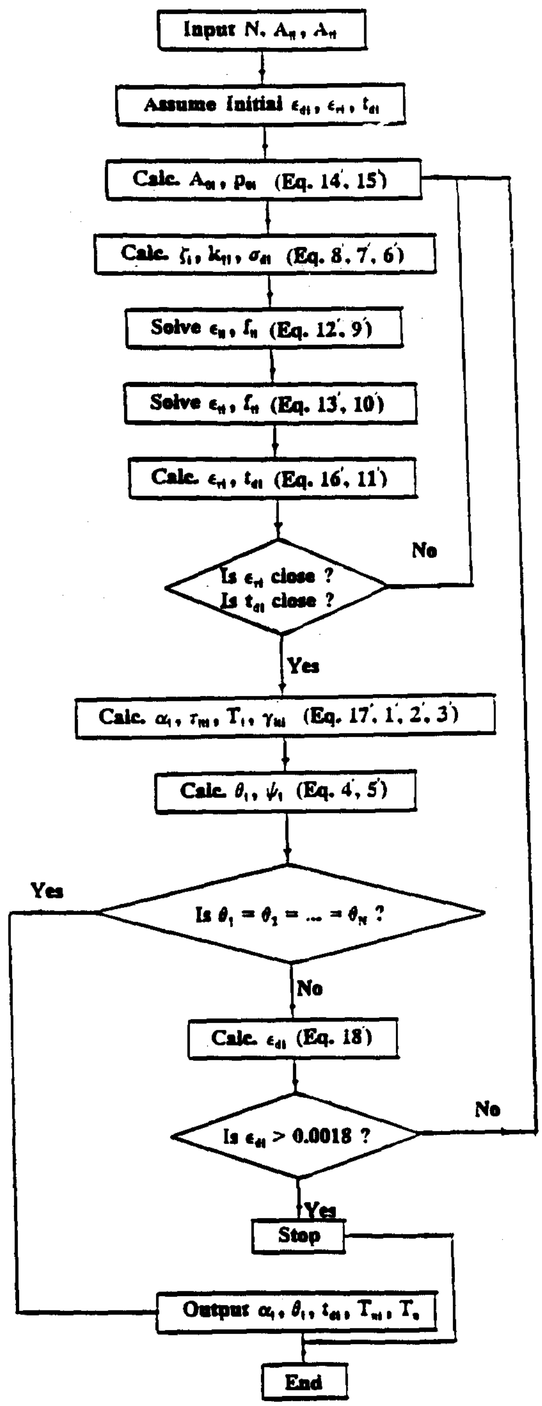


Fig. 2 Flow chart for torsion analysis.

for cell I is not the same as assumed, then the value of ε_{ri} obtained from Eq. (16') is used and Steps 3 to 6 are repeated until the desired accuracy is achieved. If t_{di}

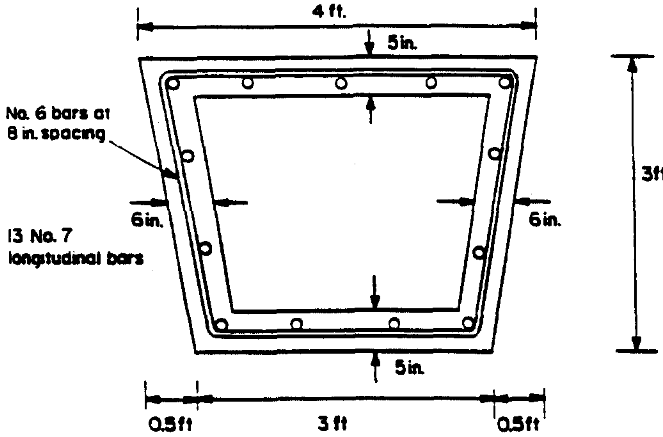


Fig. 3 Box section for example 1 (Hsu 1993).

Table 1

	T_{cr} (k-in.)	T (k-in.)	T_{max} (k-in.)	t_d (in.)	A_t (in.^{2})	A_l (in.^{2})
Proposed analysis	2391	7442	11650	4.88	0.43	7.36
Design 1 in Hsu (1993)	2391	7400	11650	4.89	0.44	7.51
Design 2	2391	7400	11650	4.89	0.43	7.35

for cell I is not the same as assumed, then the valued of t_{di} obtained from Eq. (**11'**) is used and Steps 3 to 6 are repeated until the desired accuracy is achieved.

Step 7: Repeat Step 6 from cell 1 to cell N.

Step 8: Calculate the angle α_i, the shear stress τ_{lti}, the torsional resistance T_{ni}, and the shear strain γ_{lti} from Eqs. (**17'**), (**1'**), (**2'**), and (**3'**), respectively.

Step 9: Calculate the angle of twist θ_i, and the curvature of the concrete struts ψ_i from Eqs. (**4'**) and (**5'**)

Step 10: If θ_1, θ_2, $\cdots$, θ_N are not equal, calculate ε_{di} from Eq. (**18'**). If $\varepsilon_{di}>0.0018$, stop the procedure and repeat the above solution procedure after changing some input parameters. If $\varepsilon_{di}\leq 0.0018$, Steps 4 to 11 are repeated until reaching the desired accuracy.

Step 11: Output α_i, t_{di}, and T_{ni} for each cell $I=1, 2, \cdots, N$, θ, and the total torsional resistance T_n.

A computer program was written to analyze the torsional behavior of reinforced concrete members with multiple box sections accordig to the flow chart of Fig. 2. This program can be used to analyze any type of box cells.

11.1. Example 1

The data for example 1 are adopted from Hsu (1993) for a trapezoid hollow section (see

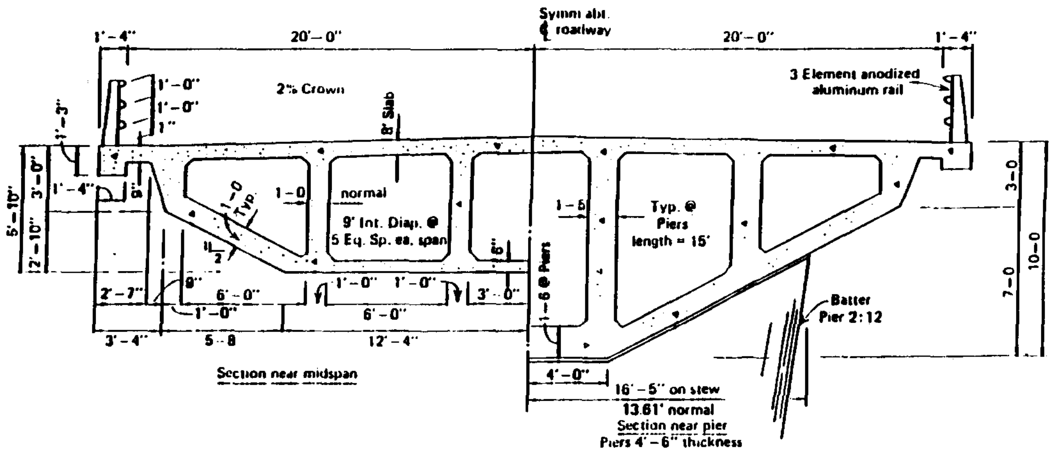


Fig. 4 Typical section for example 2 (Heins and Lawrie 1984).

Table 2.

T=55,140 (in.-kip)	Cell No.	t_{di} (in.)	q_i (p/in)	Difference for q_i	Compare
Proposed Analysis	1	1.81	262.3		
	2	3.09	733.4		
	3	3.73	1039.3		
Design in Heins and Lawrie (1984)	1	$h_{min}=8.0$	554.0	111%	Analysis with Heins (1975)
	2	$h_{min}=8.0$	852.0	16%	
	3	$h_{min}=8.0$	1179.0	13%	

Fig. 3). After using the analysis program to calculate the example, the results are compared with results from an example taken from Hsu (1993), (see Table 1).

Comparing the values from the analysis program with those from Hsu (1993), we calculate a maximum error of 2.3% which was associated with item A_t. This error of 2.3% occurred because Hsu (1993) used a simplified formula for A_0

$$A_0=A_c-\frac{1}{2}p_c t_d \tag{E-1}$$

instead of

$$A_0=A_c-\frac{1}{2}p_c t_d+t_d^2 \tag{E-2}$$

If Eq. (E-2) is used, more accurate results, which are tabulated in column "Design 2" of Table 1, can be obtained. Comparing analysis with "Design 2", it is found that a maximum error of only 0.6% occurred for T.

11.2. Example 2

The data for example 2 are adopted from example 7.1 of Heins and Lawrie (1984) for a bridge design with a section of multiple cells. The cells are numbered from the most outside to inside. The bridge cross sections near midspan and the pier to be designed are shown in Fig. 4. But, for the torsional design, only the section near the pier is used. The total applied torque T is 55,140 in.-kip. After using the analysis program to solve the example, the results are compared with results of example 7.1 from (Heins and Lawrie (1984) (see Table 2). Note that h_{min} is the minimum wall thickness used in the design. By comparing the results from analysis and Heins and Lawrie (1984), it is noted that differences for the shear flow q_i vary from cell to cell. This happens because Heins and Lawrie (1984) uses the Theory of Elasticity which assumes that a section is thin walled and thus has a uniform shear flow across its thickness. This is true when a section is made of steel. But walls made of concrete are much thicker than those made of steel and do not meet the thin wall condition. According to the Softened Truss Model Theory, the shear flow zone for a boxed section has a thickness denoted t_d. This thickness t_d is a variable determined from the equilibrium and compatibility conditions. It is not the same as the given wall thickness h. Besides, the Theory of Elasticity does not consider the softened effect due to concrete behavior. The softened Truss Model emphasizes the importance of incorporating the softened constitutive law of concrete in the analysis of reinforced Concrete Structures. So the Softened Truss Model should be used to deal with torsion of concrete structures.

12. Conclusions

(1) The Softened Truss Model Theory was first described by Hsu (1988) (1993). In this theory, torsional behavior of concrete sections was treated by combining Equilibrium, Compatibility and Constitutive Laws of Materials. However, up to now the theory has been only applied to the case of pure torsion with single cell section.

(2) Based on the Softened Truss Model Theory, a new method has been developed by the authors to model torsional behavior, especially for reinforced concrete box girder bridge superstructures with multiple cell sections.

(3) A computer analysis program has been written to solve torsional behavior. The analysis program is derived directly from the Softened Truss Model Theory.

(4) The program can be used for torsional analysis of straight and curved reinforced concrete bridges with single-cell cross sections and multiple-cell cross sections. The types of cross sections include rectangular, trapezoid and triangular sections.

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