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Vibration of electrostatically actuated microbeam by means of homotopy perturbation method

M. Bayat[*1], I. Pakar[2] and A. Emadi[3]

[1]Young Researchers and Elites Club, Science and Research Branch, Islamic Azad University, Tehran, Iran
[2]Young Researchers and Elites Club, Mashhad Branch, Islamic Azad University, Mashhad, Iran
[3]Department of Civil and Environmental Engineering, Tarbiat Modares University, Tehran, Iran

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Abstract. In this paper, it has been attempted to present a powerful analytical approach called Homotopy Perturbation Method (HPM). Free vibration of an electrostatically actuated microbeam is considered to study analytically. The effect of important parameters on the response of the system is considered. Some comparisons are presented to verify the results with other researcher's results and numerical solutions. It has been indicated that HPM could be easily extend to any nonlinear equation. We try to provide an easy method to achieve high accurate solution which valid for whole domain.

Keywords: electrostatically actuated microbeam; nonlinear vibration; homotopy perturbation method

1. Introduction

Microelectromechanical systems are widely used in many accelerometers and switches and other related systems. Many sources causes a nonlinear behavior of an electrostatically actuated microbeam. It is important to study their large deflections, electrostatic actuation and damping. The dynamic analysis and stability responses of many engineering models have become more interesting by considering the advance knowledge in micro/nanotechnology.

Generally, it is very difficult to find an exact solution for nonlinear problems. Many analytical and numerical approaches have been investigated to solve nonlinear equations such as homotopy perturbation (He 2005), energy balance method (Pakar 2013), variational iteration method (He 1999), Hamitonian approach (Bayat *et al*. 2013a), max-min approach (Zeng 2009), amplitude–frequency formulation (Ganji *et al*. 2010), and other analytical and numerical methods (He 2008, Ke *et al.* 2009, Chen *et al*. 2009, Bayat *et al*. 2013b, c, d, Sharma *et al.* 2011, Fu 2011, Ağirseven *et al.* 2010, Ganji 2006a, Vazquez-Leal 2012, Filobello-Nino *et al*. 2012, Behiry *et al*. 2007, Tsai *et al*. 2012).

Bayat *et al.* (2012) review lots of new semi-analytical approaches in their valuable review paper.

Ghasemi *et al*. (2012) provided very accurate benchmark results for further analytical analysis of engineering problems. They explicitly studied the convergence of series solutions by applying

∗Corresponding author, Researcher, E-mail: mbayat14@yahoo.com

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homotopy analysis method on forced magneto-hydrodynamic (MHD) Hiemenz flow against a flat plate with variable wall temperature in a porous medium. Their results confirmed the accuracy of the HAM method as they compared them with their numerical solutions.

Ganji *et al.* (2006b) applied homotopy perturbation to sole nonlinear problems which arises in Heat transfer nonlinear problems.

Barania *et al.* (2011) used homotopy perturbation method (HPM) and the Padé approximation a full cone subjected to wall temperature boundary conditions gives us a nonlinear ordinary differential equation (ODE). Their results also indicated that HPM-Padé can provide a convenient way to control and adjust the convergence region. They considered Nusselt number, which is an important parameter in heat transfer calculated by HPM-Padé.

In this study homotopy perturbation method is used to find analytical solutions for the large-amplitude vibration of electrostatically actuated microbeams. Some captions are presented to show the accuracy of the proposed method with exact solution and other analytical methods.

2. Nonlinear vibration of an electrostatically actuated microbeam

Fig. 1 represents a fully clamped microbeam with uniform thickness *h*, length *l*, width *b* (*b*>>5*h*), effective modulus $\bar{E} = E\big/\left(1-\nu^2\right)$, Young's modulus E , Poisson's ratio ν and density ρ . By applying the Galerkin Method and employing the classical beam theory and taking into account of the mid-plane stretching effect as well as the distributed electrostatic force, the dimensionless equation of motion for the microbeam is as follow (Fu *et al*. 2011)

$$\left(\alpha_1 q^4+\alpha_2 q^2+\alpha_3\right)\ddot{q}+\alpha_4 q+\alpha_5 q^3+\alpha_6 q^5+\alpha_7 q^7=0, \qquad q\left(0\right)=A\ , \quad \dot{q}\left(0\right)=0, \tag{1}$$

Where q is the dimensionless deflection of the microbeam, a dot denotes the derivative with respect to the dimensionless time variable $t=\tau\sqrt{\bar{E}I\big/(\rho bhl^4)}$ with *I* and *t* being the second moment of area of the beam cross-section and time, respectively.

In Eq. (1), the physical parameters α_i (i = 1−7) are given by (Fu *et al*. 2011)

$$\alpha_1=\int_0^1 \phi^6\, d\,\xi \tag{2}$$

$$\alpha_2=-2\int_0^1 \phi^4\, d\,\xi \tag{3}$$

$$\alpha_3=\int_0^1 \phi^2\, d\,\xi \tag{4}$$

$$\alpha_4=\int_0^1\left(\phi''''\phi-N\,\phi''\phi-V^{\,2}\phi\right)d\,\xi \tag{5}$$

$$\alpha_5=-\int_0^1\left(2\phi''''\phi^3-2N\,\phi''\phi^3+\lambda\phi''\phi\int_0^1\left(\phi'\right)^2 d\,\xi\right)d\,\xi \tag{6}$$

$$\alpha_6=\int_0^1\left(\phi''''\phi^5-N\,\phi''\phi^5+2\lambda\phi''\phi^3\int_0^1\left(\phi'\right)^2 d\,\xi\right)d\,\xi \tag{7}$$

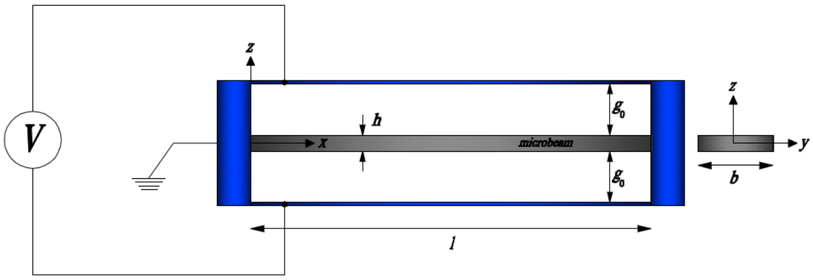


Fig. 1 Schematics of a double-sided driven clamped-clamped microbeam-based electromechanical resonator

$$\alpha_7 = -\int_0^1 \left(\lambda \phi'' \phi^5 \int_0^1 (\phi')^2 \, d\xi \right) d\xi \tag{8}$$

In which, the following nondimensional variables and parameters are introduced

$$\xi = \frac{x}{l}, \ \lambda = \frac{6 g_0^2}{h^2}, \ N = \frac{\bar{N} l^2}{\bar{E} I}, \ V^2 = \frac{24 \varepsilon_0 l^4 \bar{V}^2}{\bar{E} h^3 g_0^3} \tag{9}$$

While a prime ($'$) indicates the partial differentiation with respect to the coordinate variable ξ. The trial function is $\phi(\xi)=16\xi^2(1-\xi)^2$. The parameter $\bar{N}$ denotes the tensile or compressive axial load, g_0 is initial gap between the microbeam and the electrode, $\bar{V}$ the electrostatic load and ε_0 vacuum permittivity. The complete formulation of Eq. (1) can be referred to Fu *et al*. (2011) for details.

3. Concept of homotopy perturbation

The homotopy perturbation method is a combination of the classical perturbation technique and homotopy technique. To explain the basic idea of the homotopy perturbation method for solving nonlinear differential equations, one may consider the following nonlinear differential equation (He 2005)

$$F(q) - f(r) = 0 \quad r \in \Omega \tag{10}$$

That is subjected to the following boundary condition

$$B\left(q, \frac{\partial q}{\partial t}\right) = 0 \quad r \in \Gamma \tag{11}$$

Where F is a general differential operator, B a boundary operator, $f(r)$ is a known analytical function, Γ is the boundary of the solution domain (Ω), and $\partial q/\partial t$ denotes differentiation along the outwards normal to Γ. Generally, the operator F may be divided into two parts: a linear part L and a nonlinear part N. Therefore, Eq. (10) may be rewritten as follows

$$L(q) + N(q) - f(r) = 0 \quad r \in \Omega \tag{12}$$

Homotopy Perturbation structure is shown as follows

$$H(v,p)=(1-p)\left[L(v)-L(q_0)\right]+p\left[A(v)-f(r)\right]=0 \tag{13}$$

Where

$$v(r,p):\ \Omega\times[0,1]\rightarrow R \tag{14}$$

In Eq. (13), $p\in[0,1]$ is an embedding parameter and q_0 is the first approximation that satisfies the boundary condition. One may assume that solution of Eq. (13) may be written as a power series in p, as the following

$$v=v_0+pv_1+p^2v_2+\cdots \tag{15}$$

The homotopy parameter p is also used to expand the square of the unknown frequency of ω as follows

$$\omega_0=\omega^2-p\omega_1-p^2\omega_2-... \tag{16}$$

or

$$\omega^2=\omega_0+p\omega_1+p^2\omega_2+... \tag{17}$$

where ω_0 is the coefficient of $u(r)$ in Eq. (12) and should be substituted by the right hand side of Eq. (13). Besides, ω_i (i=1,2, …) are arbitrary parameters that have to be determined.

The best approximations for the solution and the frequency are

$$q=\lim_{p\to 1}v=v_0+v_1+v_2+\cdots \tag{18}$$

$$\omega^2=\omega_0+\omega_1+\omega_2+... \tag{19}$$

when Eq. (13) corresponds to Eq. (18) and Eq. (19) becomes the approximate solution of Eq. (10).

4. Solution using homotopy perturbation

Eq. (1) can be rewritten as the following form

$$\ddot{q}+\beta_4 q+p.\left[\beta_1 q^4\ddot{q}+\beta_2 q^2\ddot{q}+\beta_5 q^3+\beta_6 q^5+\beta_7 q^7\right]=0,\quad p\in[0,1]. \tag{20}$$

Where

$$\beta_1=\frac{\alpha_1}{\alpha_3},\ \ \beta_2=\frac{\alpha_2}{\alpha_3},\ \ \beta_4=\frac{\alpha_4}{\alpha_3},\ \ \beta_5=\frac{\alpha_5}{\alpha_3},\ \ \beta_6=\frac{\alpha_6}{\alpha_3},\ \ \beta_7=\frac{\alpha_7}{\alpha_3} \tag{21}$$

To explain the analytical solution, the unknown frequency and $q(t)$ are expanded as follows

$$q(t)=q_0(t)+p\,q_1(t)+p^2q_2(t)+\cdots \tag{22}$$

$$\beta_4=\omega^2-p\omega_1-p^2\omega_2-... \tag{23}$$

Substituting Eqs. (22) and (23) into Eq. (20) and equating the terms with identical powers of *p*, the following set of linear differential equations is obtained

$$p^0: \quad \ddot{q}_0 + \omega^2 q_0 = 0 \tag{24}$$

$$p^1: \quad \ddot{q}_1 + \omega^2 q_1 = \omega_1 q_0 - \left(\beta_1 q_0^4 \ddot{q}_0 + \beta_2 q_0^2 \ddot{q}_0 + \beta_5 q_0^3 + \beta_6 q_0^5 + \beta_7 q_0^7\right), \tag{25}$$

Solving Eq. (24) gives: $q_0(t) = A\cos(\omega t)$. Substituting $q_0(t)$ into Eq. (25), yield

$$\begin{aligned} p^1: \quad \ddot{q}_1 + \omega^2 q_1 = \omega_1 A\cos(\omega t) + \beta_1 A^5 \omega^2 \cos^5(\omega t) + \beta_2 A^3 \omega^2 \cos^3(\omega t) \\ - \beta_5 A^3 \cos^3(\omega t) - \beta_6 A^5 \cos^5(\omega t) - \beta_7 A^7 \cos^7(\omega t) \end{aligned} \tag{26}$$

For achieving the secular term, we use Fourier expansion series as follows

$$\begin{aligned} \Phi(\omega,t) &= \omega_1 A\cos(\omega t) + \beta_1 A^5 \omega^2 \cos^5(\omega t) + \beta_2 A^3 \omega^2 \cos^3(\omega t) \\ &\quad - \beta_5 A^3 \cos^3(\omega t) - \beta_6 A^5 \cos^5(\omega t) - \beta_7 A^7 \cos^7(\omega t) \\ &= \sum_{n=0}^{\infty} b_{2n+1} \cos\left[(2n+1)\omega t\right] \\ &= b_1 \cos(\omega t) + b_3 \cos(3\omega t) + \ldots \\ &\approx \left(\int_0^{\pi/2} \Phi(\omega,t)\, d(\omega t)\right)\cos(\omega t) \\ &= \left[\omega_1 A + \frac{5}{8}\beta_1 A^5 \omega^2 + \frac{3}{4}\beta_2 A^3 \omega^2 - \frac{3}{4}\beta_5 A^3 - \frac{5}{8}\beta_6 A^5 - \frac{35}{64}\beta_7 A^7\right]\cos(\omega t) \end{aligned} \tag{27}$$

Substituting Eq. (27) into Eq. (26) yields

$$\begin{aligned} p^1: \ddot{q}_1 + \omega^2 q_1 = \left[\omega_1 A + \frac{5}{8}\beta_1 A^5 \omega^2 + \frac{3}{4}\beta_2 A^3 \omega^2 - \frac{3}{4}\beta_5 A^3 - \frac{5}{8}\beta_6 A^5 - \frac{35}{64}\beta_7 A^7\right]\cos(\omega t) \\ + \sum_{n=0}^{\infty} b_{2n+1} \cos\left[(2n+1)\omega t\right] \end{aligned} \tag{28}$$

Avoiding secular term, gives

$$\omega_1 = -\frac{5}{8}\omega^2 A^4 \beta_1 - \frac{3}{4}\omega^2 A^2 \beta_2 + \frac{3}{4}A^2 \beta_5 + \frac{5}{8}A^4 \beta_6 + \frac{35}{64}A^6 \beta_7 \tag{29}$$

From Eq. (23) and setting $p = 1$, we have

$$\beta_4 = \omega^2 - \omega_1 \tag{30}$$

Substituting Eqs. (29) in (30), we can obtain

$$\omega^2 = -\frac{5}{8}\omega^2 A^4 \beta_1 - \frac{3}{4}\omega^2 A^2 \beta_2 + \frac{3}{4}A^2 \beta_5 + \frac{5}{8}A^4 \beta_6 + \frac{35}{64}A^6 \beta_7 + \beta_2 \tag{31}$$

Solving Eq. (31), and Substituting Eq. (21) in it, gives

$$\omega_{HPM} = \frac{\sqrt{2}}{4} \frac{\sqrt{64\alpha_4 + 48A^2\alpha_5 + 40A^4\alpha_6 + 35A^6\alpha_7}}{\sqrt{5A^4\alpha_1 + 6A^2\alpha_2 + 8\alpha_3}} \tag{32}$$

Solving (28) without secular term and Substituting Eq. (21) in it, we obtain

$$\begin{aligned} q_1(t) = -\frac{1}{3072}\frac{A^3}{\omega^2\alpha_3}\Big(&-128\alpha_1\omega^2A^2\cos(\omega t) - 96\alpha_2\omega^2\cos(\omega t) + 96\alpha_5\cos(\omega t) + 128\alpha_6A^2\cos(\omega t) \\ &+141\alpha_7A^4\cos(\omega t) + 120\alpha_1\omega^2A^2\cos(3\omega t) + 96\alpha_2\omega^2\cos(3\omega t) - 96\alpha_5\cos(3\omega t) \\ &-126\alpha_7A^4\cos(3\omega t) - 120\alpha_6A^2\cos(3\omega t) + 8\alpha_1\omega^2A^2\cos(5\omega t) - 8\alpha_6A^2\cos(5\omega t) \\ &-14\alpha_7A^4\cos(5\omega t) - \beta_7A^4\cos(7\omega t)\Big) \end{aligned} \tag{33}$$

Hence, we can obtain the following approximate solution

$$\begin{aligned} q(t) = A\cos(\omega t) - \frac{1}{3072}\frac{A^3}{\omega^2\alpha_3}\Big(&-128\alpha_1\omega^2A^2\cos(\omega t) - 96\alpha_2\omega^2\cos(\omega t) + 96\alpha_5\cos(\omega t) + 128\alpha_6A^2\cos(\omega t) \\ &+141\alpha_7A^4\cos(\omega t) + 120\alpha_1\omega^2A^2\cos(3\omega t) + 96\alpha_2\omega^2\cos(3\omega t) - 96\alpha_5\cos(3\omega t) \\ &-126\alpha_7A^4\cos(3\omega t) - 120\alpha_6A^2\cos(3\omega t) + 8\alpha_1\omega^2A^2\cos(5\omega t) - 8\alpha_6A^2\cos(5\omega t) \\ &-14\alpha_7A^4\cos(5\omega t) - \beta_7A^4\cos(7\omega t)\Big) \end{aligned} \tag{34}$$

5. Results and discussions

In this part, we do some comparisons between the new presented method and other analytical and exact solutions. Table 1 show the accuracy of the method for different values of constant parameters. We compare it with the Homotopy Analysis Method (HAM), Energy Balance Method (EBM) and exact solution.

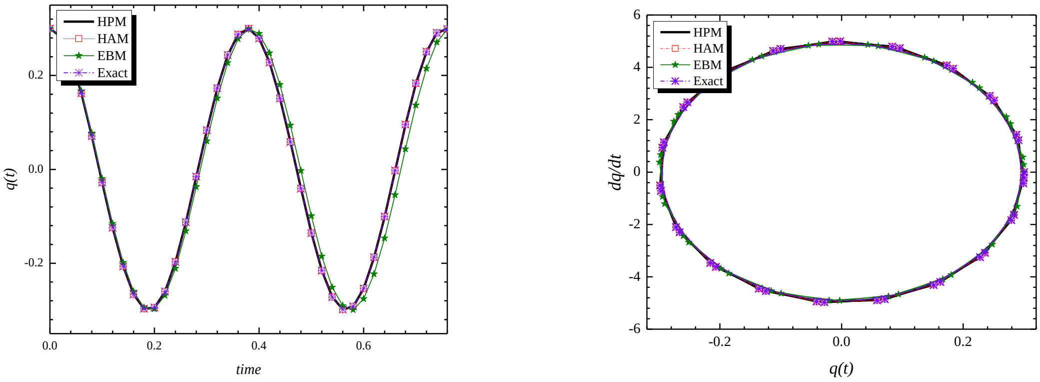
Figs. 2 and 3 show the convergence of the method comparing with HAM, EBM and Exact solution for two cases. Case 1 is time history displacement comparison and the Case 2 is phase plan. Fig. 2 is for N=10, λ=24, V=20, A=0.3 and Fig. 3 is for N=10, λ=24, V=10, A=0.6.

To have a better understand from the behavior of the system, we consider the effect of V parameter corresponded to electrostatic voltage on nonlinear frequency of electrostatically microbeam base on amplitude.

The Fig. 4(a) is shown this influence. The value of V parameters increases the nonlinear frequency by its increase. A peak point is seen in the figure in large amplitudes. Fig. 4(b) shows the effects of N parameter corresponded to axial force on nonlinear frequency of electrostatically microbeam base on amplitude. It is a similar behavior like V parameter is seen with different values of N on the nonlinear frequency.

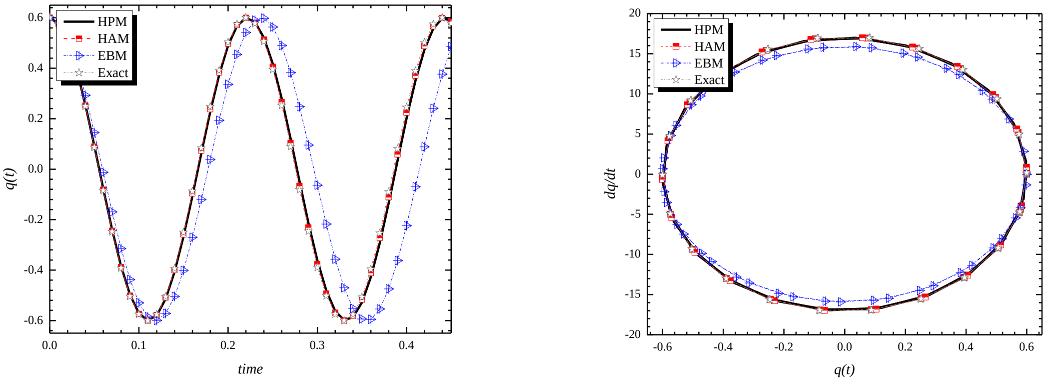
Table 1 Comparison of frequency corresponding to various parameters of system

Constant parameters				HPM	HAM	EBM	Exact solution
A	N	λ	V	ω_{HPM}	ω_{HAM} (Qian *et al*. 2012)	ω_{EBM} (Fu *et al*. 2011)	ω_{Exact} (Qian *et al*. 2012)
0.3	10	24	0	26.8262	26.8329	26.3867	26.8372
0.3	10	24	20	16.6422	16.6460	16.3829	16.6486
0.6	10	24	10	28.3441	28.4440	26.5324	28.5382
0.6	10	24	20	18.5162	18.5574	17.5017	18.5902



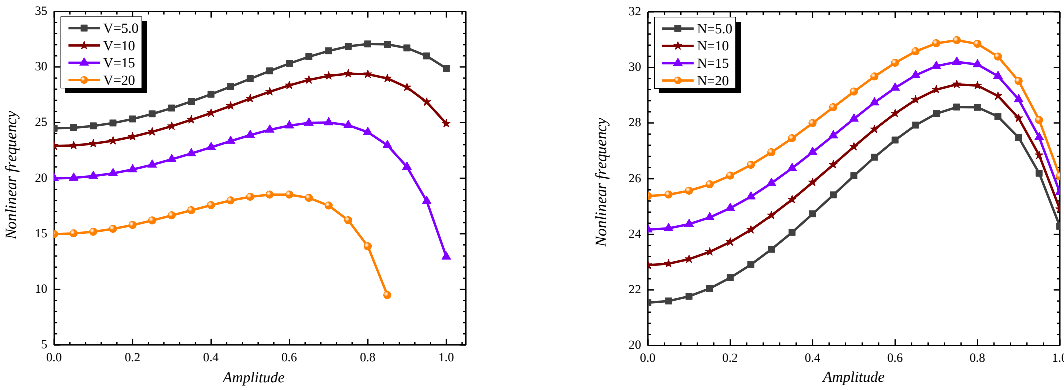
(a) (b)

Fig. 2 Comparison of homotopy perturbation method (HPM), homotopy analysis method (HAM), energy balance method (EBM) and exact solution. (a) time history response, (b) phase plane for N=10, λ=24, V=20, A=0.3



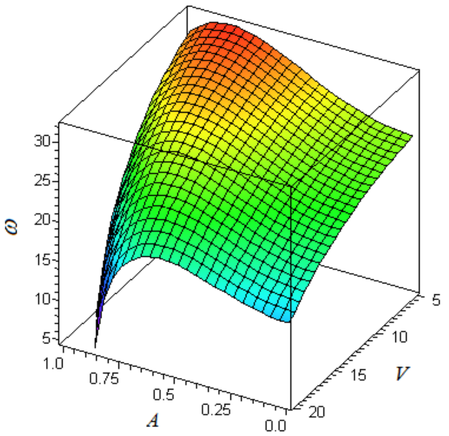
(a) (b)

Fig. 3 Comparison of homotopy perturbation method (HPM), homotopy analysis method (HAM), energy balance method (EBM) and exact solution. (a) time history response, (b) phase plane for N=10, λ=24, V=10, A=0.6



(a) (b)

Fig. 4 Effect of (a) V parameter and (b) N parameter on nonlinear frequency of electrostatically microbeam base on amplitude



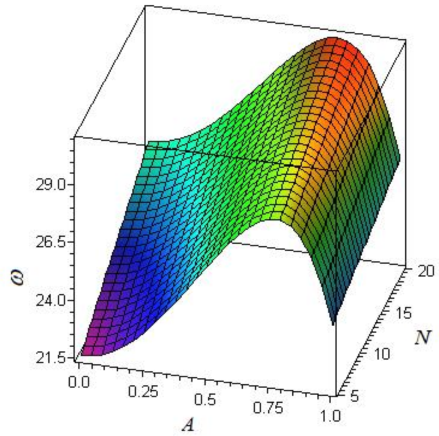


Fig. 5 Sensitivity analysis of nonlinear frequency for (a) N=10, λ=24, 5<V<20, 0<A<1 (b) 5<N<20, λ=24, V=10, 0<A<1

Fig. 5 is a Sensitivity analysis of nonlinear frequency by considering the amplitude for two important parameters of N and V. The accuracy of the HPM has been demonstrated in comparison of EBM, HAM and exact solution for different values of forces and voltages acting on the microbeam. The HPM can easily extend to any nonlinear problem with high nonlinear term. HPM is valid for a wide range of amplitude.

6. Conclusions

In this study, homotopy perturbation method was applied to nonlinear governing equation of an electrostatically actuated micro beam. The effect of different parameters on the nonlinear frequency of the systems was considered. The results of HPM compared with the HAM, EBM and exact solutions. The accuracy of the HPM shows that it could be applied to nonlinear conservative problems easily. The HPM results converge to the exact solution by only its first iteration. Therefore we can suggest the HPM as a novel method to achieve accurate results in nonlinear vibration equations.

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