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New implicit higher order time integration for dynamic analysis

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Abstract. In this paper new implicit time integration called N-IHOA is presented for dynamic analysis of high damping systems. Here, current displacement and velocity are assumed to be functions of the velocities and accelerations of several previous time steps, respectively. This definition causes that only one set of weighted factors is calculated from the Taylor series expansion which leads to a simple approach and reduce the computational efforts. Moreover a comprehensive study on stability of the proposed method i.e., N-IHOA compared with IHOA integration which is performed based on amplification matrices proves the ability of the N-IHOA in high damping vibrations such as control systems. Also, wide range of numerical examples which contains single/multi degrees of freedom, damped/un-damped, free/forced vibrations from finite element/finite difference demonstrate that the accuracy and efficiency of the proposed time integration is more than the common approaches such as the IHOA, the Wilson-*θ* and the Newmark-*β*.

Keywords: numerical dynamic analysis; higher order time integration

1. Introduction

The fundamental relationship of structural dynamic equilibrium is stated by a second order time differential equation with some initial conditions

$$[\mathrm{M}]^{n+1}\{\ddot{\mathrm{D}}\}^{n+1}+[\mathrm{C}]^{n+1}\{\dot{\mathrm{D}}\}^{n+1}+\{\mathrm{f}(\mathrm{D}^{n+1})\}=\{\mathrm{P}(\mathrm{t}^{n+1})\} \tag{1}$$

$$\{\mathrm{D}(0)\}=\{\mathrm{D}_0\} \quad , \quad \{\dot{\mathrm{D}}(0)\}=\{\dot{\mathrm{D}}_0\} \tag{2}$$

Here, $[M]^{n+1}$, $[C]^{n+1}$, $\{f(D^{n+1})\}$ and $\{P(t^{n+1})\}$ are mass matrix, damping matrix, internal and external forces vectors, respectively. Also, $\{D\}^{n+1}$is the nodal displacement vector and super dots denote differential with respect to time. All of these quantities are calculated at time t^{n+1}. Moreover, $\{D_0\}$ and $\{\dot{D}_0\}$ are displacement and velocity vectors at *t*=0, respectively. The solution of Eq. (1) can be obtained by analytical or numerical techniques. Since the applications of analytical methods such as modal analysis are restricted to linear systems, numerical schemes known as step by step time integrations are widely used in dynamic analysis. These methods are classified in three general groups: *Implicit*, *Explicit* and *Predictor-Corrector*.

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In each step of *implicit* methods, the dynamic equation is transformed to the equivalent static system such as performed in the well known Newmark-β and Wilson-θ methods. High accuracy and more stable analysis are the main advantages of the implicit integrations especially in nonlinear dynamic. However the cost and computational time may dramatically increase for large scale structures. It should be noted that there are many implicit integrations such as HHT-α (Hibler and Hughes 1977), WBZ-α (Wood 1981), generalized-α method (Chung and Hulbert 1993), Newmark multi time step approach (Kim *et al*. 1997), third order time step integration (Fung 1997), Newmark complex time step (Fung 1998), time weighted function procedure (Tamma *et al*. 2001), generalized single step integration (Modak and Sotelino 2002), Nørsett time integration (Mancuso and Ubertini 2002), composite time integration (Bathe and Baig 2005), higher order acceleration function (Keierleber and Rosson 2005), implicit integration based on conserving energy and momentum (Bathe 2007), Green function approach (Soares and Mansur 2005, Loureiro and Mansur 2010), precise integration methods (Wang and Au 2009) and higher order Newton backward extrapolation functions (Rezaiee and Sarafrazi 2010). In the recent studies, the implicit higher order time integration was presented by utilizing accelerations of several previous time steps (Rezaiee and Alamatian 2008a, Rezaiee *et al*. 2011).

In the *explicit* methods, displacement and velocity of the current step are calculated from the previous time step data just by vector operations. Substituting these quantities into the dynamics equation, and solving system of simultaneous linear equations, accelerations will be calculated (Penry and Wood 1985, Hulbert 1994, Hulber and Chung 1996). Regardless the simplicity and lower computational effort, numerical instability is the main defect of the explicit integrations. There are different explicit methods available such as generalized weighted residual approach (Zienkiewicz *et al*. 1984), the SSpj method (Wood 1984), the β_m algorithm (Katona and Zienkiewicz 1985) and the Hoff-Taylor approach (Hoff and Taylor 1990).

On the other hand, combining implicit and explicit techniques leads to an interesting approach called *predictor-corrector* algorithms. In such methods, the explicit procedures are used to obtain an estimation of answer. This estimation is corrected by the implicit relationships. As a result, calculations are in vector form and because of using the implicit correctors; the numerical instability will be limited (Zhai 1996, Zhang *et al*. 2005). For example, Zhai suggested a predictor-corrector algorithm that uses accelerations of two previous time steps (Zhai 1996). Another interesting approach is a higher order predictor-corrector integration which uses accelerations of several previous steps (Rezaiee and Alamatian 2008-b).

It is clear that many of these methods have an ability to be used as an implicit, explicit or predictor-corrector algorithm by changing their integration's parameters (Rezaiee and Alamatian 2008a, Rezaiee and Alamatian 2008b, Zuijlen and Bijl 2005, Rama Mohan Rao 2002). Also time integrations can be combined with finite element formulation which leads to a time-space model (Regueiro and Ebrahimi 2010).

In this paper, new multi time integration named N-IHOA is presented for dynamic analysis. Here the current displacement and velocity vectors are proposed to be functions of velocities and accelerations of several previous time steps, respectively. The unknown parameters are calculated in a manner that the suggested time integration has maximum accuracy. Moreover, a comprehensive study on the stability of proposed integration and its previous version i.e. IHOA is performed based on amplification matrices. Finally, some linear and nonlinear dynamic systems are analyzed to verify the numerical ability of the proposed method.

2. New Higher Order Time Integration (N-IHOA)

In many of the numerical dynamic analysis, single last time step results are used to integrate the equations of motion; i.e., displacement, velocity and acceleration of single previous step are utilized to obtain displacement, velocity and acceleration in the current time step (Zhou and Tamma 2004). Despite of simplicity, this procedure may reduce the efficiency of the numerical method, because the integration is based on a single previous time step data and the continuity of the higher order derivatives may not be satisfied. Some methods such as the β_m scheme (Katona and Zienkiewicz 1985) and the Hoff-Taylor formulation (Hoff and Taylor 1990) use higher order displacement's derivatives i.e. third order, fourth order and etc. of the single previous increment to create continuity conditions. Here, a time-polynomial function is assumed for displacement and the function's order is equal to the highest order of displacement's derivative. Also, Taylor series is applied to estimate unknown higher order derivatives. Real estimate of these derivatives increases the efficiency of such method. Therefore, personal judgments and problem specifications will affect the accuracy so that the analysis will not be automatic.

The multi-time step integrations which use the results obtained in several previous time steps is another way to satisfy the continuity of the higher order derivatives (Smolinski *et al*. 1988). In one of the recent study, the accelerations of several previous steps were utilized to integrate dynamic equilibrium equation (Rezaiee and Alamatian 2008a). This method has been called IHOA and its fundamental equations are considered as follows

$$\{D\}^{n+1} = \{D\}^{n} + \Delta t\{\dot{D}\}^{n} + \left(\frac{1}{2} - \sum_{i=0}^{m-1}\xi_i\right)\Delta t^2\{\ddot{D}\}^{n} + \xi_0\Delta t^2\{\ddot{D}\}^{n+1} + \Delta t^2\sum_{i=1}^{m-1}\xi_i\{\ddot{D}\}^{n-i} \tag{3}$$

$$\{\dot{D}\}^{n+1} = \{\dot{D}\}^{n} + \left(1 - \sum_{i=0}^{m-1}\eta_i\right)\Delta t\ \{\ddot{D}\}^{n} + \eta_0\Delta t\{\ddot{D}\}^{n+1} + \Delta t\sum_{i=1}^{m-1}\eta_i\{\ddot{D}\}^{n-i} \tag{4}$$

Where, $\{\ddot{D}\}^{n-i}$ is the acceleration vector at *n-i*th time station and can be stored in analysis memory. Also, ξ_i and η_i are weighted factors which verify stability and accuracy of time integration. These factors are calculated for maximum accuracy. The time step i.e., Δt, controls the stability of the numerical integration. At time t^{n+1}, there is only n calculated acceleration. Therefore, maximum accuracy order or integration order i.e., m, will be less than or equal to n ($m \le n$). The details of this method can be found in the references paper (Rezaiee and Alamatian 2008a).

Here, a new version of the IHOA method is presented. For this purpose, displacement and velocity of the current time step are assumed to be functions of velocities and accelerations of several previous time steps, respectively, as follows

$$\{D\}^{n+1} = \{D\}^{n} + \Delta t\left(1 - \alpha' - \sum_{i=1}^{m-1}\alpha_i\right)\{\dot{D}\}^{n} + \Delta t\alpha'\{\dot{D}\}^{n+1} + \Delta t\sum_{i=1}^{m-1}\alpha_i\{\dot{D}\}^{n-i} \tag{5}$$

$$\{\dot{D}\}^{n+1} = \{\dot{D}\}^{n} + \Delta t\left(1 - \gamma' - \sum_{i=1}^{m-1}\gamma_i\right)\{\ddot{D}\}^{n} + \Delta t\gamma'\{\ddot{D}\}^{n+1} + \Delta t\sum_{i=1}^{m-1}\gamma_i\{\ddot{D}\}^{n-i} \tag{6}$$

Here α', γ' and α_i,γ_i (i=1,2, …m−1) are weighted factors which control the accuracy and stability of integration. Eqs. (5) and (6) present the fundamental equations of the new implicit time integration, in which displacement is a function of several previous velocities and velocity is a function of several previous accelerations. To formulate this new algorithm, velocity at time t^{n+1} is obtained from Eq. (5)

$$\{\dot{D}\}^{n+1} = \frac{\{D\}^{n+1} - \{D\}^{n} - \Delta t(1-\alpha' - \sum_{i=1}^{m-1}\alpha_i)\{\dot{D}\}^{n} - \Delta t\sum_{i=1}^{m-1}\alpha_i\{\dot{D}\}^{n-i}}{\Delta t\alpha'} \tag{7}$$

If Eq. (7) is substituted in Eq. (6), acceleration at time t^{n+1} is obtained

$$\{\ddot{D}\}^{n+1} = \frac{\{D\}^{n+1} - \{D\}^{n} - \Delta t(1- \sum_{i=1}^{m-1}\alpha_i)\{\dot{D}\}^{n} - \Delta t\sum_{i=1}^{m-1}\alpha_i\{\dot{D}\}^{n-i}}{\Delta t^2\alpha'\gamma'} - \frac{(1-\gamma' - \sum_{i=1}^{m-1}\gamma_i)\{\ddot{D}\}^{n} + \sum_{i=1}^{m-1}\gamma_i\{\ddot{D}\}^{n-i}}{\gamma'} \tag{8}$$

Substituting Eqs. (7) and (8) into Eq. (1), leads to the below equivalent static system

$$[S]_{EQ}^{n+1}\{D\}^{n+1} = \{P\}_{EQ}^{n+1} \tag{9}$$

Where, $[S]_{EQ}^{n+1}$ and $\{P\}_{EQ}^{n+1}$ are equivalent secant stiffness matrix and equivalent load vector, respectively. These quantities are formulated as follows

$$[S]_{EQ}^{n+1} = \frac{1}{\alpha'\gamma'\Delta t^2}[M]^{n+1} + \frac{1}{\alpha'\Delta t}[C]^{n+1} + [S]^{n+1} \tag{10}$$

$$\begin{aligned}\{P\}_{EQ}^{n+1} = &\{P(t^{n+1})\} + \frac{1}{\Delta t^2\alpha'\gamma'}[M]^{n+1}\left(\{D\}^{n} + \Delta t(1- \sum_{i=1}^{m-1}\alpha_i)\{\dot{D}\}^{n} + \Delta t\sum_{i=1}^{m-1}\alpha_i\{\dot{D}\}^{n-i}\right)\\ &+ \frac{1}{\gamma'}[M]^{n+1}\left((1-\gamma' - \sum_{i=1}^{m-1}\gamma_i)\{\ddot{D}\}^{n} + \sum_{i=1}^{m-1}\gamma_i\{\ddot{D}\}^{n-i}\right) +\\ &\frac{1}{\Delta t\alpha'}[C]^{n+1}\left(\{D\}^{n} + \Delta t(1-\alpha' - \sum_{i=1}^{m-1}\alpha_i)\{\dot{D}\}^{n} + \Delta t\sum_{i=1}^{m-1}\alpha_i\{\dot{D}\}^{n-i}\right)\end{aligned} \tag{11}$$

The new formulation is called the N-IHOA i.e., New Implicit Higher Order Accuracy integration. In this method, the equivalent static stiffness matrix is only based on α' and γ' (weighted factors of velocity and acceleration of previous time step, respectively) and other weighted factors do not produce any effect on this matrix. First, the displacement vector is obtained by solving equivalent static system at time t^{n+1} (Eq. (9)). Then, the current velocity and acceleration vectors are calculated from Eqs. (7) and (8), respectively. This procedure is performed for all time steps. The proposed integration presents important existing methods as its special cases. When α' and γ' are null, explicit schemes are obtained in which, displacement and velocity of current time step are calculated from Eqs. (5) and (6), respectively. Then, Eq. (1) is utilized to calculate acceleration of this time step.

When m>1, there are two approaches to start the N-IHOA integration. The most elementary method is that analyst estimates initial conditions (velocities and accelerations of several artificial steps). It is clear that these assumptions do not have good accuracy and personal judgment plays an important role in the integration efficiency. Another approach to overcome this difficulty is more interesting. For this purpose, the first increment is performed by m=1. At the end of this step, two dynamic equilibrium points (n and n-1) will be available. Therefore, the second increment can be performed by m=2. At this time, three dynamic equilibrium points of previous steps (n, n-1 and n-2) exist and the next increment can be started by m=3. As a result, in each step of starting increments, integration accuracy increases one order. This procedure is repeated until accuracy order reaches to its selected value. In this method, the personal judgment does not have any effect on the integration and procedure can be performed automatically.

It is clear that the N-IHOA and IHOA algorithms have some similarities. Both of these methods use several previous time step information. Therefore, they do not depend on local specifications. Mathematically, residual errors are maximum in the last time step because of continuously increasing errors in successive increments and large number of time steps. Also, it is probable that errors grow up in one step accidentally. In the common time integrations, which use information of single previous step, the direct effect of these errors will be revealed in the next increment. As a result, the efficiency of the numerical integration is reduced because the input data has considerable error. The N-IHOA and IHOA methods insert global characteristics into the formulation by use of several previous accelerations and velocities. Thus, the effects of local and residual errors are reduced.

3. Accuracy analysis

Stability and accuracy are two main concerns about each numerical dynamic analysis. For this purpose, the integration's parameters and time step should be selected so that numerical stability is guaranteed and accuracy is maximized. Usually, time step size controls the stability of numerical integration and other weighted factors are formulated to achieve high accuracy method. On this base, the optimum weighted factors of the N-IHOA which leads to the maximum accuracy are determined so that numerical errors are minimized. Let the accuracy order be m. Therefore, there are 2m+2 free parameters in displacement and velocity relationships (Eq. (5) and (6)) which control the integration's error. The error function of displacement and velocity can be defined as follows

$$\{\mathrm{R}\}_{\mathrm{D}}^{n+1} = \{\mathrm{D}\}^{n+1} - \{\mathrm{D}\}_{\mathrm{Exact}}^{n+1} \tag{12}$$

$$\{\mathrm{R}\}_{\mathrm{V}}^{n+1} = \{\dot{\mathrm{D}}\}^{n+1} - \{\dot{\mathrm{D}}\}_{\mathrm{Exact}}^{n+1} \tag{13}$$

Where, $\{\mathrm{R}\}_{\mathrm{D}}^{n+1}$ and $\{\mathrm{R}\}_{\mathrm{V}}^{n+1}$ are displacement and velocity residuals (errors) at time t^{n+1}, respectively. The exact solution of displacement ($\{D\}^{n+1}_{Exact}$) and velocity ($\{\dot{D}\}^{n+1}_{Exact}$) are formulated based on Taylor series expansion

$$\{D\}^{n+1}_{Exact} = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} \left\{D^k\right\}^n = \{D\}^n + \Delta t\{\dot{D}\}^n + 0.5\Delta t^2\{\ddot{D}\}^n + 0.1667\Delta t^3\{D^3\}^n + 0.0417\Delta t^4\{D^4\}^n + \ldots \tag{14}$$

$$\{\dot{D}\}^{n+1}_{Exact} = \sum_{k=0}^{\infty} \frac{\Delta t^k}{k!} \left\{D^{k+1}\right\} = \{\dot{D}\}^n + \Delta t\{\ddot{D}\}^n + 0.5\Delta t^2\left\{D^3\right\}^n + 0.1667\,\Delta t^3\left\{D^4\right\}^n + 0.0417\,\Delta t^4\left\{D^5\right\}^n + \ldots \tag{15}$$

Here $\{D^{k+i}\}^n$ shows the $k+i^{th}$ order derivative of displacement at time increment n. It is clear that the exact solutions of the $n+1^{th}$ step were formulated based on displacement's derivatives of the n^{th} increment. As a result, the displacement and velocity should also be formulated as functions of the n^{th} step displacement's derivatives. For this purpose, the inverse expansions of the velocity and acceleration of several previous time steps (n-1^{th}, n-2^{th} and etc) are utilized (Rezaiee and Alamatian 2008a)

$$\{\dot{D}\}^{n-i} = \sum_{k=0}^{\infty} \frac{(-1)^k\ \Delta t^k}{k!} \left\{D^{k+1}\right\}^{n-i+1} \qquad i = 1,2,\ldots,m \tag{16}$$

$$\{\ddot{D}\}^{n-i} = \sum_{k=0}^{\infty} \frac{(-1)^k\ \Delta t^k}{k!} \left\{D^{k+2}\right\}^{n-i+1} \qquad i = 1,2,\ldots,m \tag{17}$$

Similar relationship can be written for j^{th} order of displacement's derivative of some previous time steps (n-i^{th})

$$\{D^j\}^{n-i} = \sum_{k=0}^{\infty} \frac{(-1)^k\ \Delta t^k}{k!} \left\{D^{j+k}\right\}^{n-i+1} \qquad \begin{matrix} i = 1,2,\ldots,m \\ j = 1,2,\ldots,\infty \end{matrix} \tag{18}$$

If Eqs. (16), (17) and (18) are iterated successively, previous time steps velocities and accelerations are formulated in terms of displacement's derivatives at time t^n. For example, the displacement and velocity functions for the first and second accuracy order (m=1 and m=2) are as follows

$$\{D\}^{n+1} = \{D\}^n + \Delta t\{\dot{D}\}^n + (\alpha' - \alpha_1)\Delta t^2\{\ddot{D}\}^n + (0.5\alpha' + 0.5\alpha_1)\Delta t^3\{D^3\}^n + \ldots \qquad m=1 \tag{19}$$

$$\{\dot{D}\}^{n+1} = \{\dot{D}\}^n + \Delta t\{\ddot{D}\}^n + (\gamma' - \gamma_1)\Delta t^2\{D^3\} + (0.5\gamma' + 0.5\gamma_1)\Delta t^3\{D^4\}^n + \ldots \qquad m=1 \tag{20}$$

$$\{D\}^{n+1} = \{D\}^{n} + \Delta t\{\dot{D}\}^{n} + (\alpha' - \alpha_1 - 2\alpha_2)\Delta t^2\{\ddot{D}\}^{n} + (0.5\alpha' + 0.5\alpha_1 + 2\alpha_2)\Delta t^3\{D^3\}^{n} + (0.1667\alpha' - 0.1667\alpha_1 - 1.3333\alpha_2)\Delta t^4\{D^4\}^{n} + \ldots \quad m=2 \tag{21}$$

$$\{\dot{D}\}^{n+1} = \{\dot{D}\}^{n} + \Delta t\{\ddot{D}\}^{n} + (\gamma' - \gamma_1 - 2\gamma_2)\Delta t^2\{D^3\}^{n} + (0.5\gamma' + 0.5\gamma_1 + 2\gamma_2)\Delta t^3\{D^4\}^{n} + (0.1667\gamma' - 0.1667\gamma_1 - 1.3333\gamma_2)\Delta t^4\{D^5\}^{n} + \ldots \quad m=2 \tag{22}$$

From the above equations, one can see that for a selective integration order i.e., *m,* the coefficients of weighted factors in displacement and velocity vectors are the same. In the other words, there is only one set of weighted factors (m+1 free parametres) in the proposed time integration. As a result, the weighted factors of displacement and velocity are the same, i.e., $\alpha'=\gamma'$ and $\alpha_i=\gamma_i$ i=1,2, …m. This subject causes that the N-IHOA will be simpler than the IHOA technique; because in the IHOA there is two different groups of weighted factors (ξ and η) which should be calculated and saved. Since, the similar weighted factors of displacement and velocity in the N-IHOA algorithm causes that the required memory for dynamic analysis reduces and its programming is simpler than the IHOA. Also, it is expected that the analysis time of the N-IHOA may be less than the IHOA. This subject will be verified numerically.

On the other hand, the maximum accuracy will occur if maximum possible number of elements of displacement and velocity's residuals ($\{R\}_D^{n+1}$ and $\{R\}_V^{n+1}$) are zero. This procedure leads to a linear system of equations, in which weighted factors are unknown

$$[Z]_{m\times m}\begin{Bmatrix}\alpha' \\ \alpha_1 \\ \cdot \\ \cdot \\ \cdot \\ \alpha_{m-1}\end{Bmatrix} = \begin{Bmatrix}\frac{1}{2!} \\ \frac{1}{3!} \\ \cdot \\ \cdot \\ \cdot \\ \frac{1}{(m+1)!}\end{Bmatrix} \tag{23}$$

In these equations, [Z]m×m is a constant matrix, calculated from Eqs. (16), (17) and (18). This matrix can be found for integration's order 1, 2, 3, 4and 5 in Appendix. By solving these linear systems, optimum weighted factors which have the same values for both displacement and velocity are obtained. If the first non-zero derivative's order is defined as the mathematical accuracy order, the calculated optimum weighted factors present both displacement and velocity by an accuracy order as Δt^{m+2}, respectively. In the linear Newmark-β, central finite difference and Zhai method, accuracy order related to displacement and velocity are Δt^3 and Δt^2, respectively. The least integration's order of the N-IHOA technique is 1. If integration's order is at least (m=1), the accuracy order of both displacement and velocity will be Δt^3. Here, the accuracy order of velocity

Table 1 Optimum values weighted factors of the N-IHOA

m	α’	α_1	α_2	α_3	α_4	α_5
1	0.5					
2	0.416666667	-0.083333333				
3	0.375	-0.208333333	0.041666667			
4	0.348611111	-0.366666667	0.147222222	-0.026388889		
5	0.329861111	-0.554166667	0.334722222	-0.120138889	0.01875	
6	0.315591931	-0.768204365	0.62010582	-0.334176587	0.104365079	-0.01426918

is higher than the well-known methods such as Newmark scheme. Therefore, higher accuracy of the proposed method is proved mathematically. Moreover, comparing the IHOA and N-IHOA show that for integration order as m (m previous time steps are used), the accuracy of displacement is Δt^{m+3} and Δt^{m+2} in the IHOA and N-IHOA algorithms, respectively. From mathematical point of view, the displacement’s accuracy of the IHOA is one order higher than the N-IHOA. However, both methods present the velocity by accuracy order as Δt^{m+2}.

On the other hand, the optimum weighted factors, presented here, are unique and they are not dependent on the problem specification. These linear systems have been solved for accuracy order between 0 and 6, and the optimum values of weighted factors have been found and inserted in Table 1. Therefore, the proposed formulation is self-beginning and does not need personal judgments, which reduce the technique ability.

4. Stability analysis

The stability of IHOA integration has been previously studied by employing the Routh-Hurwitz criterion (Rezaiee and Alamatian 2008a). This method has a significant limitation so that it does not able to study the effect of some important specifications such as damping. In the other words, the effect of damping has not been considered on the stability of the IHOA. To overcome this difficulty and provide a better comparison, the stability of both N-IHOA (the proposed method) and IHOA methods are studied based on their amplification matrices. These studies help to make a comprehensive comparison between IHOA and N-IHOA. It should be noted that the most common approach to investigate numerical stability of step by step time integrations is performed by using the amplification matrix (Wieberg and Li 1993, Gobat and Grosenbaugh 2001, Pegon 2001), defined as below for *free vibration of single degree of freedom system*

$$\begin{Bmatrix} \mathrm{D} \\ \Delta \mathrm{t}\dot{\mathrm{D}} \end{Bmatrix}^{\mathrm{n}+1} = [A]_{2\times 2} \begin{Bmatrix} \mathrm{D} \\ \Delta \mathrm{t}\dot{\mathrm{D}} \end{Bmatrix}^{\mathrm{n}} \tag{24}$$

Here $[A]_{2\times2}$ is amplification matrix. The numerical integration will be stable if the highest spectral radius of the amplification matrix is less than one

$$\left|\rho_{\max}\right| < 1 \tag{25}$$

Where $\rho_{\max}$ is the highest eigenvalue of $[A]$. For multi time step integrations such as IHOA and N-IHOA with accuracy order m, Eq. (24) can be written as below

$$\begin{Bmatrix} \mathrm{D} \\ \Delta t\dot{\mathrm{D}} \end{Bmatrix}^{n+1} = [A]^{(0)} \begin{Bmatrix} \mathrm{D} \\ \Delta t\dot{\mathrm{D}} \end{Bmatrix}^{n} + [A]^{(1)} \begin{Bmatrix} \mathrm{D} \\ \Delta t\dot{\mathrm{D}} \end{Bmatrix}^{n-1} + [A]^{(2)} \begin{Bmatrix} \mathrm{D} \\ \Delta t\dot{\mathrm{D}} \end{Bmatrix}^{n-2} + \cdots + [A]^{(m-1)} \begin{Bmatrix} \mathrm{D} \\ \Delta t\dot{\mathrm{D}} \end{Bmatrix}^{n-(m-1)} \tag{26}$$

In multi time step integrations, several steps are used for numerical integration of the current stage (n+1). Each previous step (n, n-1 … n-j) has different amplification matrix ($[A]^{(j)}$ j=0,1,2, …m−1). Therefore, the stability condition can be written as follows

$$\left| \rho_{\max}^{j} \right| < 1 \qquad j = 0,1,2,\cdots m-1 \tag{27}$$

Where $\rho_{\max}^{j}$ is the highest eigenvalue of $[A]^{(j)}$ It should be note that here, set of amplification matrices defines between successive steps. Compared with other existing methods (Rezaiee *et al*. 2011), the proposed approach is more compatible with multi time step integration in which several previous steps are used to integrate the equation of motion. For this reason, the stability limits, presented here are few different with results of other approaches (Rezaiee *et al*. 2011, Keierleber and Rosson 2005).

First, the amplification matrices of the IHOA and N-IHOA integrations are formulated. For this purpose, the accelerations should be removed from fundamental relationships of the IHOA and N-IHOA and replaced with equal displacements and velocities; i.e.

$$\ddot{\mathrm{D}}^{n+1} = -\omega^2 \mathrm{D}^{n+1} - 2\zeta\omega\dot{\mathrm{D}}^{n+1} \tag{28}$$

Where ζ is viscous damping ratio of structure. By substituting Eq. (28) into Eqs. (3), (4), (5) and (6), the amplification matrices of the IHOA and N-IHOA integrations are obtained as follows

$$[A]^{(0)} = Z_1 \begin{bmatrix} (1+2\eta_0\zeta\Omega)\left[1-\left(\frac{1}{2}-\sum_{i=0}^{m-1}\xi_i\right)\Omega^2\right] + 2\xi_0\zeta\left(1-\sum_{i=0}^{m-1}\eta_i\right)\Omega^3 & (1+2\eta_0\zeta\Omega)\left[1-2\left(\frac{1}{2}-\sum_{i=0}^{m-1}\xi_i\right)\zeta\Omega\right] - 2\xi_0\zeta\Omega\left[1-2\left(1-\sum_{i=0}^{m-1}\eta_i\right)\zeta\Omega\right] \\ -\eta_0\Omega^2\left[1-\left(\frac{1}{2}-\sum_{i=0}^{m-1}\xi_i\right)\Omega^2\right] - (1+\xi_0\Omega^2)\left(1-\sum_{i=0}^{m-1}\eta_i\right)\Omega^2 & (1+\xi_0\Omega^2)\left[1-2\left(1-\sum_{i=0}^{m-1}\eta_i\right)\zeta\Omega\right] - \eta_0\Omega^2\left[1-2\left(\frac{1}{2}-\sum_{i=0}^{m-1}\xi_i\right)\zeta\Omega\right] \end{bmatrix} \quad \text{IHOA}$$

$$[A]^{(j)} = Z_1 \begin{bmatrix} -(1+2\eta_0\zeta\Omega)\xi_j\Omega^2 + 2\xi_0\eta_j\zeta\Omega^3 & -2(1+2\eta_0\zeta\Omega)\xi_j\zeta\Omega + 4\xi_0\eta_j\zeta^2\Omega^2 \\ \eta_0\xi_j\Omega^4 - (1+\xi_0\Omega^2)\eta_j\Omega^2 & 2\eta_0\xi_j\zeta\Omega^3 - 2(1+\xi_0\Omega^2)\eta_j\zeta\Omega \end{bmatrix} \tag{29}$$

$$j = 1,2,\cdots m-1$$

$$[A]^{(0)} = Z_2 \begin{bmatrix} \left(1+2\gamma_0\zeta\Omega\right) - \alpha_0\left(1-\sum_{i=0}^{m-1}\gamma_i\right)\Omega^2 & \left(1+2\gamma_0\zeta\Omega\right)\left(1-\sum_{i=0}^{m-1}\alpha_i\right)+\alpha_0\left[1-2\left(1-\sum_{i=0}^{m-1}\gamma_i\right)\zeta\Omega\right] \\ -\gamma_0\Omega^2 - \left(1-\sum_{i=0}^{m-1}\gamma_i\right)\Omega^2 & \left[1-2\left(1-\sum_{i=0}^{m-1}\gamma_i\right)\zeta\Omega\right] - \gamma_0\left(1-\sum_{i=0}^{m-1}\alpha_i\right)\Omega^2 \end{bmatrix} \quad \text{N-IHOA}$$

$$[A]^{(j)} = Z_2 \begin{bmatrix} -\alpha_0\gamma_j\Omega^2 & \left(1+2\gamma_0\zeta\Omega\right)\alpha_j - 2\alpha_0\gamma_j\zeta\Omega \\ -\gamma_j\Omega^2 & -\gamma_0\alpha_j\Omega^2 - 2\gamma_j\zeta\Omega \end{bmatrix} \qquad j = 1,2,\cdots m-1 \qquad (30)$$

Where Ω is natural frequency i.e., $\Omega = \omega\Delta t$. Also parameters Z_1 and Z_2 are defined as below

$$Z_1 = \frac{1}{1+2\eta_0\zeta\Omega+\xi_0\Omega^2} \qquad (31)$$

$$Z_2 = \frac{1}{1+2\gamma_0\zeta\Omega+\alpha_0\gamma_0\Omega^2} \qquad (32)$$

Here, the weighted factors i.e., α are used from Table 1. It should be mentioned that γ=α. Also the values of quantities ξ and η are selected from the reference (Rezaiee and Alamatian 2008a). Figs. 1, 2, 3, 4 and 5 show the variation of spectral radius for different accuracy order and damping ratio. These figures show that by increasing accuracy order, the stability bounds of both IHOA and N-IHOA decrease (in constant damping ratio). Also, the undamped vibrations of both IHOA and N-IHOA methods (ζ=0.0) become unstable for any accuracy order except the first order of N-IHOA (m=1) which is *unconditionally stable*. On the other hand, Figs. 1(a), 2(a), 3(a), 4(a) and 5(a) demonstrate that by increasing damping ratio of system the stability bound of the IHOA decreases continuously. However, by increasing damping of dynamic system the stability domain of each accuracy order of the N-IHOA increases so that the highest stability is achieved when structural damping is critical (ζ=1.0). Also, the over damping vibrations i.e., ζ>1.0, cause a reduction in the stability domain of the N-IHOA (Figs. 1(b), 2(b), 3(b), 4(b) and 5(b)). As a result, the proposed integration method (N-IHOA) will be more stable and efficient than the IHOA for high damping models such as control systems. For example, if accuracy order is assumed to be 4 (m=4), the stability condition for the N-IHOA and IHOA will be Ω<3.644 and Ω<1.504, respectively when damping is critical. It is clear that in low damping systems the IHOA will be more stable than the N-IHOA.

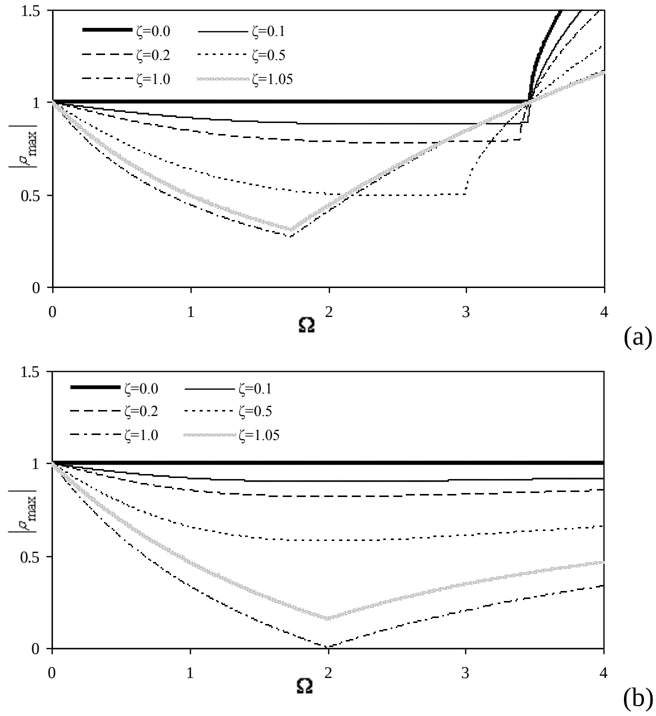


Fig. 1 Variation of spectral radius of (a) IHOA, (b) N-IHOA methods for *m*=1

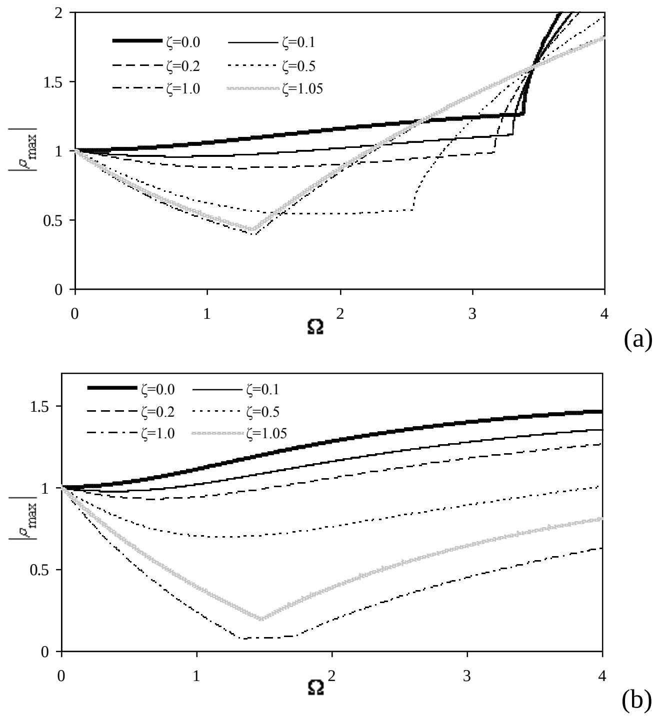
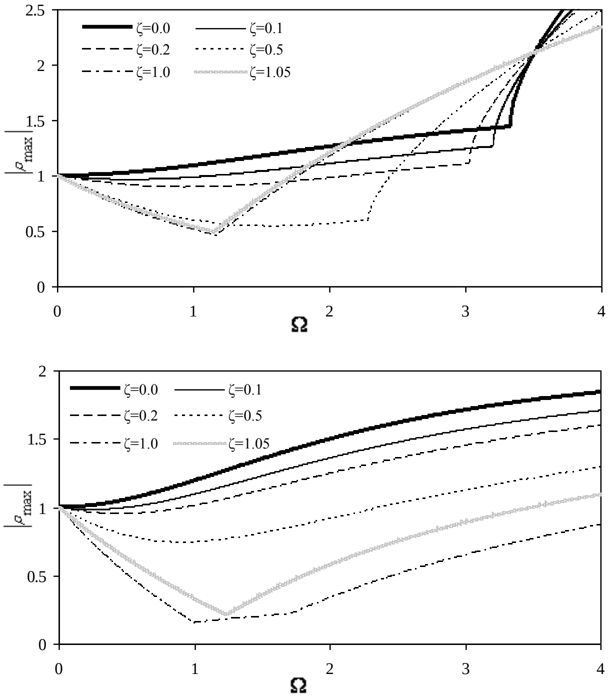


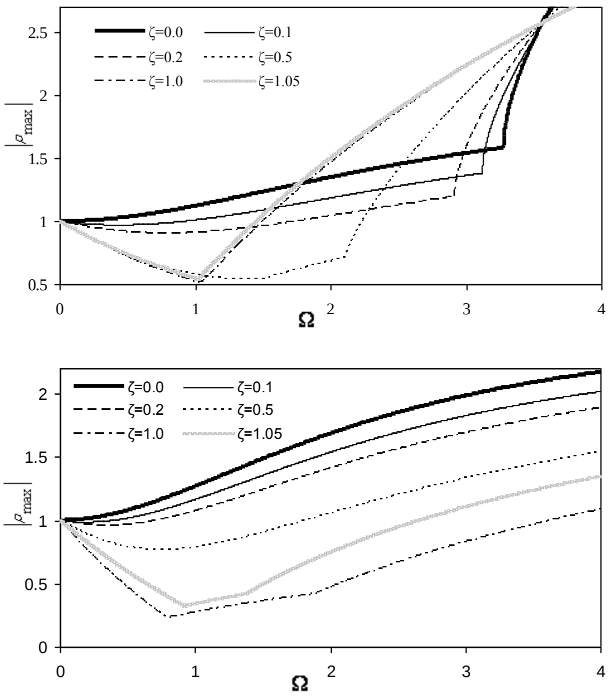
Fig. 2 Variation of spectral radius of (a) IHOA, (b) N-IHOA methods for *m*=2



(a)

(b)

Fig. 3 Variation of spectral radius of (a) IHOA, (b) N-IHOA methods for *m*=3



(a)

(b)

Fig. 4 Variation of spectral radius of (a) IHOA, (b) N-IHOA methods for *m*=4

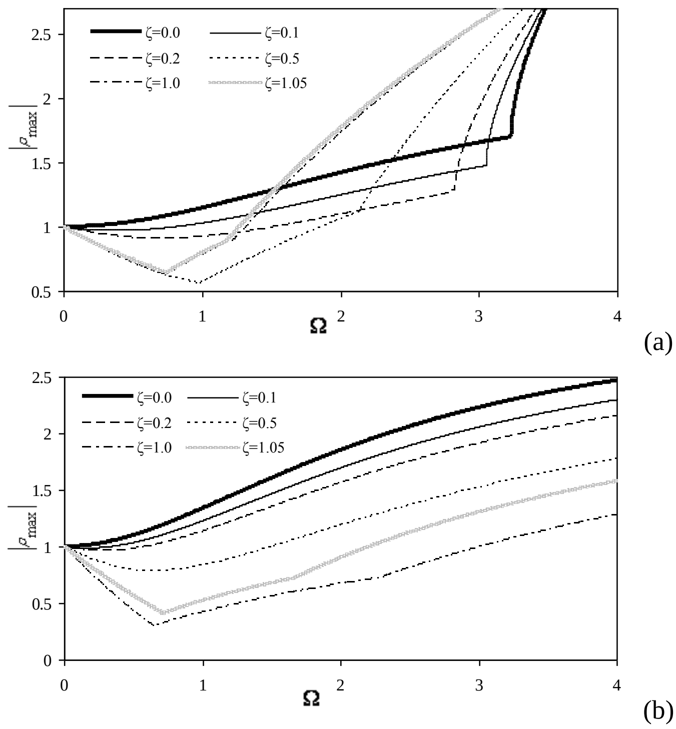


Fig. 5 Variation of spectral radius of (a) IHOA, (b) N-IHOA methods for m=5

Like the IHOA technique, in the N-IHOA formulation, the time step is assumed to be constant. If time step is variable, weighted factors should be recomputed. This procedure is very complicated and time consuming so that simplicity of the N-IHOA is destroyed. Using weighted factors of Table 1 (constant time step) for the case of variable time steps causes predictable local instabilities in the vicinity of the times in which time step varies. These local instabilities have a small effect on the overall answer, because the N-IHOA uses several previous data.

5. Numerical examples and discussion

In this section, the N-IHOA algorithm is utilized for analysis of some dynamic systems. For this propose, a computer program, using Fortran Power Station software, has been written by the author. Some bench mark problems which their exact solutions are available are solved to verify the validity of computer's program and numerical method. Wide range of dynamic systems such as linear and nonlinear, single and multi degree of freedom, damped and un-damped, free and forced from finite element and finite difference are used to compare the proposed N-IHOA integration with other existing methods. For this purpose, the results of the N-IHOA (NI) are compared with some well-known methods such as Newmark linear acceleration approach (LA), Wilson-θ (WT), trapezoidal method (CA) and also IHOA scheme.

It should be noted that in nonlinear dynamic analyses, system of Eq. (9) will be nonlinear. The explicit Dynamic Relaxation (DR) method is an efficient approach which can be utilized for

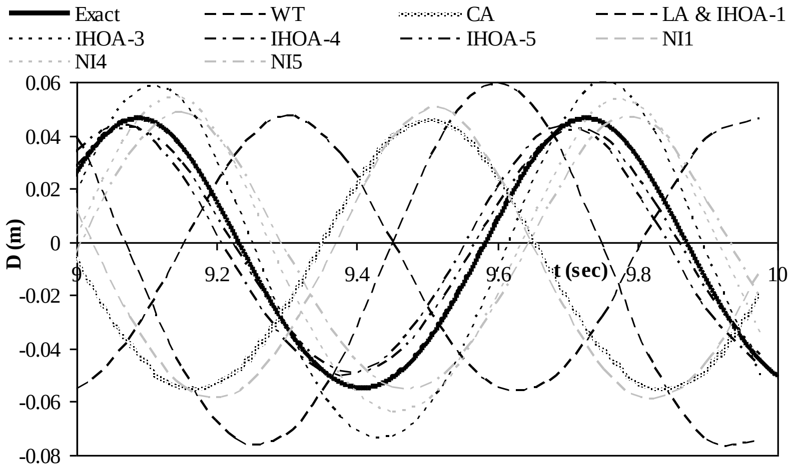


Fig. 6 Response of the nonlinear free vibration for time step 0.075 sec

solving such nonlinear systems (Kadkhodayan *et al.* 2008). As described in the recent papers, this method has been successfully combined with implicit time integrations so that it causes a considerable reduction in numerical errors (Rezaiee and Alamatian 2008c, Alamatian 2012). Simplicity, vector operators and higher efficiency in nonlinear systems are the other advantages of DR method. Here, Dynamic Relaxation method is employed to solve system of simultaneous equations (Eq. (9)) in each time step (Alamatian 2012).

5.1 The nonlinear free vibration

The nonlinear free vibration of a dynamic system with the following equation of motion and initial values is going to be solved (Mickens 2005)

$$\begin{aligned} &\ddot{D} + 100D + 500D^2 + 1000D^3 = 0 \\ &D(0) = 0 \quad \dot{D}(0) = 0.5 \end{aligned} \tag{33}$$

The quasi-exact solution is obtained by using higher order integrations with very small time step (0.0005 sec). Because the estimation of the exact period is about 0.64 second, two time steps as 0.075 and 0.025 second are utilized for the numerical dynamic analysis. For time step as 0.075 second, the IHOA-2, NI2 and NI3 are unstable however the errors of Wilson-θ (WT) and IHOA-3 are higher than other integrations (Fig. 6). Also, some methods such as LA, CA, IHOA-1 and NI1 cause a delay in vibration path. In this time step, the accuracy of the IHOA-4 and IHOA-5 integrations is more than the NI4 and NI5 methods, however, the errors of the proposed method (NI) is less than common dynamic analysis approaches such as Newmark and Wilson-θ (LA, CA and WT). This subject can also be concluded when time step reduces to 0.025 second (Fig. 7). In this case, the proposed NI3, NI4 and NI5 integrations come near to the quasi-exact solution however the IHOA-3, IHOA-4 and IHOA-5 have few more accuracy. The lack of damping in vibration of this system is the reason for this behavior. As proved mathematically (section of stability analysis), the most efficiency of the proposed NI method will be in damped vibrations.

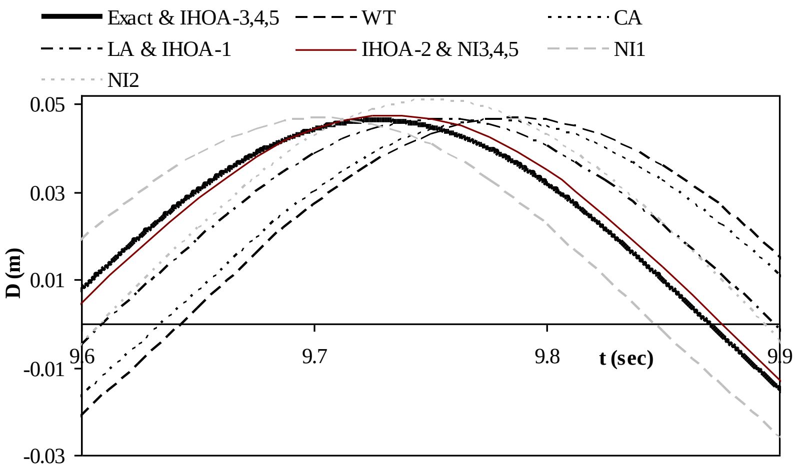


Fig. 7 Response of the nonlinear free vibration for time step 0.025 sec

5.2 Mass-spring system

A two dimensional oscillator with original length 3.0443 m has a stiffness of 328.4827 N/m (Fig. 8). It is pinned at its origin A and a mass 20.55 kg is attached to its extremity B. This extremity has a vertical initial velocity of 7.72 m/s. By using Lagrange principle (Clough and Penzien 1993), fundamental motion relationship is obtained as follows (Rezaiee and Alamatian 2008a)

$$M\ddot{x} + K\frac{\sqrt{(L+y)^2 + x^2} - L}{\sqrt{(L+y)^2 + x^2}}x = 0$$

$$M\ddot{y} + K\frac{\sqrt{(L+y)^2 + x^2} - L}{\sqrt{(L+y)^2 + x^2}}(L+y) = 0 \tag{34}$$

The quasi exact solution is obtained by selecting small time step (0.00025 second) in higher order implicit integrations (Bathe and Baig 2005). The IHOA method has been previously used to analyze this system (Rezaiee and Alamatian 2008a) so that it is possible to compare the IHOA with the N-IHOA. Here, three time step as 0.2, 0.1 and 0.05 second are utilized. Figs. 9, 10 and 11 show the displacement-time responses in the *X*, *Y* space for times between 36.4 and 40 second. For large time step (0.2 second), the NI2 and NI3 methods become unstable; however the NI4 algorithm has the less error (Fig. 9). By decreasing time step to 0.1 second, all proposed algorithms except NI2 have more accuracy than Wilson-θ (WL) and Newmark (CA and LA) approaches (Fig. 10). If time step reduced to 0.05 sec, the proposed integrations present the quasi exact solution (Fig. 11). On the other hand, Fig. 12 shows the response of both IHOA and NI algorithms for time step 0.1 second. It is clear that for high accuracy order such as fourth (IHOA-4 and NI4) and fifth (IHOA-5 and NI5), both methods approximately leads to same response however the IHOA-2 and IHOA-3 are more accurate than the NI2 and NI3, respectively. In this case, the absolute value of numerical error of both NI1 and IHOA-1 are approximately the same.

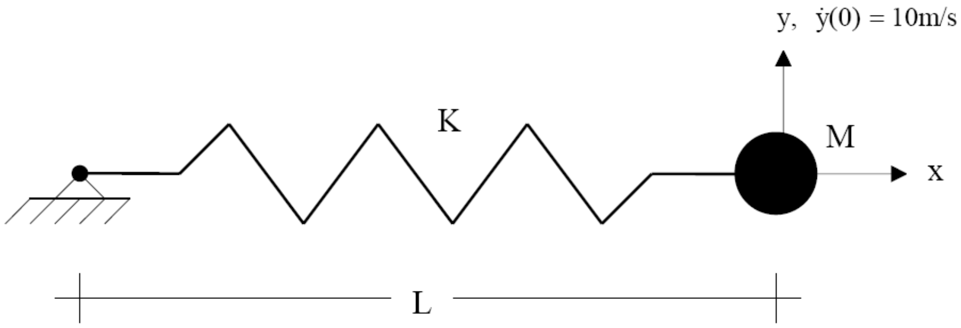


Fig. 8 Mass-spring system

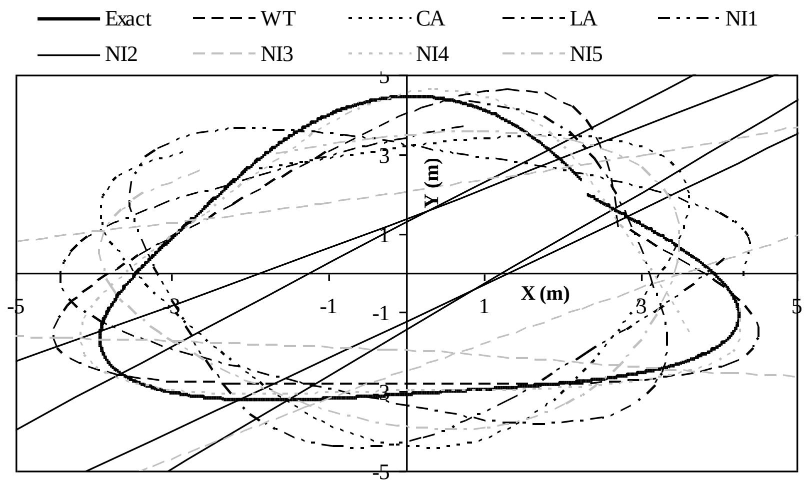


Fig. 9 Response of mass-spring system in the X-Y space for time step 0.2 sec

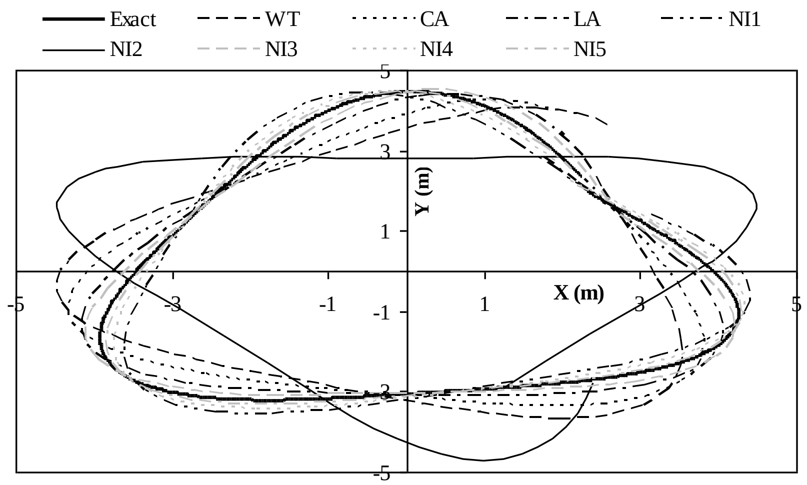


Fig. 10 Response of mass-spring system in the X-Y space for time step 0.1 sec

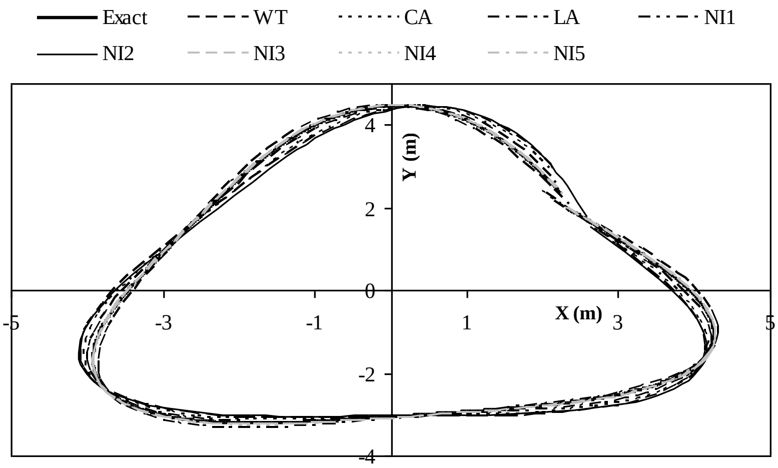


Fig. 11 Response of mass-spring system in the X-Y space for time step 0.05 sec

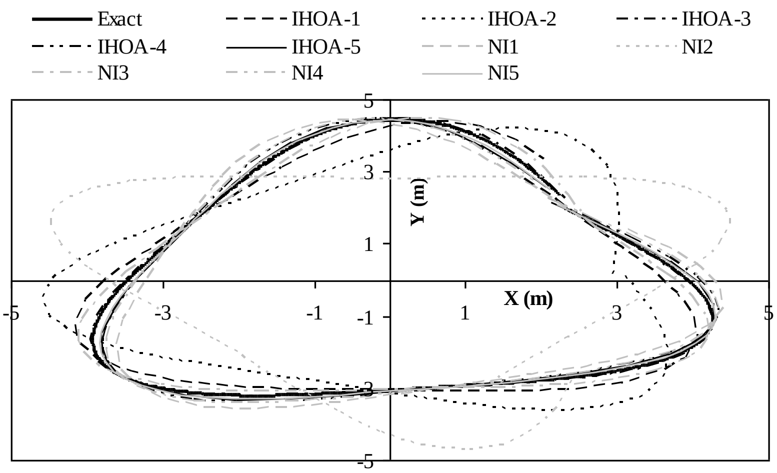


Fig. 12 Response of mass-spring system with IHOA and N-IHOA for time step 0.1 sec

5.3 Portal frame under based excitation

A concrete plane frame with two bays and three stories which shown in Fig. 13 is analyzed under El Centro based acceleration plotted in Fig. 14 (Liu *et al*. 2010). This structure has elastic geometrically nonlinear behavior and the co-rotational finite element formulation is used to model this nonlinearity (Felippa 1999). The cross section and moment of area of beams and columns are 0.40 m^2, 0.03333 m^4, 0.64 m^2 and 0.03413 m^4, respectively. In order to construct the consistent mass matrix of each beam and column of frame (Paz 1979), the mass density of concrete is assumed to be 2500 kg/m^3. This structure is analyzed in two cases, i.e., undamped and damped vibrations. In damped case, the damping factor of the first, second and third vibration modes are 20%, 15% and 10%, respectively. Figs. 15 and 16 show the quasi exact response of the horizontal

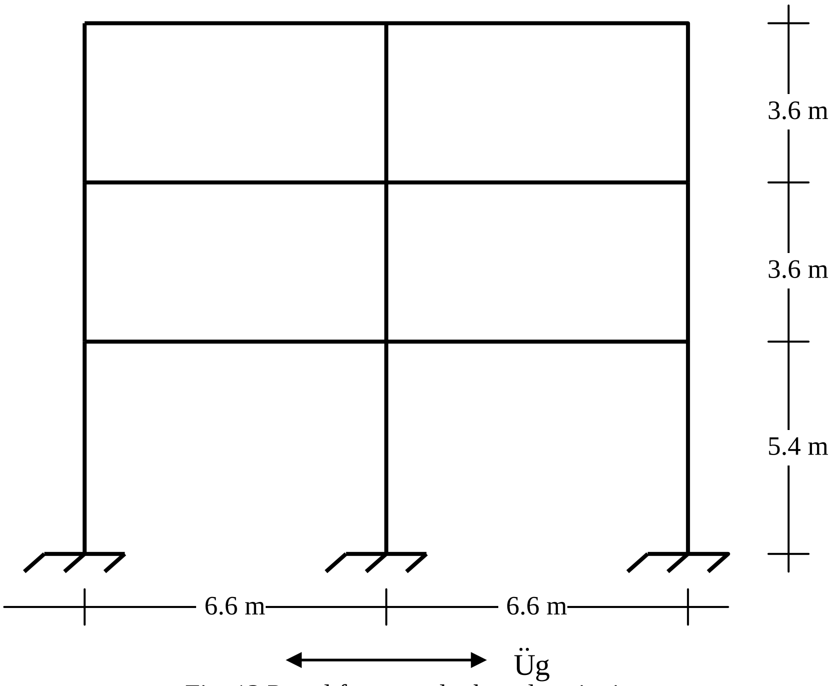


Fig. 13 Portal frame under based excitation

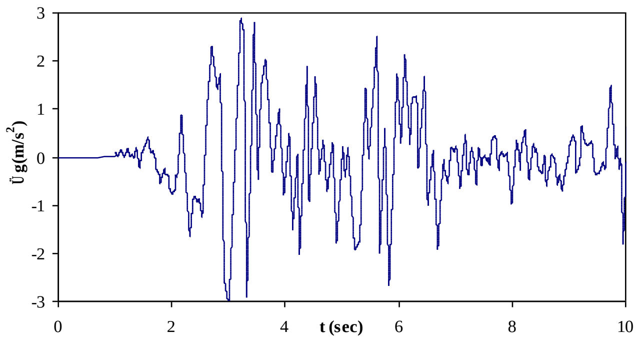


Fig. 14 El Centro ground acceleration

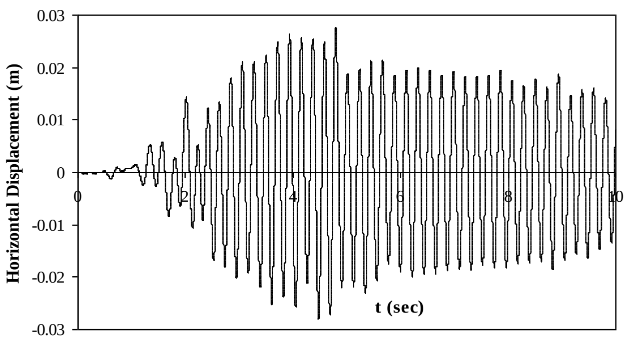


Fig. 15 Horizontal displacement of undamped vibration for top of the frame

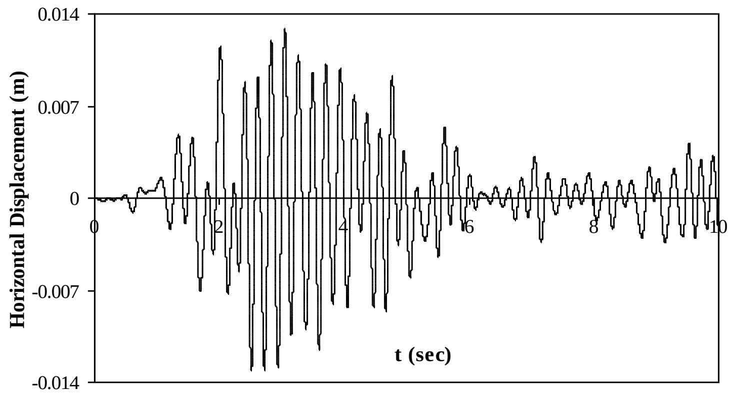


Fig. 16 Horizontal displacement of damped vibration for top of the frame

Table 2 The required time for dynamic analysis of the portal frame

Time step (sec)	Type of Vibrations	Converged Integration Method	Analysis Time (±1 sec)
0.00075	Undamped	CA, IHOA-1	73
		NI1	42
	Damped	CA, IHOA-1	73
		NI1	41
		NI4	68
0.0005	Undamped	CA, IHOA-1, 4, 5	111
		NI1	64
		NI4, 5	108
	Damped	CA, IHOA-1, 4, 5	110
		NI1	63
		NI4, 5	107

displacement of top of the frame in undamped and damped analyses, respectively. This example is utilized to compare the required analysis time of the proposed method (N-IHOA) with other existing approaches such as IHOA, Newmark-β and Wilson-θ. Since the minimum period of this structure is 0.003313 second, two time steps as 0.00075 and 0.0005 second is considered for nonlinear dynamic analysis within duration of 10 seconds. By running numerical analyses, the methods which can present the quasi exact solution are selected and the required time of each analysis is obtained. Results of these analyses have been inserted in Table 2. When time step is 0.00075 second, only three integrations i.e., CA, IHOA-1 and NI1 could present the quasi exact response of undamped vibrations. In this case, the analysis time of the IHOA-1 and NI1 is 73 and 42 seconds. Therefore, the NI1 scheme which is unconditionally stable runs with the least analysis time (the most rapid integration) compared with the other methods. This subject clearly proves the high efficiency of the proposed method especially its first accuracy order (NI1).

On the other hand, when the structure has damping, the ability of the N-IHOA is more than the IHOA. The reason for this subject can be explained when time step is 0.00075 second. In this case, the forth order of the proposed method (NI4) could present the quasi exact vibrations however the same order of the IHOA integration i.e., IHOA-4 could not. This conclusion is completely consistent with the stability analysis of the N-IHOA method presented in the previous section. By reducing time step to 0.0005 second, some integrations such as CA. IHOA-1,4,5 and NI1,4,5 could present the quasi exact solution. It is clear that in the same accuracy order, the required analysis time of proposed N-IHOA is less than IHOA. The reason for this subject is that the proposed method (NI) has only one set of integration parameters which reduces computational time. By increasing the scale of structure, the reduction of analysis time will be considerable. It should be noted that the least analysis time belongs to the NI1 method.

5.4 Euler beam

Here the vibration of clamped Euler beam is analyzed by the proposed method. The governing equation for the Euler beam's motion is as follows (Kim *et al*. 1997)

$$\rho A \frac{\partial^2 D}{\partial t^2} + \frac{\partial^2}{\partial x^2}\left(EI \frac{\partial^2 D}{\partial x^2} \right) = P(x, t) \tag{35}$$

The boundary conditions of clamped Euler beam can be written in the following form

$$D(0,t) = 0, \quad \left.\frac{\partial D}{\partial x}\right|_{x=0} = 0, \quad \left. EI\frac{\partial^2 D}{\partial x^2}\right|_{x=L} = 0, \quad \left.\frac{\partial}{\partial x}\left(EI\frac{\partial^2 D}{\partial x^2}\right)\right|_{x=L} = 0, \tag{36}$$

Also the initial conditions are as follows

$$D(x,0) = D_0(x), \quad \dot{D}(x,0) = \dot{D}_0(x) \tag{37}$$

The length of beam, material density, modulus of elasticity, cross section and moment of area are 0.508 m, 2768 kg/m^3, 6.897×10^{10} N/m^2, 6.4516×10^{-4} m^2 and 3.4686×10^{-8} m^4, respectively. The finite differences approach is utilized to obtain the dynamic equilibrium equations of Euler beam. By using one dimensional mesh and central finite differences, the dynamic equilibrium equation for i^{th} node of mesh is as follows

$$\rho A \frac{\partial^2 D_i}{\partial t^2} + EI \frac{D_{i+2} - 4D_{i+1} + 6D_i - 4D_{i-1} + D_{i-2}}{(\Delta x)^4} = P(x_i, t) \tag{38}$$

Here Δx is the distance between mesh nodes which is assumed to be constant. Here a mesh with eleven nodes (Δx=0.0508m) is considered. All boundary conditions are also expressed by central finite differences. As a result, a linear system of dynamic equations is obtained. At this stage, numerical time integrations are used to calculate the time response of beam. This structure is analyzed under a harmonic load which is applied to the free end of beam as follows

$$P(L,t) = 88.99632\sin(30t) \qquad N \tag{39}$$

The initial displacement and initial velocities are set zero. Fig. 17 shows the quasi exact vibration of free end displacement of beam achieved by very small time step i.e., 0.00001 second. Since the minimum period of beam is 0.002226 second, the analysis is started by time step 0.001 second. By this time step, a dramatically growth in numerical errors which leads to unstable

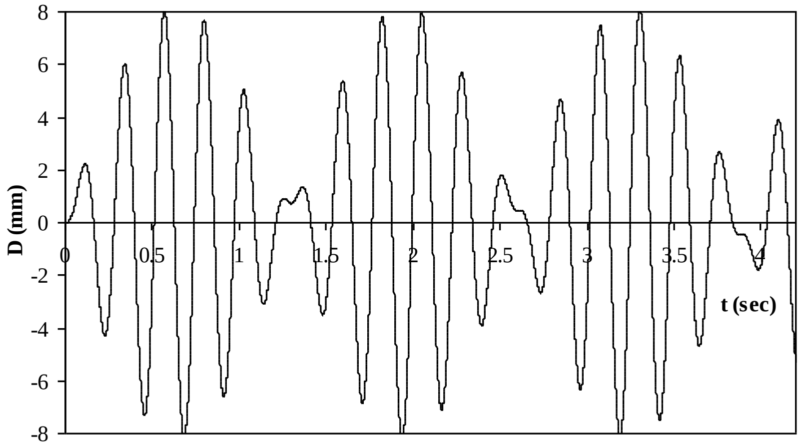


Fig. 17 Tip displacement of the Euler beam

vibrations has been occurred in all methods. Therefore, time step reduces so that each integration scheme can present the quasi exact solution. If time step is 0.0007 second, the NI1, IHOA-1, CA (Newmark Constant Acceleration) and LA (Newmark Linear Acceleration) methods present the quasi exact solution of Fig. 17. Regardless the same efficiency, the analysis time of proposed NI1 scheme is less than other techniques (as described in the portal frame). By reducing time step to 0.0005 second, furthermore the previous integrations (NI1, IHOA-1, CA and LA); the IHOA-5 also leads to the quasi exact solution. Since the NI5 method could not converge to the solution here, it is concluded that for the same accuracy order and time step size, the IHOA may be few effective than the N-IHOA in *undamped vibrations*. When time step is 0.0004 second, all methods can present the quasi exact response except the IHOA-2,3 and NI2,3; however smaller time steps such as 0.0003 second solve this problem. These analyses show that the first, fourth and fifth accuracy order of the proposed integration (NI1, NI4 and NI5) have more ability than other orders however undamped vibrations which do not appear in real systems cause a few reduction in efficiency of the N-IHOA. Moreover, this example shows that the proposed time integration can be successfully used for dynamic analysis of systems which are modeled by finite differences methods.

5.5 Elastic pendulum

Fig. 18 shows an elastic pendulum which is modeled by a two nodes truss element (Bathe 2007). This structure has large deflection nonlinearity. Here, total LaGrange finite element approach is utilized to form the nonlinear equilibrium equations. The mass matrix is consistent (Paz 1979) and the axial rigidity (AE) and material density per element length (ρA) are 10^4 N and 6.57 kg/m, respectively. By using two time steps as 0.05 sec and 0.01 sec, Bathe has been analyzed this structure in a small time domain between 0 to 5 seconds. Here, the analysis time domain is extended to 50 seconds and time step is considered as 0.05 sec. Fig. 19 shows the response of the horizontal displacement of the pendulum. The exact solution has been obtained by very small time step (0.0001 sec). By this time step, some methods such as the Wilson-θ, NI2 and NI3 have many fluctuate and they will be unstable. From Fig. 19, it is clear that the proposed NI1, NI4 and NI5

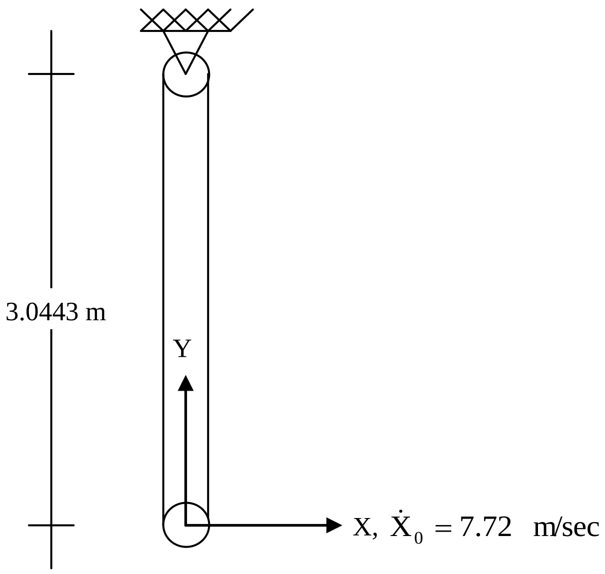


Fig. 18 Elastic pendulum

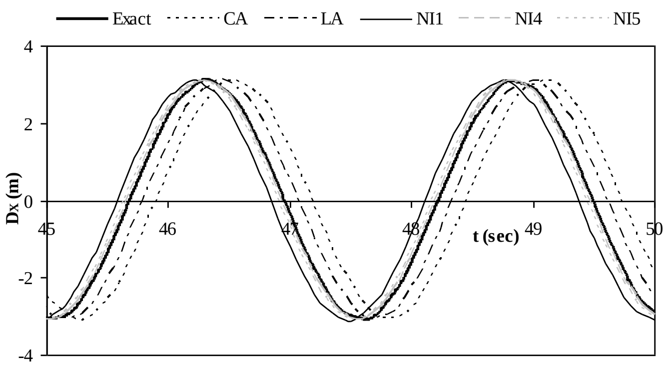


Fig. 19 Response of horizontal displacement of the elastic pendulum for time step 0.05 sec

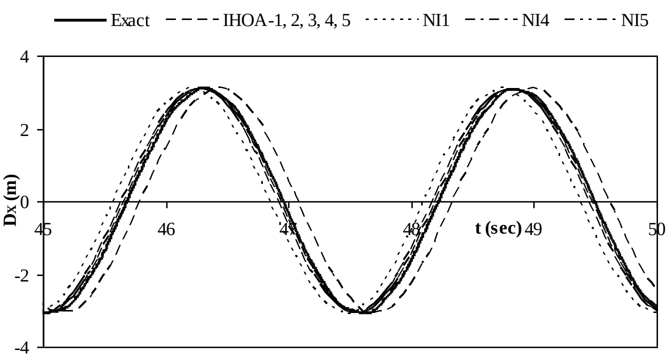


Fig. 20 Response of horizontal displacement of elastic pendulum with IHOA and N-IHOA for time step 0.05 sec

producers are more accurate than the Newmark methods so that the NI4 and NI5 integrations present the quasi-exact solution. Therefore, the proposed time integration has very good efficiency in nonlinear dynamic analysis. Moreover, this example is utilized to compare the accuracy of the IHOA and NI integrations. By using time step as 0.05 sec, all accuracy order of the IHOA procedure (IHOA-1,2,3,4 and 5) present unique response which has been shown in Fig. 20. From Fig. 20 it is concluded that the proposed NI1, NI4 and NI5 procedures have more accuracy than the IHOA integrations. It should be noted that by increasing time step, i.e., 0.075 sec., the proposed time integrations (NI methods) will be unstable, however the IHOA procedures are stable and can present the response by some few errors.

6. Conclusions

The new implicit higher order accuracy method i.e. N-IHOA which followed by a comprehensive study on its stability and accuracy was proposed here for dynamic analysis with constant time step. This scheme uses the velocities and accelerations of previous time steps to integrate the displacement and velocity of the current step, respectively. Because of existing only one set of weighted factors, the algorithm of the N-IHOA is very simpler than the previous implicit higher order integration, IHOA, which has two groups of integration's parameters. Moreover, the stability analysis which was performed based on the amplification matrices shows that the proposed method has high efficiency in damped vibrations such as control systems. This study also proves that the first order of the proposed integration i.e. NI1 is unconditionally stable. Wide range of numerical examples (single/multi degrees of freedom, damped/un-damped, free/forced vibrations from finite element/finite difference) show that the efficiency and accuracy of the proposed integration with orders one (NI1), four (NI4), and five (NI5) are more than the other orders. Also, in the same accuracy order, the analysis time of the N-IHOA is less than the IHOA however their numerical stability and accuracy are approximately the same. Another important point is that the proposed NI1 method is the most rapid integration so that its analysis time to achieve the same accuracy is very less than other schemes. It should be noted that the N-IHOA integration has more accuracy than the common methods such as Wilson-θ and Newmark-β.

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Appendix. The matrix [Z] for the integration's orders 1, 2, 3, 4 and 5

$$[Z]_{1\times1}=\begin{bmatrix}1\end{bmatrix} \qquad [Z]_{2\times2}=\begin{bmatrix}1 & -1\\ 0.5 & 0.5\end{bmatrix} \qquad [Z]_{3\times3}=\begin{bmatrix}1 & -1 & -2\\ 0.5 & 0.5 & 2\\ 0.1667 & -0.1667 & -1.3333\end{bmatrix}$$

$$[Z]_{4\times4}=\begin{bmatrix}1 & -1 & -2 & -3\\ 0.5 & 0.5 & 2 & 4.5\\ 0.1667 & -0.1667 & -1.3333 & -4.5\\ 0.0417 & 0.0417 & 0.6667 & 3.375\end{bmatrix} \qquad [Z]_{5\times5}=\begin{bmatrix}1 & -1 & -2 & -3 & -4\\ 0.5 & 0.5 & 2 & 4.5 & 8\\ 0.1667 & -0.1667 & -1.3333 & -4.5 & -10.6667\\ 0.0417 & 0.0417 & 0.6667 & 3.375 & 10.6667\\ .0083 & -.0083 & -.2667 & -2.025 & -8.5333\end{bmatrix}$$